A GENERALIZATION OF THE TRISTRAM-LEVINE KNOT SIGNATURES AS A SINGULAR FURUTA-OHTA INVARIANT FOR TORI (ANNOTATED VERSION)

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Abstract. Given a knot K inside an integer homology sphere Y, the Casson-Lin-Herald invariant can be interpreted as a signed count of conjugacy classes of irreducible representations of the knot complement into $SU(2)$ mapping the meridian of the knot to a fixed conjugacy class, which are picked out by the Alexander polynomial $\triangle_K$ of the knot. It has the interesting feature that it determines the Tristram-Levine signature of the knot associated to the conjugacy class chosen.

Turning things around, given a 4-manifold X with the integral homology of $S^1 \times S^3$ and an embedded torus T inside X such that $H_1(T;\mathbb{Z})$ surjects onto $H_1(X;\mathbb{Z})$, we define a signed count of conjugacy classes of irreducible representations of the torus complement into $SU(2)$ which satisfy an analogous fixed conjugacy class condition to the one mentioned above for the knot case, and which are also picked out by the Alexander polynomial of the torus $\triangle_T$. Our count recovers the Casson-Lin-Herald invariant of the knot in the product case, i.e, when $X = S^1 \times Y$ and $T = S^1 \times K$, whenever we choose an allowable conjugacy class. Therefore, our invariant can be regarded as implicitly defining a Tristram-Levine signature for tori.

This invariant can also be considered as a singular Furuta-Ohta invariant. In fact, it can be regarded as a special case of a larger family of Donalson type invariants we also define. In particular, when the pair (X, T) is obtained from a self-concordance of a knot (Y, K) which satisfies a certain admissibility condition, these collection of invariants can be determined in terms of a Frøyshov type invariant and the Lefschetz number of an Instanton Floer homology for knots we also define in this paper.

1. Introduction

An observation of some analogies: defining $\lambda_{FO}(X, T, \alpha)$.

The main impetus behind this work stems from the following observation: given an integer homology sphere Y, Casson defined [1] an invariant $\lambda_C(Y) \in \mathbb{Z}$ which morally can be regarded as $1/2$ of a signed count of conjugacy classes of irreducible representations $\pi_1(Y) \to SU(2)$. Moving one dimension up, if X is a four manifold with the same integral homology as $S^1 \times S^3$, i.e, $H_*(X;\mathbb{Z}) \simeq H_*(S^1 \times S^3;\mathbb{Z})$, Furuta and Ohta defined [28] a Casson-type invariant $\lambda_{FO}(X) \in \mathbb{Z}$ which again can morally be interpreted as a signed count of conjugacy classes of irreducible representations $\pi_1(X) \to SU(2)$. Strictly speaking, X must satisfy additional conditions besides being a homology $S^1 \times S^3$, but it will always be the case that if one takes $X = S^1 \times Y$, for Y an arbitrary integer homology sphere, then $\lambda_{FO}(X)$ is always well defined, and modulo rescaling and orientation conventions, can be made to agree with $\lambda_C(Y)$.

Now, if we consider the case of a knot K inside Y, one can follow the same strategy and try to define a signed count of conjugacy classes of irreducible representations $\pi_1(Y\backslash K) \to SU(2)$. As will become clear soon, in this case it is natural to fix the conjugacy class where the meridian of the knot μ_K is being sent, regarded as the generator of the reducible representations $H_1(Y\backslash K) \simeq \mathbb{Z}[\mu_K] \to SU(2)$. This conjugacy class is specified by a (holonomy) parameter $\alpha \in (0, 1/2)$, and whenever $\triangle_K(e^{-4\pi i\alpha}) \neq 0$, where $\triangle_K$ is the Alexander polynomial of the knot, Herald showed [32, Theorem 0.1] that a Casson-type count of representations can be made, which we will denote as $\lambda_{CLH}(Y, K, \alpha) \in \mathbb{Z}$. Here the L in λ_{CLH} stands for Lin, who had studied before Herald [54] the case where the meridian is sent to a trace zero matrix, which corresponds to the parameter $\alpha = 1/4$ (and using a symplectic rather than gauge theoretic approach).

The invariant λ_{CLH} recovers the Tristram-Levine knot signatures σ_K [75, 52], in the sense that (with our orientation conventions) $\lambda_{CLH}(Y, K, \alpha) = 4\lambda_C(Y) + \frac{1}{2}\sigma_K(e^{4\pi i\alpha})$. This clearly begs the question:

Question 1. *Given an embedded torus T inside a four manifold X with the integer homology of $S^1 \times S^3$ and a number $\alpha \in (0, 1/2)$, which conditions must be imposed on T and α so that one can define an invariant $\lambda_{FO}(X, T, \alpha) \in \mathbb{Z}$ that satisfies the property that for any knot $K \subset Y$ and α such that $\triangle_K(e^{-4\pi i\alpha}) \neq 0$, $\lambda_{FO}(S^1 \times Y, S^1 \times K, \alpha)$ coincides with $\lambda_{CLH}(Y, K, \alpha)$ (perhaps up to rescaling)?*

As the reader may expect, our answer to this question is affirmative given some conditions on the embedding of T and the Alexander polynomial of the torus $\triangle_T$.

Before stating our result, we need to explain some additional context and notation to make the statement more natural, although the reader familiarized with the singular instanton Floer homology defined by Kronheimer and Mrowka in [45] and the Furuta-Ohta papers of Ruberman and Saveliev [68, 67] may safely skip the next paragraphs and read immediately the statement of Theorem 4 for an answer to this question.

Recall that in order to define the Casson invariant $\lambda_C(Y)$, one needs first to understand the reducible representations of $\pi_1(Y) \to SU(2)$, given that they correspond to singular points in the character variety $\mathcal{R}(Y, SU(2)) = \hom(\pi_1(Y), SU(2))/SU(2)$ (or the space of flat $SU(2)$ connections mod gauge depending on one's preference).

Reducible representations factorize through the abelianization of $\pi_1(Y)$, so in other words one needs to understand first $\hom(H_1(Y), SU(2))/SU(2)$. In the case of an integer homology sphere, up to conjugacy there will be only one such reducible representation, which we will denote as θ, and we will think of it as representing the trivial $SU(2)$ connection associated to the trivial $SU(2)$ bundle $E = Y \times \mathbb{C}^2 \to Y$. In general, every representation $\rho : \pi_1(Y) \to SU(2)$ gives rise to a local coefficient system , which we denote as $\mathfrak{g}_\rho$, but in the case of the trivial connection this is the same as the trivial coefficient system $\mathfrak{g}_\theta \simeq \mathbb{R}^3$. Since the Zariski tangent space of ρ inside $\mathcal{R}(Y, SU(2))$ can be identified with $H^1(Y; \mathfrak{g}_\rho)$, in the case of the trivial representation it will be zero dimensional since

$$H^1(Y; \mathfrak{g}_\theta) = H^1(Y; \mathbb{R}^3) = H^1(Y; \mathbb{R}) \otimes \mathbb{R}^3 = 0$$

Moreover, by Poincare duality $H^2(Y; \mathfrak{g}_\theta) \simeq H^1(Y; \mathfrak{g}_\theta) = 0$, that is, the trivial connection is non-degenerate as well. This means that trivial representation ρ_θ is isolated from the subset of irreducible representations $\mathcal{R}^*(Y, SU(2)) \subset \mathcal{R}(Y, SU(2))$. Since the latter is a compact space, the isolation of ρ_θ means that $\mathcal{R}^*(Y, SU(2))$ is compact as well, making it possible to define $\lambda_C(Y)$ (after introducing perturbations if necessary in order to deal with the fact that the irreducible representations may not be isolated from each other).

Moving one dimension up, for X a homology $S^1 \times S^3$, we have that $H_1(X; \mathbb{Z}) = \mathbb{Z}$, so

$$\hom(H_1(X), SU(2))/SU(2) = \hom(\mathbb{Z}, SU(2))/SU(2) \simeq [-2, 2]$$

where the last identification uses the fact that conjugacy classes of $SU(2)$ are completely specified by the trace $-2 \leq \mathrm{tr}(A) \leq 2$ of a matrix $A \in SU(2)$.

The endpoints of this interval correspond to the trivial representation and a twist of it which will be discussed later: these have an $SO(3)$ stabilizer. The points in the interior $(-2,2)$ can be represented by non-trivial $U(1)$ reducible representations ρ (i.e, a reducible representation whose stabilizer is not $SO(3)$).

The moral of this story is that for X the space of reducible representations does not consist of isolated points, but rather appear in moduli. In particular, if we want to define the quantity $\lambda_{FO}(X)$, X must have the property that for every non-trivial $U(1)$ reducible representation ρ its Zariski tangent space $H^1(X;\mathfrak{g}_\rho)$ is one dimensional (so that no infinitesimal deformations near ρ can move it near to the space of irreducible representations).

For a $U(1)$ representation the $SU(2)$ bundle $E \to X$ decomposes as $E = L_\rho \oplus L_\rho^{-1}$, where each summand is parallel with respect to the flat connection A_ρ (associated to ρ via the holonomy representation), and the local system decomposes as $\mathfrak{g}_\rho \simeq \mathbb{R} \oplus L_\rho^{\otimes 2}$, so the desire that

$$H^1(X;\mathfrak{g}_\rho) \simeq H^1(X;\mathbb{R}) \oplus H^1(X;L_\rho^{\otimes 2}) = \mathbb{R} \oplus H^1(X;L_\rho^{\otimes 2})$$

be one-dimensional translates into the following condition[1] [36, Remark 7.4]:

Claim 2. Suppose that X is a compact, oriented 4-manifold with the integral homology of $S^1 \times S^3$. If for every non-trivial $U(1)$ representation ρ one has that $H^1(X;L_\rho^{\otimes 2})$ vanishes, then the space of reducible representations $\mathcal{R}^{red}(X,SU(2))$ is isolated from $\mathcal{R}^*(X,SU(2))$ inside $\mathcal{R}(X,SU(2))$. Hence, since $\mathcal{R}(X,SU(2))$ is compact so will be $\mathcal{R}^*(X,SU(2))$ and an invariant $\lambda_{FO}(X)$ can be defined as a signed count of the elements inside $\mathcal{R}^*(X,SU(2))$, after adding perturbations if necessary in order to have only finitely many elements to count, and choosing an orientation convention (determined by an orientation of $H^1(X;\mathbb{R})$) which attaches a sign to the representations.

As an example, Furuta and Ohta observed [28] that the previous condition is always satisfied when the infinite cyclic cover $\tilde{X}$ of X is homologically a 3-sphere, i.e, $H_*(\tilde{X};\mathbb{Z}) = H_*(S^3;\mathbb{Z})$.

It is important to observe that the perturbations being used are slightly different from those one might expect. Namely, an $SU(2)$ bundle E over X is classified topologically just by one number, the second Chern number $k = c_2(E)$ of the bundle. If $\mathcal{A}(E_k)$ denotes the space of $SU(2)$ connections associated to E_k ,which is affine over $\Omega^1(X,\mathfrak{g}_E)$, then the Chern-Weil formula says that for $A \in \mathcal{A}(E_k)$

$$8k = \frac{1}{8\pi^2}\int_X \operatorname{tr}(F_A \wedge F_A) = \frac{1}{8\pi^2}\left(\|F_A^-\|_{L^2(X)} - \|F_A^+\|_{L^2(X)}\right)$$

so for the trivial $SU(2)$ bundle $E_0 = X \times \mathbb{C}^2 \to X$, which corresponds to $k=0$, we have that

$$(k=0) \implies \|F_A^-\|_{L^2(X)} = \|F_A^+\|_{L^2(X)}$$

In particular, for $k=0$, A is anti-self-dual (i.e, $F_A^+ \equiv 0$) if and only if A is a flat connection (i.e, $F_A \equiv 0$). Therefore, we should really think of $\lambda_{FO}(X)$ as a degree-zero Donaldson invariant $D_0(X)$, in the sense that the equation that is being perturbed in order to make the count required for $\lambda_{FO}(X)$ is the ASD equation $F_A^+ = 0$, not the flat equation $F_A = 0$. Before continuing to the case of knots, it is useful to observe that the condition $H^1(X;L_\rho^{\otimes 2}) = 0$ needed to define $\lambda_{FO}(X)$ can also be described in terms of the Alexander polynomial of X:

[1]Strictly speaking we should also mention the non-degeneracy condition but we will defer this discussion until section 6, where we will analyze this in the context of $\lambda_{FO}(X,T,\alpha)$, which is what we really care about.

Remark 3. Suppose that X is a compact, oriented 4-manifold with the integral homology of $S^1 \times S^3$. Let $\hat{\rho} \in \hom(X, \mathbb{C}^*)$ be the character determined by a non-trivial $U(1)$ representation ρ. Then $\lambda_{FO}(X)$ can be defined whenever each such ρ satisfies the condition $\triangle_X(\hat{\rho}) \neq 0$, where $\triangle_X$ denotes the Alexander polynomial of X.

In the appendix we recall the definition of the Alexander polynomial of an arbitrary topological space X, and how it is related to the cohomology groups $H^1(X; \mathbb{C}^{\hat{\rho}})$ determined by a monodromy map (character) $\hat{\rho} \in \hom(X, \mathbb{C}^*)$. The interpretation of $\triangle_X(\hat{\rho}) \neq 0$ is also explained in the appendix, where we considering $\triangle_X$ as the greatest common divisor of the Alexander ideal, so that $\triangle_X(\hat{\rho}) \neq 0$ means that $\hat{\rho}$ does not belong to the zero locus (variety) of this ideal.

Returning to the case of a knot K inside Y, since $H_1(Y \backslash K) \simeq \mathbb{Z}[\mu_K]$, the space of reducible representations $\pi_1(Y \backslash K) \to SU(2)$ is non-isolated, so it seems that we are in a situation which is more like that of X. Therefore, to make the analogy with the case of Y hold, we impose a condition on the conjugacy class where the meridian μ_K is mapped, namely, we want to consider representations $\rho : \pi_1(Y \backslash K) \to SU(2)$ which send the meridian to the conjugacy class of the matrix

$$\mu_K \to \begin{pmatrix} e^{-2\pi i\alpha} & 0 \\ 0 & e^{2\pi i\alpha} \end{pmatrix}$$

Notice that this matrix has trace $2\cos(2\pi\alpha)$, so for $0 < \alpha < 1/2$, we exhaust all conjugacy classes, expect for those which have trace equal to ± 2, but which we exclude from our analysis since those are equivalent to representations of $\pi_1(Y)$ into $SU(2)$ (or $SO(3)$), and thus not interesting if we want the knot to play an important role. Denote $\mathcal{R}_\alpha(Y \backslash K, SU(2))$ the space of such representations.

Notice that it is important to have oriented the meridian μ_K, since $-\begin{pmatrix} e^{-2\pi i\alpha} & 0 \\ 0 & e^{2\pi i\alpha} \end{pmatrix}$ is not conjugate to $\begin{pmatrix} e^{-2\pi i\alpha} & 0 \\ 0 & e^{2\pi i\alpha} \end{pmatrix}$, unless $\alpha = 1/4$. Since we already assumed that Y has been oriented, once an orientation for K has been chosen, there is a natural choice of orientation for μ_K, as will discuss in greater detail in the next section.

In any case, the point is that our choice of α means that there is only one reducible representation θ_α (up to conjugacy) inside $\mathcal{R}_\alpha(Y \backslash K, SU(2))$, and in order to try to define a count $\lambda_{CLH}(Y, K, \alpha)$ we need to guarantee that θ_α is isolated. The reducible θ_α is a non-trivial $U(1)$ representation, in other words, for the trivial $SU(2)$ bundle $E = (Y \backslash K) \times \mathbb{C}^2 \to Y \backslash K$ it induces a parallel splitting $E = L_\alpha \oplus L_\alpha^{-1}$ and a corresponding local system $\mathfrak{g}_{\theta_\alpha} \simeq \mathbb{R} \oplus L_\alpha^{\otimes 2}$. In this case [2]

$$H^1(Y \backslash K; \mathfrak{g}_{\theta_\alpha}) \simeq H^1(Y \backslash K; \mathbb{R}) \oplus H^1(Y \backslash K; L_\alpha^{\otimes 2}) = \mathbb{R} \oplus H^1(Y \backslash K; L_\alpha^{\otimes 2})$$

Now, the first $\mathbb{R}$ factor is not a concern, since it arises from deforming the value of α, which we are not allowed to do given that working within $\mathcal{R}_\alpha(Y \backslash K, SU(2))$. Therefore, to guarantee that θ_α is isolated inside $\mathcal{R}_\alpha(Y \backslash K, SU(2))$ it suffices to require that $H^1(Y \backslash K; L_\alpha^{\otimes 2})$ vanishes (an alternative justification of this fact is given in Lemma 21 and the remarks before that). In any case, the vanishing $H^1(Y \backslash K; L_\alpha^{\otimes 2})$ is equivalent to $\triangle_K(e^{-4\pi i\alpha}) \neq 0$. Under these conditions $\lambda_{CLH}(Y, K, \alpha)$ can be defined, which as we mentioned before agrees with $4\lambda_C(Y) + \frac{1}{2}\sigma_K(e^{-4\pi i\alpha})$.

To complete the analogy, let's return to the case of an embedded torus T inside X. To study the representations of $\pi_1(X \backslash T)$ into $SU(2)$, we will use the framework of Kronheimer and Mrowka's papers [41, 43] on singular gauge theory. We will review the novel features of this approach in the next section, but the key points are as follows. For pairs (X, Σ) consisting of a closed, oriented four

[2] For this analysis having worked with $Y \backslash \text{nbd}(K)$ instead of $Y \backslash K$ would have led to exactly the same conclusion.

manifold X and an oriented surface Σ, the $SU(2)$ bundles E over (X, Σ) are now classified by a pair of two integers k, l, called the instanton and monopole numbers respectively.

Write $E(k,l)$ for the corresponding $SU(2)$ bundle associated to the pair (k,l). Then for each such $E(k,l)$, we can study the space of connections $\mathcal{A}(E(k,l),\alpha)$ which have a prescribed singular behavior along Σ. As before $0 < \alpha < 1/2$, and the case of the anti-self-dual connections $F_A^+ = 0$ for $\mathcal{A}(E(0,0),\alpha)$ (mod gauge) can again be interpreted as corresponding to representations of $\pi_1(X \backslash T) \to SU(2)$ which map the "meridian" of the torus μ_T to a specific conjugacy class, if we take $\Sigma = T$. In general, the solutions of $F_A^+ = 0$ for $A \in \mathcal{A}(E(k,l),\alpha)$ will be called α-ASD connections. The moduli space of (perturbed) α-ASD connections modulo gauge will be denoted $\mathcal{M}(X,\Sigma,k,l,\alpha)$. Finally, a gauge equivalence class will normally be denoted as $[A]$, instead of A.

In order to have the appropriate compactness theorems, for example, that a sequence of α-ASD connections $[A_i]$ on $\mathcal{M}(X,\Sigma,k,l,\alpha)$ will converge weakly (due to Uhlenbeck bubbling) to a connection $[A_\infty]$ on a bundle $E(k',l')$ whose corresponding moduli space $\mathcal{M}(X,\Sigma,k',l',\alpha)$ has *smaller* expected dimension than that of $\mathcal{M}(X,\Sigma,k,l,\alpha)$, we need to use an *orbifold* metric (or a more general conical metric) along Σ. This means that there will be an integer parameter ν such that the orbifold metric along Σ has a cone angle of $2\pi/\nu$ (again, more details are given in the next section). Each α determines a set of allowable cone parameters $\{\nu\}$, but there is no cone parameter that works simultaneously for all values of α.

Moreover, a particular choice of cone parameter ν determines what a priori is a different moduli space $\mathcal{M}(X,\Sigma,k,l,\alpha,\nu)$, although as we will state in the final section of the introduction, we expect the invariants we define to be independent of the particular cone parameter ν used (so long as it is compatible with α), so throughout the paper we will just write this as $\mathcal{M}(X,\Sigma,k,l,\alpha)$. Also, for reasons which will become more clear as the paper progresses, we will also choose a *rational* value of α, i.e, $\alpha \in \mathbb{Q} \cap (0,1/2)$.

Back to the torus T, we need to understand its complement $X \backslash T$, since $H_1(X\backslash T;\mathbb{Z})$ will determine the reducible representations. The natural condition in our context is to assume that $H_1(T;\mathbb{Z})$ surjects onto $H_1(X;\mathbb{Z})$, so that homologically T is like S^1 times a knot, in the sense that $H_*(X\backslash \mathrm{nbd}(T);\mathbb{Z}) \simeq H_*(S^1 \times S^1 \times D^2;\mathbb{Z})$, where $\mathrm{nbd}(T)$ denotes a small tubular neighborhood of T (this can be seen from Mayer-Vietoris).

Now that the stage has been finally been set we can state the answer to our first question 1:

Theorem 4. *(Definition of the singular Furuta-Ohta invariant) Suppose that X is a closed oriented four manifold with the integral homology of $S^1 \times S^3$. Let T denote an embedded and oriented torus such that $H_1(T;\mathbb{Z}) \twoheadrightarrow H_1(X;\mathbb{Z})$ is a surjection. Consider a rational value $\alpha \in \mathbb{Q} \cap (0,1/2)$ such that for all $U(1)$-representations $\rho : \pi_1(X\backslash T) \to SU(2)$ satisfying the holonomy condition 9 (explained in the next section), we have that $H^1(X\backslash T; L_\rho^{\otimes 2})$ vanishes, where $E = L_\rho \oplus L_\rho^{-1}$ is the decomposition induced by the corresponding connection A_ρ, and $E = E(0,0)$ is the $SU(2)$ bundle over (X,T) corresponding to vanishing instanton and monopole numbers.*

Choose an orbifold metric along T with cone angle $2\pi/\nu$, where ν is a cone parameter compatible with α. Then we can define a "degree-zero" Donaldson invariant, denoted the ***singular Furuta-Ohta invariant*** *$\lambda_{FO}(X,T,\alpha,\nu)$, which is as a signed-count of points inside the moduli space $\mathcal{M}(X,T,0,0,\nu)$, a zero dimensional oriented compact manifold.*

It has the property that for $X = S^1 \times Y$, $T = S^1 \times K$, for K inside an integer homology sphere Y, and a value of α such that $\triangle_K(e^{-4\pi i \alpha}) \neq 0$, then $\lambda_{FO}(S^1 \times Y, S^1 \times K, \alpha, \nu)$ can be defined, and we have that $\lambda_{FO}(S^1 \times Y, S^1 \times K, \alpha, \nu) = 2\lambda_{CLH}(Y,K,\alpha)$.

Remark 5. a) The condition that $H^1(X\backslash T; L_\rho^{\otimes 2})$ vanishes can be equivalently stated in terms of the Alexander polynomial $\triangle_T = \triangle_{X\backslash T}$ of the knot complement. Namely, we would require that $\triangle_T(\hat{\rho}) \neq 0$ for all the characters associated to the $U(1)$ -representations ρ which satisfy the holonomy condition determined by α.

b) To be precise in our theorem we stated the dependence of the invariants on ν, however, as we mentioned before we expect them to be independent of the particular value of cone angle chosen, which is why we will continue to write them as $\lambda_{FO}(X, T, \alpha)$ from now on. In those cases where no perturbations are needed, i.e, all irreducible representations $\pi_1(X\backslash T) \to SU(2)$ satisfying the holonomy condition specified by α are isolated and non-degenerate, then $\lambda_{FO}(X, T, \alpha)$ truly corresponds to a signed count of these representations. While the set itself does not depend on ν, the sign assigned to the representation could a priori depend on ν, although again we do not expect this to be the case.

c) To explain the factor of 2 in our formula, it is useful to compare this to the case where the knot and torus are not present, and also no perturbations are needed. In this situation the Casson invariant $\lambda_C(Y)$ is $\frac{1}{2}$ the signed count of conjugacy classes of irreducible representations $\pi_1(Y) \to SU(2)$. On the other hand, using the normalization conventions of Ruberman and Saveliev for $\lambda_{FO}(X)$ in [68, Section 4.3], we have that $\lambda_{FO}(X)$ equals $\frac{1}{4}$ the signed count of conjugacy classes of irreducible representations $\pi_1(X) \to SU(2)$.

When $X = S^1 \times Y$ one has that $\lambda_{FO}(X) = \lambda_C(Y)$ using their normalization [68, eq. 5], which means that there are twice as many (conjugacy classes of) irreducible representations $\pi_1(X) \to SU(2)$ as there are for $\pi_1(Y) \to SU(2)$. The same happens in our situation (as we will explain in Section 6 when we define $\lambda_{FO}(X, T, \alpha)$), but we chose not to normalize neither $\lambda_{FO}(X, T, \alpha)$ nor $\lambda_{CLH}(Y, K, \alpha)$, which accounts for the difference of 2.

d) To orient the moduli spaces we need a choice of homology orientation. As we will explain in Section 6, in our case a homology orientation is determined by an orientation of $H^1(X; \mathbb{R})$. Since we assumed that T is oriented and $H_1(T; \mathbb{Z}) \twoheadrightarrow H_1(X; \mathbb{Z})$ is a surjection, there is a natural orientation of $H^1(X; \mathbb{R})$ coming from that of T.

e) In section 8 we will give some examples of $\lambda_{FO}(X, T, \alpha)$, which arise from the mapping tori associated to finite group actions on homology spheres, as well as certain circle bundles over a 3-manifold with the homology of $S^1 \times S^2$. It will become clear that these are the orbifold versions of the calculations of $\lambda_{FO}(X)$ done in [68, 67].

An observation of more analogies: defining $HI(Y, K, \alpha)$ and the splitting formula.

Now we describe the second motivation for this project. It is well known that for an integer homology sphere Y, the instanton Floer homology $HI(Y)$ [20, 73] categorifies the Casson invariant in the sense that $\chi(HI(Y)) = 2\lambda_C(Y)$.

Moreover, Frøyshov defined [21] a refinement of $HI(Y)$ which is known as the reduced instanton Floer homology $HI_{red}(Y)$. From it one can define the Frøyshov h-invariant $h(Y) = \frac{1}{2}(\chi(HI_{red}(Y)) - \chi(HI(Y)))$, which is the precursor to the d-invariant in Heegaard Floer homology and Frøyshov invariant in monopole Floer homology. We should notice that our conventions are slightly different from those of Frøyshov, since we are working with the homology (not cohomology) version of instanton Floer homology.

By the TQFT-like features of instanton Floer homology, a homology cobordism $W : Y_1 \to Y_2$ between two integer homology spheres will induce a map between the corresponding Floer homologies $HI(W) : HI(Y_1) \to HI(Y_2)$. In particular, for a homology cobordism $W : Y \to Y$ from Y to itself, there are maps $HI(W) : HI(Y) \to HI(Y)$ and $HI_{red}(W) : HI_{red}(Y) \to HI_{red}(Y)$ for the unreduced and reduced instanton Floer homologies. This is an interesting case since closing up W one obtains a four manifold X which is a homology $S^1 \times S^3$ and for which $\lambda_{FO}(X)$ can be defined. In this case, Anvari proved [2, Theorem A] a splitting formula for $\lambda_{FO}(X)$ in terms of the Lefschetz number $\text{Lef}(HI(W))$ of the cobordism map, which reads (in our conventions)

$$\lambda_{FO}(X) = \frac{1}{2}\text{Lef}(HI(W)) = \frac{1}{2}\text{Lef}(HI_{red}(W)) - h(Y)$$

and is the analogue of the splitting formula [53, Theorem A] Lin, Ruberman and Saveliev proved for a similarly-constructed invariant $\lambda_{SW}(X)$ which is defined using the Seiberg-Witten equations instead. Moreover, an argument due to Frøyshov [24, Theorem 8] for the case of $\lambda_{SW}(X)$ (and which is readily adapted to $\lambda_{FO}(X)$) shows that if X is obtained as the closure of a different homology cobordism $W' : Y' \to Y'$ then $h(Y) = h(Y')$ and thus it makes sense to talk about the h-invariant of X, which we denote as $h(X)$. In this way the splitting formula for $\lambda_{FO}(X)$ reads

$$\lambda_{FO}(X) + h(X) = \frac{1}{2}\text{Lef}(HI_{red}(W))$$

This naturally leads to the following question:

Question 6. *Is it possible to define instanton Floer homologies $HI(Y, K, \alpha)$, $HI_{red}(Y, K, \alpha)$ for a value of α satisfying $\triangle_K(e^{-4\pi i\alpha}) \neq 0$ and corresponding Frøyshov h-invariants $h(Y, K, \alpha)$ for the knot K so that whenever $(W, \Sigma) : (Y, K) \to (Y, K)$ is a self-concordance of K, there is a splitting formula involving $\lambda_{FO}(X, T, \alpha)$ and Lef$(HI(Y, Y, \alpha))$ (respectively Lef$(HI_{red}(Y, K, \alpha))$, $h(Y, K, \alpha)$)? Moreover, can one define $HI(Y, K, \alpha)$ in such a way that $\chi(HI(Y, K, \alpha)) = \lambda_{CLH}(Y, K, \alpha)$?*

As the reader may expect, the answers to all of these questions are mostly in the affirmative, however, we need to point out some details first. Our analytical framework will be based again on the singular gauge theory developed by Kronheimer and Mrowka. In particular this means that we will use a metric with a cone angle along the knot which will give rise to the structure of an orbifold.

The reader may be aware that Collin and Steer [11] had developed almost two decades ago a similar version of instanton Floer homology for knots. Many aspects of our construction are identical to theirs, however, as Kronheimer and Mrowka point out in [45, Section 1.2.5], unless $\alpha = 1/4$ (which corresponds to the case $k/n = 1/4$ in Collin and Steer's paper), the differential needed to define the chain complex which gives rise to the Floer groups may be ill-defined, since certain energy bounds for the moduli spaces cannot be guaranteed, which are needed to appeal directly to some compactness theorems which will be recalled in Section 4. This is usually referred as saying that for $\alpha \neq 1/4$ we are in a non-monotone situation. Fortunately, as Kronheimer and Mrowka also point out in that same section of their paper, there is a way to get out of this conundrum provided one is willing to work with an appropriate local coefficient system. The finite dimensional analogue of this situation was first studied by Novikov in [62], where an analogue of Morse theory for the case of circle valued functions on a finite dimensional manifold was developed.

Although the Chern-Simons functional is typically circle valued (on the space of connections mod gauge that is), the reason why instanton Floer homology is typically referred as an infinite dimensional Morse theory, rather than an infinite dimensional Morse-Novikov theory, is that the monotonicity condition usually holds, so the behavior of the Chern Simons functional is more similar to the case of ordinary Morse theory and not the slightly more complicated Morse-Novikov theory. On the other hand, the use of Floer-Novikov theories is certainly not a new thing on the symplectic versions of Floer homologies, and was first investigated by Hofer and Salamon [35] (see also [51, 78, 26] for more recent, but in no way exhaustive references).

As will be explained in Section 3, there are at least three natural choices of local (Novikov) systems we could use in our situation, although most of the time we will stick with what we call the Universal Novikov/Local system, since it seems to require the fewest amount of extra choices (for functoriality purposes, that is). The end result will be that rather than defining the instanton chain complex over a vector space, like $\mathbb{Q}$ or $\mathbb{C}$, we will need to define it over a much larger vector space Λ (the Novikov field), but in the end the groups $HI(Y,K,\alpha)$ we will produce continue to be finite dimensional over Λ, so formally many statements continue to hold. For example, as we will soon state precisely, the Euler characteristic $\chi_\Lambda(HI(Y,K,\alpha))$ of $HI(Y,K,\alpha)$ with respect to the field Λ recovers $\lambda_{CLH}(Y,K,\alpha)$.

What seems to be new is the idea that one can also define a reduced version $HI_{red}(Y,K,\alpha)$ of these groups, although for the case of $\alpha = 1/4$, Christopher Scaduto and Aliakbar Daemi had also realized this independently (and a bit earlier as well [14]). In fact, in their work in progress they have already proven many interesting properties for $HI(Y,K,1/4)$ and $HI_{red}(Y,K,1/4)$, as well as computed several examples. In any case, here is the answer to Question 6.

Theorem 7. *(Construction of the instanton Floer-Novikov homologies for knots) Suppose that K is an oriented knot inside an integer homology sphere Y and that a parameter $\alpha \in \mathbb{Q} \cap (0,1/2)$ is chosen so that $\triangle_K(e^{-4\pi i\alpha}) \neq 0$. Choose an orbifold metric along K with cone angle $2\pi/\nu$, where ν is a cone parameter compatible with α.*

*Then there is a family of vector spaces $HI_i(Y,K,\alpha,\nu)$ for $i \in \mathbb{Z}/4\mathbb{Z}$, which are finite dimensional over a Novikov field Λ, and which we will call the **instanton Floer-Novikov knot homology groups of the knot** K. These vector spaces have the property that they recover the Casson-Lin-Herald invariant (and hence the Tristram-Levine knot signatures), in the sense that*

$$\chi_\Lambda(HI(Y,K,\alpha,\nu)) = \lambda_{CLH}(Y,K,\alpha) = 4\lambda_C(Y) + \frac{1}{2}\sigma_K(e^{-4\pi i\alpha})$$

*where we are using an absolute $\mathbb{Z}/2\mathbb{Z}$ grading of these groups in order to compute the Euler characteristic with respect to Λ. Moreover, each such $HI_i(Y,K,\alpha,\nu)$ admits a refinement $HI_{red,i}(Y,K,\alpha,\nu)$, which again will be finite dimensional vector spaces over the Novikov field Λ, and which we will call the **reduced instanton Floer-Novikov knot homology groups of the knot** K. Given these two groups one can define the **knot Frøyshov h-invariants***

$$h(Y,K,\alpha,\nu) = \chi_\Lambda(HI^{red}(Y,K,\alpha,\nu)) - \chi_\Lambda(HI(Y,K,\alpha,\nu))$$

For the case of $\alpha = 1/4$, no Novikov field is needed and in fact the Floer groups can be defined over $\mathbb{Q}$ (for example).

Remark 8. a) In order to be completely accurate, notice that we specified the cone parameter being used in the resulting groups. However, we expect the groups (and h-invariants) to be independent of ν, which is why we will continue to denote them as $HI(Y,K,\alpha)$, $HI_{red}(Y,K,\alpha)$ and $h(Y,K,\alpha)$ throughout the paper. In fact, as we will discuss in the last section of the introduction, we also expect some independence regarding the values of α being used.

b) Rather than working over a Novikov field, one could also use a Novikov ring, so that the groups become modules over this ring. This would open the possibility of studying torsion in the Floer groups (especially when $\alpha = 1/4$), but given that they are already difficult to compute over the Novikov field, we will not bother ourselves with this generalization.

Some examples of these groups and the knot h-invariants will be exhibited in Section 8, but now we have to answer the other part of Question 6, namely, what is the relation between $\lambda_{FO}(X,T,\alpha)$ and $\mathrm{Lef}(HI(Y,K,\alpha))$ in the case of a self-concordance of a knot?

First of all, we must observe that currently the functoriality properties of the groups $HI(Y,K,\alpha)$ are weaker than their non-singular counterparts $HI(Y)$. By this we mean that a homology concordance $(W,\Sigma):(Y_1,K_1)\to(Y_2,K_2)$ for which both of $HI(Y_1,K_1,\alpha)$ and $HI(Y_2,K_2,\alpha)$ are defined, may not induce a cobordism map between the groups. The reason for this has to do once again with the reducible connections on the cobordism. In the case where Σ is not present, there are maps between $HI(Y_1)$ and $HI(Y_2)$ essentially because the trivial connection θ_W on the cobordism, which is the unique reducible up to gauge since $H_1(W;\mathbb{Z})=0$, is automatically isolated and non-degenerate since $H^1(W;\mathfrak{g}_{\theta_W})=H^1(W;\mathbb{R})\otimes\mathbb{R}^3=0$ and $H^{2,+}(W;\mathfrak{g}_{\theta_W})=H^{2,+}(W;\mathbb{R})\otimes\mathbb{R}^3=0$.

In the case of a homology concordance there is still a unique reducible connection $\theta_{W,\alpha}$ after we have fixed a choice of α since $H_1(W\backslash\Sigma;\mathbb{Z})=\mathbb{Z}$, but now it is a priori not immediate that it will be isolated or non-degenerate. The cobordisms for which this will happen will be called **α-admissible,** and after we explain more of the setup we are using in sections 2 and 4 it will become clear that the condition we are after is the following.

Definition 9. A homology concordance $(W,\Sigma):(Y_1,K_1)\to(Y_2,K_2)$ between two knots $K_1\subset Y_1$ and $K_2\subset Y_2$ is called **α-admissible** for $\alpha\in(0,1/2)$ if $\triangle_{K_1}(e^{-4\pi i\alpha})\neq 0$, $\triangle_{K_2}(e^{-4\pi i\alpha})\neq 0$ and moreover $H^1(W\backslash\Sigma;L^{\otimes 2}_{\theta_{W,\alpha}})=0$. Here $\theta_{W,\alpha}$ denotes the unique reducible (up to gauge) compatible with the holonomy condition α, and $E=L_{\theta_{W,\alpha}}\oplus L^{-1}_{\theta_{W,\alpha}}$ denotes the decomposition of the trivial $SU(2)$ bundle over $W\backslash\Sigma$ induced by $\theta_{W,\alpha}$.

We will discuss to what extent this condition on the cobordism is really needed at the end of this section, but for now let's assume that it holds. In that case it is straightforward to see that we have cobordism maps $HI(W,\Sigma,\alpha)$ between $HI(Y_1,K_1,\alpha)$ and $HI(Y_2,K_2,\alpha)$ as explained in Section 4.

For obtaining an analogous splitting formula to the one $\lambda_{FO}(X)$ satisfies we need an additional piece of data. As we mentioned before, it is more accurate to regard the Furuta-Ohta invariant $\lambda_{FO}(X)$ as a degree zero Donaldson invariant. Likewise, $\lambda_{FO}(X,T,\alpha)$ can be regarded as the degree zero Donaldson invariant associated to the moduli space of $\mathcal{M}(X,T,0,0,\alpha)$ of α-ASD connections on the bundle $E(0,0)$ (that is, the bundle with vanishing instanton and monopole numbers).

Interestingly enough, in the case that $\alpha\neq 1/4$, there are other moduli spaces $\mathcal{M}(X,T,k,l,\alpha)$ whose expected dimension is zero (and are a priori non-empty), which means that there are additional candidates for degree zero Donaldson invariants. As we will explain in more detail in Section 6, whenever k is an integer such that $k(1-4\alpha)\geq 0$, the moduli space $\mathcal{M}(X,T,k,-2k,\alpha)$ is of expected dimension zero, and can in fact be used to define a Donaldson-type invariant $D_0(X,T,\alpha,k)$ (see definition 39 for a precise statement). In particular, we can create a formal power series

$$\sum_{k\in\mathbb{Z}} D_0(X,T,\alpha,k)T^{-\mathcal{E}_{top}(X,T,k,-2k,\alpha)} \tag{1}$$

where $\mathcal{E}_{top}(X,T,k,-2k,\alpha)$ denotes the topological energy of the moduli space $\mathcal{M}(X,T,k,-2k,\alpha)$. In fact it is equal to the quantity $k(1-4\alpha)$ which explains the restriction $k(1-4\alpha)\geq 0$, since negative energy moduli spaces of α-ASD instantons are always empty, just as in the ordinary case where no holonomy condition is present. Now, the formal power series 1 is in fact the sort of object a Novikov field is equipped to handle, in other words, we can think of the series 1 as an element of Λ.

This is a good thing, since the Lefschetz number of a degree-preserving linear transformation $L:V\to V$ between two finite dimensional vector spaces over some field $\mathbb{F}$ will be an element of $\mathbb{F}$. In the case of a self-concordance $(W,\Sigma):(Y,K)\to(Y,K)$ which is α-admissible we should think of V as being either $HI(Y,K,\alpha)$ (or $HI_{red}(Y,K,\alpha)$), $\mathbb{F}$ as the Novikov field Λ and L as the map on corresponding Floer groups induced by the cobordism. Therefore, one would expect that the Lefschetz number equals 1.

That will be the case, modulo a final caveat. There is an action of $H^1(X;\mathbb{Z}/2)$ on the moduli spaces $\mathcal{M}(X,T,k,-2k,\alpha)$, which has been studied ad nauseam in other situations involving Instanton Floer homology [5, 66, 67, 69, 70, 44]. Unless the action of $H^1(X;\mathbb{Z}/2)$ is free, one cannot expect a relationship between the formal power series and the Lefschetz number to hold, since there will be an ambiguity when solving the gluing problem, as explained in great detail in [44, Section 5]. In fact, it will turn out that the action of of $H^1(X;\mathbb{Z}/2)$ on $\mathcal{M}^*(X,T,k,-2k,\alpha)$ is free, though the case of $\alpha=1/4$ requires some extra care, in which case the splitting (or Lefschetz) formula 2 is essentially a consequence of [44, Proposition 5.5], which in Kronheimer and Mrowka's situation comes from the assumption that the subgroup $\phi^*\subset H^1(W^*,S^*,\mathbf{P}^*)$ (in their notation) satisfies a non-integral condition.

With these remarks in place, we can finish answering Question 6.

Theorem 10. *(Splitting formula for singular Furuta-Ohta invariant in the case of a self-concordance) Let $K\subset Y$ an oriented knot inside an oriented integer homology sphere and $(W,\Sigma):(Y,K)\to(Y,K)$ a self-concordance of K. Consider $\alpha\in\mathbb{Q}\cap(0,1/2)$ such that $\triangle_K(e^{-4\pi i\alpha})\neq 0$ and a cone parameter ν compatible with α.*

If (W,Σ) is α-admissible then there are degree-preserving maps

$$HI(W,\Sigma,\alpha,\nu):HI(Y,K,\alpha,\nu)\to HI(Y,K,\alpha,\nu)$$
$$HI_{red}(W,\Sigma,\alpha,\nu):HI(Y,K,\alpha,\nu)\to HI(Y,K,\alpha,\nu)$$

induced by the cobordism (W,Σ).

If (X,T) is the pair obtained by closing up (Y,K) then for $k\neq 0$ the invariants $D_0(X,T,\alpha,k)$ are always well defined. Moreover, $\lambda_{FO}(X,T,\alpha,\nu)=D_0(X,T,\alpha,0)$ can be defined if and only if (W,Σ) is α-admissible, and the following splitting formula holds

$$\sum_{k\in\mathbb{Z}}D_0(X,T,\alpha,k,\nu)T^{-\mathcal{E}(X,T,k,-2k,\alpha)} = 2\mathit{Lef}(HI(W,\Sigma,\alpha,\nu)) = 2\mathit{Lef}(HI_{red}(W,\Sigma,\alpha,\nu))-2h(Y,K,\alpha,\nu) \tag{2}$$

Finally, if (X,T) arises as the closure of another self-concordance $(W',\Sigma'):(Y',K')\to(Y',K')$ and $\triangle_{K'}(e^{-4\pi i\alpha})\neq 0$ as well, then

$$\mathit{Lef}(HI_{red}(W,\Sigma,\alpha,\nu))=\mathit{Lef}(HI_{red}(W',\Sigma',\alpha,\nu)) \tag{3}$$

and hence

$$h(Y,K,\alpha,\nu)=h(Y',K',\alpha,\nu)$$

In particular,we can define an h-invariant $h(X,T,\alpha,\nu)$ for the embedded torus as $h(Y,K,\alpha,\nu)$ regardless of the "slice" chosen.

Remark 11. a) Again, the dependence on the cone angle will be suppressed from now on.

b) The statement regarding the equality of the Lefschetz numbers 3 will follow the strategy employed by Frøyshov in the case of $\lambda_{SW}(X)$, which as we mentioned before can be found in [24, Theorem 8]. The argument is essentially the same, the only thing one need to needs to verify is a suitable version of [24, Lemma 10] for the case of matrices with coefficients in a Novikov field.

c) Since $h(Y,K,\alpha)$ is always an integer, the splitting formula has the non-obvious consequence that $2\mathrm{Lef}(HI_{red}(W,\Sigma,\alpha,\nu))-\sum_{k\in\mathbb{Z}}D_0(X,T,\alpha,k,\nu)T^{-\mathcal{E}(X,T,k,-2k,\alpha)}$ is always an integer, and not just an element of the Novikov field Λ (there is a natural embedding $\mathbb{Z}\hookrightarrow\Lambda$). In fact, we expect that more can be said about $\mathrm{Lef}(HI_{red}(W,\Sigma,\alpha,\nu))$ and each of the $D_0(X,T,\alpha,k,\nu)$, but we will hold our thoughts until the final section of the introduction.

d) Besides the case of $\lambda_{FO}(X)$ and $\lambda_{SW}(X)$, there are other instances where a Lefschetz formula has appeared in a similar context, in fact, in some ways more closely related to our situation. In [37] Juhász and Zemke compute the effect of concordance surgery on the Ozsváth-Szabó 4-manifold invariant. One is given a smooth closed oriented 4-manifold X with $b_2^+(X) \geq 2$ and a homologically essential torus $T \subset X$ with trivial self-intersection. If $(I \times Y, \Sigma)$ is a self-concordance of a knot $K \subset Y$, then there is a natural torus T_C inside $S^1 \times Y$ which they use to construct a 4-manifold X_C generalizing the Fintushel and Stern knot surgery [19]. Namely, one forms the 4-manifold $X_C = (X \backslash N(T)) \cup_\phi W_C$ where $N(T)$ denotes a tubular neighborhood of T and $W_C = (S^1 \times Y)\backslash N(T_C)$, while ϕ is a gluing diffeomorphism. In (knot) Heegaard-Floer homology there is a concordance map

$$\hat{F}_C : \widehat{HFK}(Y,K) \to \widehat{HFK}(Y,K)$$

and an associated graded Lefschetz number

$$\mathrm{Lef}_t(C) = \sum_{i\in\mathbb{Z}} \mathrm{Lef}\left(\hat{F}_C \mid_{\widehat{HFK}(Y,K,i)} : \widehat{HFK}(Y,K,i) \to \widehat{HFK}(Y,K,i)\right) t^i$$

Then Theorem 1.1 in [19] shows that $\Phi_{X_C;\omega} = \mathrm{Lef}_{t_1}(C)\Phi_{X;\omega}$, where $\Phi_{X;\omega}$ denotes a version of the Ozsváth-Szabó 4-manifold invariant twisted by a certain collection of closed 2-forms.

e) Another situation very close to our splitting formula is the one Kronheimer and Mrowka found for the Seiberg-Witten invariants on a closed 4-manifold [48, Section 32.1]. More precisely, one is given a closed oriented 4-manifold with $b_2^+(X) \geq 2$ and within it a separating hypersurface Y, so that $X = X_1 \cup X_2$, $\partial X_1 = Y$ and $\partial X_2 = -Y$. If ω_X is a 2-form used to perturb the Seiberg Witten equations on X and ω is the restriction of ω_X to X, then for a torsion spin-c structure $\mathfrak{s}$ on Y, in general ω will induce a non-balanced perturbation in the sense of [48, Chapter 32], which essentially means that a Novikov system Λ is required to define the corresponding monopole Floer homology groups $HM(Y,\mathfrak{s},\omega)$. Proposition 32.1.1 in [48] shows that

$$\sum_{\mathfrak{s}_X|_Y=\mathfrak{s}} T^{-\mathcal{E}^{top}_{\omega_X}(\mathfrak{s}_X)} SW(X,\mathfrak{s}_X) = \langle \psi_+, \psi_- \rangle_{\omega_\mu} \tag{4}$$

Here the sum is taking place over all spin-c structures on X which restrict to the given one on Y, $SW(X,\mathfrak{s}_X)$ denotes the Seiberg-Witten invariant associated to such a spin-c structure, $\mathcal{E}^{top}_{\omega_X}$ is a perturbed topological energy, and the right hand side denotes a pairing of two relative invariants ($\psi_\pm$ being an element of $HM(\pm Y,\mathfrak{s},\omega)$), the pairing taking place with respect to the Novikov ring. Notice that one can interpret our splitting formula 2 as an analogue of the pairing formula 4 when the 3-manifold is non-separating (rather than separating), and with the Donaldson invariants (rather than the Seiberg-Witten invariants). From this perspective, the different bundles $E(k,-2k)$ are playing a role analogous to the one the different isomorphism classes of spin-c structures play in the Seiberg-Witten context.

In section 8 we will give some applications of the splitting formula 2, including a proof that $\lambda_{FO}(S^1 \times Y, S^1 \times K, \alpha) = 2\lambda_{CLH}(Y,K,\alpha)$. But as a way to entice the reader, we now prove the following.

Theorem 12. *Suppose that $K \subset Y$ and $K' \subset Y'$ are knots and there exists a concordance (C,A) : $(Y,K) \to (Y',K')$ where C is a homology $[0,1] \times S^3$ and A an embedded annulus. Then for any $\alpha \in \mathbb{Q} \cap (0,1/2)$ such that (C,A) is α-admissible, we have that $h(Y,K,\alpha) = h(Y',K',\alpha)$, i.e, the knot h-invariants are α-concordance invariants.*

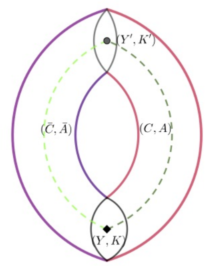

Figure 1. Obtaining (X, T) from a concordance (C, A)

In particular, $h(Y', K', \alpha)$ will vanish whenever K' is α-slice in the sense that there is an α-concordance $(C, A) : (S^3, \circ) \to (Y', K')$, where $\circ$ is the unknot.

Proof. Observe that we can form a self-concordance of the knot (Y, K) obtained by "stacking" (C, A) with the opposite concordance $(\bar{C}, \bar{A}) : (Y', K') \to (Y, K)$ obtained by reversing orientations,

$$(W, \Sigma) = (\bar{C}, \bar{A}) \circ (C, A) : (Y, K) \to (Y, K)$$

From (W, Σ) we can close it up as shown in Figure 1 to obtain (X, T).

The corresponding closed 4-manifold (X, T) can also be obtained from doing the staking in the opposite order, namely $(C, A) \circ (\bar{C}, \bar{A})$, so the last statement in the splitting formula (Theorem 2) implies that the knot h-invariants are the same, i.e,

$$h(Y, K, \alpha) = h(Y', K', \alpha)$$

The last claim follows from the fact that for the unknot, for all $\alpha \in \mathbb{Q} \cap (0, 1/2)$, we have $HI(S^3, \circ, \alpha) = HI_{red}(S^3, \circ, \alpha) = 0$, and hence $h(S^3, \circ, \alpha) = 0$.

□

We conclude this section by mentioning the "flip symmetry" property our Floer groups enjoy (in the context of singular gauge theory on 4-manifolds it was introduced by Kronheimer and Mrowka, [41, Lemma 2.12]).

Theorem 13. *(Flip symmetry for instanton Novikov-Floer groups for knots and the singular Furuta-Ohta invariant) Let $K \subset Y$ be an oriented knot inside an oriented integer homology sphere and choose $\alpha \in \mathbb{Q} \cap (0, 1/2)$ such that $\triangle_K(e^{-4\pi i \alpha}) \neq 0$. Choose a cone parameter ν which is compatible with both α and $\frac{1}{2} - \alpha$. Then there is a flip isomorphism*

$$\mathcal{F} : HI(Y, K, \alpha, \nu) \to HI\left(Y, K, \frac{1}{2} - \alpha, \nu\right)$$

which is grading preserving, with an analogous isomorphism for the reduced groups $HI_{red}(Y, K, \alpha, \nu)$. In particular,

$$h(Y, K, \alpha, \nu) = h\left(Y, K, \frac{1}{2} - \alpha, \nu\right)$$

Likewise, whenever $\lambda_{FO}(X, T, \alpha, \nu)$ can be defined, we have the equality

$$\lambda_{FO}(X, T, \alpha, \nu) = \lambda_{FO}\left(X, T, \frac{1}{2} - \alpha, \nu\right)$$

and more generally there is an equality

$$D_0(X, T, k, \alpha, \nu) = D_0\left(X, T, -k, \frac{1}{2} - \alpha, \nu\right)$$

Some Speculations and Further Directions of Work. We finish this introduction by mentioning remaining problems related to the invariants and Floer homologies we have described so far. The author hopes to address many of these questions in the near future.

(1) Independence of cone angle: Perhaps the most obvious question we would like to answer is how the invariants $HI(Y, K, \alpha, \nu)$, $HI_{red}(Y, K, \alpha, \nu)$ and $\lambda_{FO}(X, T, \alpha, \nu)$ depend on the cone parameter ν being used. That will probably require a deformation argument where one is allowed to vary ν, which in particular means one would need to set up the theory without using orbifolds, since now the cone angle parameter ν can no longer remain an integer. To a large extent the analytical machinery from the original papers by Kronheimer and Mrowka on singular gauge theory [41, 43] was developed without using the orbifold language, but as we mentioned before these became necessary for analyzing the compactness of the moduli spaces and allowing the use of stronger Sobolev norms. However, it is useful to point out that Gerard [29] developed some of the technical aspects needed if one wanted to construct the Floer groups with an arbitrary cone angle, using the theory of elliptic edge operators [57]. In any case, from a philosophical point of view we expect the invariants to be independent of ν, even though we do not have a way to prove that at the moment.

(2) Irrational values of holonomy parameter α: the other natural question is trying to develop versions of $HI(Y, K, \alpha, \nu)$, $HI_{red}(Y, K, \alpha, \nu)$ and $\lambda_{FO}(X, T, \alpha, \nu)$ for the case of irrational values of the holonomy α. At least for the case of $\lambda_{FO}(X, T, \alpha, \nu)$, the author is quite certain this can be done without much problems. The reason why we avoided this case in the paper is because we framed certain computations in terms of orbifold cohomology groups (Lemma 34 being the relevant one for this discussion). However, using the approach from [29], it should be possible to recast the necessary computation in ways that avoids talking about the orbifold cohomology group. We have in mind something like [29, Proposition 2.10], which essentially finds the same deformation complex as the one that implicitly appears in our Lemma 21. So it seems feasible to try to find the four dimensional analogue of this result by Gerard. The situation with $HI(Y, K, \alpha, \nu)$ seems a bit more complicated, given that in general defining Floer homologies is more difficult than defining a counting invariant, but it may still be possible to rephrase things in such a way that avoid a direct appeal to orbifold arguments, which involve taking a local cover of the manifold where the singular connection pulls back to a non-singular connection upstairs (and which cannot be used if the holonomy parameter α is irrational). However, the definition of $HI_{red}(Y, K, \alpha, \nu)$ seems conceptually a lot more subtle. As we will explain in Section 5, the reduced Floer groups require defining maps that use a singular version of Donaldson's μ-map. In particular, we have two take the μ-map of a point $x \in K$, which in the singular case is a degree two cohomology class, instead of a degree four cohomology class, as is usually the case. That on its own seems definable for the case of irrational holonomy, but verifying certain properties of $\mu(x)$ require taking the holonomy of connections along the line $\mathbb{R} \times \{x\} \subset \mathbb{R} \times K \subset \mathbb{R} \times Y$. For rational values of α one can still make sense of what that means, but it is not clear to the author what we would be the meaning of this for irrational values.

(3) Dependence of the invariants on the holonomy parameter α: as Leibniz would put it, if this is the best of all possible worlds[3] and the invariants $HI(Y, K, \alpha, \nu)$, $HI_{red}(Y, K, \alpha, \nu)$ and $\lambda_{FO}(X, T, \alpha, \nu)$ are in fact independent of the cone angle and can be defined for irrational values of α as well (provided of course that $\triangle_K(e^{-4\pi i \alpha}) \neq 0$), then it is reasonable to expect that $HI(Y, K, \alpha)$, $HI_{red}(Y, K, \alpha)$ remain the same on the subintervals of $(0, 1/2)$ determined by the solutions of the equation $\triangle_K(e^{-4\pi i \alpha}) = 0$. At least $\lambda_{CLH}(Y, K, \alpha)$ does enjoy this property. A similar behavior should also hold true for $\lambda_{FO}(X, T, \alpha, \nu)$, namely it should remain the same on subintervals specified by the Alexander polynomial of the torus $\triangle_T$. The only exception might be the case of $\alpha = 1/4$, where maybe we should not expect to be able to compare $HI(Y, K, \alpha)$ to $HI(Y, K, 1/4)$, even if α is arbitrarily closed to $1/4$ and even if we regard $HI(Y, K, 1/4)$ as being defined over the Novikov field (which again is not necessary in this case).

(4) Geometric meaning of $\lambda_{FO}(X, T, \alpha)$ and $D_0(X, T, k, \alpha)$ (when $k \neq 0$): in the case that no perturbations are needed, the meaning of $\lambda_{FO}(X, T, \alpha)$ is clear: it represents a signed count of flat connections satisfying some holonomy condition. The meaning of the $D_0(X, T, k, \alpha)$ is less clear. This setup is not the only one where degree-zero Donaldson-type invariants can be defined, and where their meaning is not obvious [74]. In fact, if we could carry out the necessary deformation arguments on α and ν, we would expect that for $k \neq 0$, all the additional Donaldson invariants $D_0(X, T, k, \alpha)$ vanish. The reason for this, as we explain in Section 6, is that the bundles $E(k, -2k)$ used to define the invariants $D_0(X, T, k, \alpha)$ carry no reducibles α-ASD connections. Therefore, one would expect to be able to deform these moduli spaces to the case of holonomy $\alpha = 1/4$, where the $D_0(X, T, k, 1/4)$ vanish automatically (for $k \neq 0$), so the $D_0(X, T, k, \alpha)$ would have to vanish as well. A similar deformation argument was used by Kronheimer and Mrowka in order to obtain constraints on the singular Donaldson polynomials they defined, and which were crucial for the proof of the structure theorem for the Donaldson polynomials [41, 43, 42]. One could also wonder if there is a symplectic interpretation of the invariant $\lambda_{FO}(X, T, \alpha)$ (and $\lambda_{FO}(X)$ as well), analogous to the original definition Casson gave for $\lambda_C(Y)$ [1] and Lin gave for $\lambda_{CLH}(S^3, K, 1/4)$ [54]. An interpretation in terms of Heegaard-Floer homology or monopole Floer homology would also be very interesting.

(5) Künneth Formula and Skein Relations: the ordinary instanton Floer homology groups $HI(Y)$ satisfy a Künneth formula under the connected sum of 3-manifolds [25]. We expect that a similar property will also hold for $HI(Y, K, \alpha)$, which Scaduto and Daemi have proven in the case that $\alpha = 1/4$. Another property one might hope for is an skein exact sequence when three knots in S^3 differ by simple moves (like Figure 6 in [44]). Now, the skein relations Kronheimer and Mrowka prove in [44, 46] involve cobordism maps induced by non-orientable surfaces. Their version of instanton Floer homology for knots is different from ours, in that get rid of reducibles by taking connected sum with an “atom”. However, this does not impact the question of whether or not we can use non-orientable surfaces in our case. The important aspect to keep in mind is that Kronheimer and Mrowka work with holonomy $\alpha = 1/4$, where the flip symmetry $\alpha \to 1/2 - \alpha$ is an involution when $\alpha = 1/4$. In particular, this means that one can recast things using unoriented knots, and non-orientable surfaces can be used to define cobordism maps. Therefore, for $\alpha \neq 1/4$ there might not be an useful skein sequence one can use.

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[3] We should still remember that Schopenhauer could be right and this end up being the worst of all possible worlds.

(6) Instanton thin knots: one can also ask if it is possible to characterize those knots with vanishing $HI(Y,K,\alpha)$ (or vanishing $HI_{red}(Y,K,\alpha)$). Following the analogy with Heegaard Floer homology [56], we will call these knots *instanton thin knots.* This question is mostly interesting from two extreme positions: which knots, besides the unknot $\circ \subset S^3$, have the property that $HI(Y,K,\alpha)$ (or $HI_{red}(Y,K,\alpha)$) vanishes for all values of α where these Floer homologies can be defined? Likewise, which knots, besides the unknot $\circ \subset S^3$, have the property that $HI(Y,K,1/4)$ (or $HI_{red}(Y,K,1/4)$) vanish? The specific value of $\alpha = 1/4$ is interesting not only because Novikov systems can be avoided but also because for knots inside S^3 we always have $\triangle_K(e^{-4\pi i(1/4)}) = \triangle_K(-1) \neq 0$ which means that $HI(S^3,K,1/4)$ is well defined for any knot. As we will see in Section 8 for the case of the figure-eight knot 4_1, $HI(S^3,4_1,1/4)$ vanishes identically (and so does $HI_{red}(S^3,4_1,1/4)$), despite the fact that there are two irreducible representations satisfying this holonomy condition. So, $HI(Y,K,1/4)$ **is not an unknot detector,** as opposed to the case of the different knot Floer homology $I^\natural(K)$ Kronheimer and Mrowka defined in [44].

(7) Relation between $\lambda_{FO}(X,T,\alpha)$ and $\lambda_{FO}(X)$: Recall that $\lambda_{CLH}(Y,K,\alpha) = 4\lambda_C(Y) + \frac{1}{2}\sigma_K(e^{-4\pi i\alpha})$. This can be rewritten as $\lambda_{FO}(S^1\times Y, S^1\times K,\alpha) = 8\lambda_{FO}(S^1\times Y)+\sigma_K(e^{-4\pi i\alpha})$. This suggests that if one could find a non-gauge theoretic definition of a tori signature σ_T for $T \subset X$, then the formula

$$\lambda_{FO}(X,T,\alpha) = 8\lambda_{FO}(X) + \sigma_T(e^{-4\pi i\alpha}) \tag{5}$$

should hold [assuming that σ_T is normalized so that $\sigma_{S^1\times K} = \sigma_K$]. Now, there are two problems with this approach. For every example where we can compute $\lambda_{FO}(X,T,\alpha)$, the underlying 4-manifold X has a well defined Furuta-Ohta invariant $\lambda_{FO}(X)$. We expect this will always be the case, but in general we do not have an argument saying that if there is an embedded torus T and a parameter α for which $\lambda_{FO}(X,T,\alpha)$ can be defined, then $\lambda_{FO}(X)$ can be defined as well. If there are cases where this does not happen, then the equation 5 would not make much sense to being with. However, one could still take equation 5 as a *definition* of what we mean by σ_T, at least when $\lambda_{FO}(X,T,\alpha)$ and $\lambda_{FO}(X)$ can be both computed. Then the more important problem would be to give a topological interpretation of σ_T (if possible) to facilitate computations of the invariant. It should be pointed out that the expression Herald found for $\lambda_{CLH}(Y,K,\alpha)$ in terms of $\lambda_C(Y)$ and $\sigma_K(e^{-4\pi i\alpha})$ relied on the fact that the Tristram-Levine signatures enjoy multiple interpretations [12], so the question becomes if the same could be done for σ_T.

(8) Defining the invariants for other embedded surfaces: as we discuss in section 6 it is possible to define analogues of the invariants $D_0(X,T,k,\alpha)$ when $k \neq 0$ and $\alpha \neq 1/4$ for embedded spheres and null-homologous tori inside of X. This is perhaps another reason we expect our invariants $D_0(X,T,k,\alpha)$ to vanish in the case $k \neq 0$: they seem to good to be true since they can be adapted for computing information for these other embedded surfaces.

(9) α-admissible cobordisms: we would also like to know to what extent one can relax condition that the cobordism is α-admissible. As we mention in the statement of Theorem 10, having an α-admissible cobordism in the case of a self-concordance $(W,\Sigma):(Y,K)\to(Y,K)$ is equivalent to having a well defined singular Furuta-Ohta invariant $\lambda_{FO}(X,T,\alpha)$ when we close the concordance up. Therefore, if it were possible to introduce holonomy perturbations (or other techniques) in order to deal with the degeneracy of the reducible connection on the cobordisms, it might also be possible to relax the conditions needed for defining $\lambda_{FO}(X,T,\alpha)$, perhaps by adding some correction terms similar to the ones needed for the definition of $\lambda_{SW}(X)$ [61] in the case of the Seiberg-Witten analogue of $\lambda_{FO}(X)$.

Outline of the paper: In section 2 we review the main features of singular gauge theory as developed by Kronheimer and Mrowka. They have used slightly different variations of this setup over the years, and our approach follows closely the papers [41, 43, 45], and relies less on ideas from [44, 49, 47].

After that the paper bifurcates in two main parts: sections 3, 4, 5 discuss the definition of the instanton Floer-Novikov homology groups $HI(Y, K, \alpha)$ and $HI_{red}(Y, K, \alpha)$. Section 3 discusses how the monotonicity issue arises in this particular version of Floer homology and what are the reasonable candidates for local systems one can use. Section 4 discusses the construction of $HI(Y, K, \alpha)$ while section 5 discusses the construction of $HI_{red}(Y, K, \alpha)$.

The second part of the paper includes sections 6, 7 and 8 where the construction of $\lambda_{FO}(X, T, \alpha)$ is given as well as some of its properties. This part can be read independently from the first one, as long as the reader takes from granted the properties of the groups $HI(Y, K, \alpha)$ and $HI_{red}(Y, K, \alpha)$. Section 6 defines $\lambda_{FO}(X, T, \alpha)$ as well as its relatives $D_0(X, T, \alpha)$. Section 7 discusses the splitting formula $\lambda_{FO}(X, T, \alpha)$ and $D_0(X, T, \alpha)$ satisfy. Finally, section 8 includes some calculations of $\lambda_{FO}(X, T, \alpha)$ for certain tori inside mapping tori as well as some properties of the groups $HI(Y, K, \alpha)$ and $HI_{red}(Y, K, \alpha)$, like the usual duality isomorphisms under orientation reversal as well as the flip symmetry property.

The Appendix 9 includes a summary of cohomology with local coefficients and the Alexander polynomial for CW complexes. Everything we say here is probably well known to the experts, but thought it would be useful to have a go to summary of some key facts involving the relation between the Alexander polynomial and the character variety $\hom(X, \mathbb{C}^*)$ of the space X, since most discussions on the first topic focuses on knots in S^3, while the second topic is usually discussed more in the algebraic geometry circles.

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2. Review of the Orbifold Setup

As we mentioned in the introduction, the Floer homology we will construct for knots is based on an orbifold approach, which was pioneered by Kronheimer and Mrowka in their papers on gauge theory and embedded surfaces [41, 43]. Slightly different versions of this setup have been employed by them over the years [47, 45, 44], so the main purpose of this section is to give a brief review of their orbifold construction, as well as discussing the advantages and disadvantages of different strategies for building Floer homologies for knots.

Let X be a smooth, oriented, closed four manifold and Σ any oriented, smoothly embedded surface inside X. Similarly, suppose that Y is a smooth, oriented, closed three manifold and K is an oriented knot or link inside Y. For either of the pairs (X, Σ), (Y, K), we want to do gauge theory using connections on an $SU(2)$ bundle which are singular along Σ (or K). The nature of the singularity is precisely what distinguishes one approach from the other.

A natural strategy would be to work on either of the complements $X\backslash\Sigma$, $Y\backslash K$ and to choose a riemannian metric which is simply the restriction to one of these complements of a smooth metric g defined on all of X (or Y). Analytically, this means that one is working on an open manifold with an incomplete metric, which from the Sobolev package point of view is not a nice situation. However, provided one works with weighted Sobolev spaces, one can still define a reasonable family of function spaces [41, Section 3 (i)].

The next issue is whether one wants to work with an $SU(2)$ bundle E which is defined only on the complement $X\backslash\Sigma$ (respectively $Y\backslash K$). In this case the major problem is how to recover useful topological information. This was called the *extension problem* by Kronheimer and Mrowka, and discussed in some detail in [41, Section 2 (iv)].

The setup used in [41, 43, 45], which will follow as well, is one for which the bundle E is defined over the entire manifold X (or Y), subject to the condition that *topologically* the bundle E admits a reduction to a $U(1)$ bundle along Σ (or K). For example, this means that for (X, Σ) an $SU(2)$ bundle E should decompose as

$$E\mid_{\nu(\Sigma)} = L \oplus L^* \tag{6}$$

where $\nu(\Sigma)$ is a closed tubular neighborhood of Σ , and $L \to \nu(\Sigma)$ some complex line bundle compatible with the hermitian metric. In particular, this allows us to define two topological invariants, the **instanton number** k

$$k = c_2(E)[X] \tag{7}$$

as well as the **monopole number** l

$$l = -c_1(L\mid_\Sigma)[\Sigma] \tag{8}$$

Notice that the latter invariant is not present when $\Sigma = \emptyset$. The reader may be interested in knowing that for their "Khovanov is an unknot-detector" paper Kronheimer and Mrowka do work with bundles which in principle are not defined over the entire manifold [44, Section 2.1].

For the 3-manifold case, the bundle topology is not very interesting in our situation so we can consider E simply as the trivial $SU(2)$ bundle over Y, i.e, $E = Y \times SU(2)$, together with the trivial $U(1)$ sub-bundle corresponding to the diagonal embedding of $U(1)$ in $SU(2)$. In both cases what we will be more interesting is the space of connections we will use, which we now describe.

These connections have the following singular behavior: after choosing a riemannian metric we consider $\nu(\Sigma)$ as being diffeomorphic to the unit disk bundle of the normal bundle, and we choose a connection 1-form $i\eta$ for the circle bundle $\partial\nu(\Sigma)$, so that it coincides with the 1-form $d\theta$ on each circle fibre, where (r, θ) are polar coordinates in some local trivialization of the disk bundle ($dr \wedge d\theta$ orients the normal planes). By radial projection η is extended to $\nu(\Sigma)\backslash\Sigma$.

For each real number $0 < \alpha < 1/2$, we consider the matrix 1-form with values in $\mathfrak{su}(2)$ which behaves near Σ as

$$i\begin{pmatrix} \alpha & 0 \\ 0 & -\alpha \end{pmatrix} d\theta$$

Locally, the holonomy of a connection A on $X\backslash\Sigma$ whose matrix connection coincides with the previous matrix 1-form near Σ on the positively-oriented small circles of constant r is approximately

$$\exp\begin{pmatrix} -2\pi i\alpha & 0 \\ 0 & 2\pi i\alpha \end{pmatrix} \tag{9}$$

We are excluding the values $\alpha = 0, 1/2$ because those cases give no new information, i.e, they essentially correspond to the situation where the connections extend smoothly across the singularity (as $SU(2)$ or $SO(3)$ connections more generally).

For a more global description, using the reduction (6), choose any smooth $SU(2)$ connection A^0 on E which reduces in the same way, i.e,

$$A^0 \mid_{\nu(\Sigma)} = \begin{pmatrix} b & 0 \\ 0 & -b \end{pmatrix}$$

where b is a smooth connection on L. Define the **model connection** A^α on $E \mid_{X \backslash \Sigma}$ as

$$A^\alpha = A^0 + i\beta(r) \begin{pmatrix} \alpha & 0 \\ 0 & -\alpha \end{pmatrix} \eta \tag{10}$$

where β is a smooth cut-off function equal to 1 in a neighborhood of 0 and equal to 0 for $r \geq 1/2$. The model connection has holonomy around small linking circles asymptotically equal to (9). The connections Kronheimer and Mrowka consider can be written as $A = A^\alpha + a$, where $a \in \Omega^1(X \backslash \Sigma, \mathfrak{su}(2))$, and as always necessary Sobolev spaces are needed [41, Eq. 2.2, 2.3].

Remark 14. In the 3-manifold case we can take A^0 to be the (usual) trivial connection θ on the trivial $SU(2)$ bundle E from before. In the nomenclature of [44, Section 3.6], our situation corresponds to one where $\triangle$ is trivial, where $\triangle$ is a local system determining the possible extensions of the bundle across the singularity.

Most of the usual gauge theory technology can be extended to this setup: for example, one can define moduli spaces of ASD connections with the singular behavior described before, compute the expected dimension of the moduli space, as well as the energy of the connection in terms of topological quantities. These moduli spaces are indexed by the instanton and monopole number, so we will write them typically as $\mathcal{M}(X, \Sigma, k, l, \alpha)$ or $\mathcal{M}(k, l)$, depending on the context.

We will remind the reader about the precise formulas once we explain the monotonicity condition usually required before defining Floer homologies. For now, it suffices to say that there is one key aspect missing by working in the previous setup described. Namely, Kronheimer and Mrowka were not able to show that if one has a sequence of ASD connections A_i belonging to some moduli space $\mathcal{M}(X, \Sigma, k, l, \alpha)$, then the limiting connection A_∞, which in principle belongs to a different moduli space $\mathcal{M}(X, \Sigma, k', l', \alpha)$ because of Uhlenbeck bubbling, must belong to a moduli space of lower expected dimension. The fact that this should happen, that is, that after bubbling one should land in a moduli space of smaller dimension, would follow if Conjecture 8.2 in [41] were proven true. Fortunately, they were able to prove this fact by modifying slightly the previous setup and working instead with *orbifolds*.

On a first level, this means that instead of using the restriction of a smooth metric on X (or Y) to $X \backslash \Sigma$ (or $Y \backslash K$), we consider metrics which have a cone-like singularity along the surface (knot) [41, Section 2, iii)]. This means that near the surface Σ, the metric is modeled on

$$ds^2 = du^2 + dv^2 + dr^2 + \left(\frac{1}{\nu^2}\right) r^2 d\theta^2$$

where u, v are coordinates on Σ, and $\nu \geq 1$ is a real parameter. To obtain a global metric on $\nu(\Sigma)$ of this form replace $du^2 + dv^2$ by the pull-back of any smooth metric on Σ, and replace the form $d\theta$ by the 1-form η. The metric is then patched to a smooth one on the complement of $\nu(\Sigma)$ and extended to the rest of X. The resulting metric has a cone-angle of angle $2\pi/\nu$ in the normal planes to Σ. When ν is an integer greater than 1 the metric is an orbifold metric: locally there is a ν-fold branched cover on which the metric is smooth.

An advantage of the orbifold perspective is that it allow us to compute certain quantities (like gradings for the generators of the Floer complex) in terms of equivariant indices on appropriate branched covers along the knot (and/or surface). Moreover, in the orbifold setup one can still use a Coulomb slice determined explicitly by $\ker \check{d}_A^*$, where $\check{}$ emphasizes that we are thinking of this operator as being defined on an orbifold. If we were to use instead the non-orbifold setup (with the smooth metric and the weighted Sobolev spaces mentioned before), then the construction of the slices is more subtle because of the lack of the usual d_A^* operator [41, Lemma 5.5].

However, as we pointed out in the introduction, a drawback of the orbifold approach is that currently there is no way to show that the groups one obtains are independent of the choice of orbifold structure (cone angle) one uses. Moreover, each value of holonomy α determines an allowable set of possible orbifold cone angles ν compatible with α one can use for defining the Floer groups. These values are described in Proposition 4.8, Lemma 4.9 (and the remark after it), as well as Proposition 4.17 of [41].

In practice we can think of the allowable ν in the following way [45, Remark p.894]. If our holonomy parameter is a rational number $\alpha \in \mathbb{Q} \cap (0, 1/2)$, then $\nu > 0$ can be taken to be any integer satisfying the property that

$$\exp \begin{pmatrix} -2\pi i\alpha\nu & 0 \\ 0 & 2\pi i\alpha\nu \end{pmatrix} \in Z(SU(2)) = \pm Id_{2\times 2} \tag{11}$$

Therefore, if we write α as $\frac{p}{q}$, where p, q are relatively prime, it is not difficult to see from this description that taking $\nu = q$ suffices. However, if q happens to be even, then in fact $\nu = \frac{q}{2}$ also satisfies the condition 11. This is why for $\alpha = \frac{1}{4}$ one can choose $\nu = 2$, so that the metric has a cone angle π along the knot (a *bifold* in the terminology of Kronheimer and Mrowka [47]). If one is annoyed by this dependence on the cone angle, it is reasonable to choose the smallest cone parameter ν that works for α as the "canonical" cone angle. Incidentally, notice that the this canonical cone angle works simultaneously for α and $\frac{1}{2} - \alpha$ so that the flip symmetry $\alpha \to \frac{1}{2} - \alpha$ can be analyzed with this common choice.

Remark 15. Besides describing the singular metric in the orbifold setup, one can also adapt the notion of bundle and connections to this situation. The definitions given in [47] are particularly convenient for our purposes (it also has the advantage of being suitable for the case that the bundle does not extend across the singularity, which as we said is more general than what we will consider).

Every point in an orbifold has a neighborhood U which is the codomain of an orbifold chart

$$\phi : \tilde{U} \to U$$

The map ϕ is a quotient map for an action of a finite group, which in our case will be either trivial or $\mathbb{Z}_\nu$. A C^∞ $SU(2)$ **orbifold connection** with respect to (X, Σ) (or (Y, K)) means an oriented $\mathbb{C}^2$ vector bundle E over $X \backslash \Sigma$ (respectively $Y \backslash K$) with an $SU(2)$ connection A having the property that the pull-back of (E, A) via any orbifold chart $\phi : \tilde{U} \to U$ extends to a smooth pair $(\tilde{E}, \tilde{A})$ on $\tilde{U}$.

If we have two $SU(2)$ bifold connections (E, A) and (E', A'), then an **isomorphism** between them is a bundle map $\tau : E \to E'$ over $X \backslash \Sigma$ (respectively $Y \backslash K$) such that $\tau^*(A') = A$. The group $\Gamma_{E,A}$ of automorphisms of (E, A) can be identified with the group of parallel sections of the associated bundle. It is isomorphic to either the trivial group, the circle $U(1)$ embedded as a maximal torus inside $SU(2)$, or all of $SU(2)$.

Now that we have explained the geometric setup, we will describe of the space of connections and gauge transformations that we will be working with, in the spirit of [45]. Suppose there is an oriented link K inside an integer homology sphere Y. For $\alpha \in \mathbb{Q} \cap (0, 1/2)$ consider the model connection B^α described before (10). We will typically use B when referring to connections defined on three-dimensional manifolds from now on. At the same time we choose an orbifold metric g^ν compatible with α in the sense we mentioned before.

We will usually follow the convention of Kronheimer and Mrowka, and use the ˇ notation when we want to emphasize that the orbifold metric is being used in a specific construction. Hence, $\check{L}^p_{m,B^\alpha}$ will denote the Sobolev spaces using the Levi-Civita derivative of g^ν and the covariant derivative of B^α on $\mathfrak{g}_P$. Taking $p = 2$ and an integer[4] $m > 2$, the space of connections to be considered is

$$\mathcal{C}(Y, K, \alpha) = \left\{ B = B^\alpha + b \mid b \in \check{L}^2_{m,B^\alpha}(T^*Y \mid_{Y \setminus K} \otimes \mathfrak{g}_P) \right\}$$

The gauge group $\mathcal{G}(Y, K, \alpha)$ consists of those automorphisms of E which preserve the reducibility of the model connection B^α. In fact, $\mathcal{G}(Y, K, \alpha)$ turns out to be independent of α (and cone angle used) so we will just denote it as $\mathcal{G}(Y, K)$. Here is an useful way to think about $\mathcal{G}(Y, K)$ [43, Appendix I]:

Let $\mathcal{G}^K(Y, K)$ denote the subgroup of $\mathcal{G}(Y, K)$ consisting of those gauge transformations which are the identity over K. Let $\mathcal{G}_K(Y, K)$ denote the space of gauge transformations of the line bundle $L \to K$, that is, the space of maps from K to $U(1)$. Then there is an exact sequence

$$1 \to \mathcal{G}^K(Y, K) \to \mathcal{G}(Y, K) \to^r \mathcal{G}_K(Y, K) \to 1 \tag{12}$$

In particular, we can think of $\mathcal{G}(Y, K)$ as the space of maps $g : Y \to SU(2)$ satisfying $g(K) \subset U(1) \subset SU(2)$. Moreover, $\mathcal{G}^K(Y, K)$ is weakly homotopy equivalent to the space of smooth gauge transformations which are the identity over a sufficiently small neighborhood of the knot K. From this we can see that for knots the topological information of the map g is carried by two integers, corresponding to the degrees of $g : Y \to SU(2)$ and $g \mid_K : K \to U(1)$.

More generally, if the link K has r components, the space of components $\pi_0(\mathcal{G}(Y, K))$ is isomorphic to [45, Section 3]

$$\mathbb{Z} \oplus \mathbb{Z}^r$$

Also, there is a preferred homomorphism

$$d : \mathcal{G}(Y, K) \to \mathbb{Z} \oplus \mathbb{Z} \tag{13}$$

where the map to the second factor is obtained by taking the sum of all the entries in the factor $\mathbb{Z}^r$. After introducing Sobolev completions, we will take our gauge group to be

$$\mathcal{G}(Y, K) = \left\{ g \mid g \in \check{L}^2_{m+1,B^\alpha}(\mathrm{Aut}(P)) \right\}$$

The space of connections $\mathcal{C}(Y, K, \alpha)$ is an affine space, and on the tangent space $T_B\mathcal{C}(Y, K, \alpha)$ we define an inner product (independent of B) by

$$\langle b, b' \rangle_{L^2} = \int_{\check{Y}} -\mathrm{tr}(\mathrm{ad}(*b) \wedge \mathrm{ad}(b'))$$

where we are using the Killing form to contract the Lie algebra indices, and the Hodge star on Y and the wedge product to contract the form indices. It is important to notice that *the Hodge star is the one defined by the orbifold metric* g^ν.

The **Chern-Simons functional** on $\mathcal{C}(Y, K, \alpha)$ is defined to be the unique function

$$CS : \mathcal{C}(Y, K, \alpha) \to \mathbb{R}$$

[4] We denote this integer by m instead of k to avoid confusion with the instanton number k.

satisfying $CS(B^\alpha)=0$ and having gradient (with respect to the above inner product)

$$(\text{grad}CS)_B = *F_B$$

From this one obtains[5]

$$CS(B^\alpha+b) = < *F_{B^\alpha}, b >_{L^2} + \frac{1}{2} < *d_{B^\alpha}b, b >_{L^2} + \frac{1}{6} < *[b\wedge b], b >_{L^2}$$

The critical points B of CS then satisfy $F_B=0$. When restricted to $Y\backslash K$, this gives rise to a representation (here y_0 is a base-point in the complement of the knot or link)

$$\rho : \pi_1(Y\backslash K, y_0) \to SU(2)$$

satisfying the constraint that for each oriented meridian μ_K, $\rho(\mu_K)$ is conjugate to

$$\rho(\mu_K) \sim \begin{pmatrix} \exp(-2\pi i\alpha) & 0 \\ 0 & \exp(2\pi i \alpha) \end{pmatrix} \tag{14}$$

There is one such conjugacy class for each component of K, once the components are oriented.

As usual in the gauge theory context, we will need to perturb the flatness equation (in fact, the Chern-Simons functional) to obtain the necessary transversality properties for the moduli spaces. The perturbations that we will use are described in section 3.2 of [45]. As we will see soon, the support of these perturbations must stay away from the knot if we want to appeal to the relationship Herald found between the count these flat connections (modulo conjugacy) and the knot signatures [32]. Kronheimer and Mrowka had to address a similar problem in their paper on Tait colorings [47], although for different purposes.

Before discussing this support condition, we will explain next how the monotonicity issue arises when one tries to define a version of Instanton Floer homology for an arbitrary value of α.

3. Monotonicity and Novikov Systems

As usual, we would like to do "Morse theory" on

$$\mathcal{B}(Y,K,\alpha) = \mathcal{C}(Y,K,\alpha)/\mathcal{G}(Y,K)$$

Notice that since $\mathcal{G}(Y,K)$ is independent of α,ν and $\mathcal{C}(Y,K,\alpha)$ is contractible, the homotopy type of the space $\mathcal{B}(Y,K,\alpha)$ is independent of α,ν. As in the case when K is not present, the Chern-Simons functional is no longer single-valued on $\mathcal{B}(Y,K,\alpha)$. In fact CS is invariant only under the identity component of the gauge group. If $B\in\mathcal{C}(Y,K,\alpha)$ is a connection and g a gauge transformation, we can use the latter to form the bundle $S^1\times_g E$ over $S^1\times Y$ together with its reduction over $S^1\times K$, defined by B^α. Then the map d from (13) becomes

$$d(g) = (k,l) \in \mathbb{Z}\oplus\mathbb{Z}$$

where k and l are the instanton and monopole numbers respectively. In this case we have [45, p. 874]

$$CS(B) - CS(g(B)) = 32\pi^2(k+2\alpha l) \tag{15}$$

More generally, we define for a path $\gamma:[0,1]\to\mathcal{C}(Y,K,\alpha)$ the **topological energy** as twice the drop in the Chern Simons functional, so that the last equation implies that a path from B to $g(B)$ has topological energy

$$\mathcal{E}(k,l) = 64\pi^2(k+2\alpha l) \tag{16}$$

[5]Notice that there is a factor of $\frac{1}{2}$ missing in the last term of the formula in [45, Eq. 67].

For a path that formally solves the downward gradient-flow equation for CS on $\mathcal{C}(Y,K,\alpha)$, the topological energy coincides with the modified path energy

$$\int_0^1 (\|\dot{\gamma}(t)\|^2 + \|\text{grad}CS(\gamma(t))\|^2)dt$$

Before comparing $\mathcal{E}(k,l)$ with the formula for the grading of two flat connections, we discuss the perturbations needed to achieve transversality, which involve using **holonomy perturbations**. These will be the same as in [45, Section 3.2], and we will give more details about them in the next section, where we will need to keep track on certain estimates involving their norms. The basic idea is that each holonomy perturbation gives rise to a **cylinder function**

$$f : \mathcal{C}(Y,K,\alpha) \to \mathbb{R}$$

which depends on an l-tuple

$$\mathbf{q} = (q_1, \cdots, q_l)$$

of immersions $q_i : S^1 \times D^2 \to Y \backslash K$, $i = 1, \cdots, l$ and an $SU(2)$-invariant function h on $SU(2)^{\times l}$ via the formula

$$f(B) = \int_{D^2} h(\text{Hol}_{\mathbf{q}}(B))\mu$$

Here μ is a non-negative 2-form supported in the interior of D^2 and having integral 1. We assume that h is invariant under the action of $SU(2)$ on each of the l factors separately (see the remark after definition 3.3 in [45]). It follows that *each cylinder function is invariant under the full gauge group, not just the identity component.*

The space of perturbations one uses to achieve transversality involves taking a countable collection of cylinder functions with an l^1 notion of convergence [45, Definition 3.6]. If $\mathcal{P}$ denotes the Banach space of perturbations, then $\mathfrak{p} \in \mathcal{P}$ induces a function $f_{\mathfrak{p}}$, and the perturbed Chern-Simons functional we consider is

$$CS_{\mathfrak{p}} = CS + f_{\mathfrak{p}}$$

The critical points are now connections satisfying the equation

$$*F_B + V_{\mathfrak{p}}(B) = 0$$

where $V_{\mathfrak{p}}$ is the formal gradient of $f_{\mathfrak{p}}$ with respect to the L^2 inner product. Proposition 3.10 in [45] states that there is a residual subset of $\mathcal{P}$ such that, for all $\mathfrak{p}$ all the irreducible critical points of $CS + f_{\mathfrak{p}}$ are non-degenerate (we will return to the reducibles momentarily). The **perturbed topological energy** is then defined for a path $\gamma : [0,1] \to \mathcal{C}(Y,K,\alpha)$ as

$$\mathcal{E}_{\mathfrak{p}}(\gamma) = 2\left((CS + f_{\mathfrak{p}})(B(t_0)) - (CS + f_{\mathfrak{p}})(B(t_1))\right) \tag{17}$$

Notice that because the perturbations are invariant under the full gauge group, for a path from B to $g(B)$, the term $f_{\mathfrak{p}}(B) - f_{\mathfrak{p}}(g(B))$ will cancel, *so the perturbed topological energy $\mathcal{E}_{\mathfrak{p}}$ on loops has the same expression as the unperturbed case, namely, equation (16).*

Returning to the grading question, define the (perturbed) **Hessian** of a connection $B \in \mathcal{C}(Y,K,\alpha)$ as

$$\mathbf{Hess}_{B,\mathfrak{p}}(b) = *d_B b + DV\mid_B(b) \tag{18}$$

where $b \in T_B\mathcal{C}(Y,K,\alpha)$ and the (perturbed) **extended Hessian** [45, p. 881]

$$\widehat{\mathbf{Hess}}_{B,\mathfrak{p}} = \begin{pmatrix} 0 & -d_B^* \\ -d_B & \mathbf{Hess}_{B,\mathfrak{p}} \end{pmatrix} : \check{L}_j^2(Y;\mathfrak{g}_P) \oplus \mathcal{T}_j \to \check{L}_{j-1}^2(Y;\mathfrak{g}_P) \oplus \mathcal{T}_{j-1} \tag{19}$$

To clarify the notation, here $j \le m$ is an integer, $\check{L}^2_j(Y;\mathfrak{g}_P)$ is a shorthand for $\check{L}^2_{j,B^\alpha}(Y\backslash K;\mathfrak{g}_P\mid_{Y\backslash K})$, and $\mathcal{T}_j$ is shorthand for sections of $T_B\mathcal{C}(Y,K,\alpha)$ with regularity Sobolev $\check{L}^2_{j-1,B^\alpha}$.

For two irreducible, non-degenerate (perturbed) flat connections $B_0, B_1 \in \mathcal{C}(Y,K,\alpha)$, the **relative grading**

$$\mathrm{gr}(B_0,B_1) \in \mathbb{Z}$$

will be defined by taking a path $B(t)$ in $\mathcal{C}(Y,K,\alpha)$ from B_0 to B_1 and letting $\mathrm{gr}(B_0,B_1)$ be equal to the spectral flow of the 1-parameter family $\widehat{\mathrm{Hess}}_{B(t),\mathfrak{p}}$. This number only depends on the endpoints and not the path since $\mathcal{C}(Y,K,\alpha)$ is contractible. If

$$[\beta_0]=[B_0], \qquad [\beta_1]=[B_1]$$

are the corresponding gauge equivalence classes in $\mathcal{B}=\mathcal{B}(Y,K,\alpha)$, then $[B(t)]$ determines a path ζ from $[\beta_0]$ to $[\beta_1]$, defining a relative homotopy class $z \in \pi_1([\beta_0],\mathcal{B},[\beta_1])$. This relative homotopy class only depends on B_0, B_1. Conversely, for a relative homotopy class of paths from $[\beta_0]$ to $[\beta_1]$, we define

$$\mathrm{gr}_z([\beta_0],[\beta_1]) \in \mathbb{Z}$$

via the above procedure.

For the case of a closed loop z based at a point $[\beta] \in \mathcal{B}(Y,K,\Phi)$, we obtain an element inside $\pi_1(\mathcal{B}(Y,K,\alpha)) \simeq \pi_0(\mathcal{G}(Y,K,\alpha))$. Lemma 3.14 in [45] shows that

$$\mathrm{gr}_z([\beta],[\beta]) = 8k+4l \tag{20}$$

where we think of z as being specified by a gauge transformation g as before, and $d(g)=(k,l)$.

From this grading formula we can see that $\mathrm{gr}_z([\beta],[\beta]) \equiv 0 \mod 4$, which means that *the Floer homology we will define is $\mathbb{Z}/4\mathbb{Z}$ graded.* We will eventually show how to promote this relative grading to an absolute grading. More importantly, comparing the energy and grading formulas on loops we can see that

$$\mathcal{E}(k,l) = 8\pi^2 \mathrm{gr}_z([\beta],[\beta]) + 32\pi^2 l(4\alpha-1) \tag{21}$$

which means that *the only way for $\mathcal{E}(k,l)$ and $gr_z([\beta],[\beta])$ to be proportional regardless of the value of (k,l) is if $\alpha=\frac{1}{4}$.*

To explain what happens when $\alpha \neq 1/4$, we recall Proposition 3.23 in [45]. Let $\mathcal{M}_z([\beta_0],[\beta_1])$ denote a connected component of the moduli space of ASD connections on $\mathbb{R}\times Y$ limiting to $[\beta_0]$ as $t\to-\infty$ and $[\beta_1]$ as $t\to\infty$ (with suitable perturbations thrown into the picture to guarantee transversality).

Proposition 16. *(*[45, Proposition 3.23]*) Given any $C>0$, there are only finitely many $[\beta_0],[\beta_1]$ and z for which the moduli space $\mathcal{M}_z([\beta_0],[\beta_1])$ is non-empty and has topological energy $\mathcal{E}_\mathfrak{p}(\gamma)$ at most C.*

Now suppose one wants to define the differential via the usual formula

$$\partial[\beta_0] = \sum_{[\beta_1]\mid \mathrm{gr}([\beta_0],[\beta_1])=1} \sum_z n_z([\beta_0],[\beta_1])[\beta_1]$$

where we are taking the sum over all moduli spaces $\mathcal{M}_z([\beta_0],[\beta_1])$ which are one-dimensional and $n_z([\beta_0],[\beta_1]) \in \mathbb{Z}$ denotes the number (with orientations included) of trajectories inside the zero-dimensional (compact) space $\check{\mathcal{M}}_z([\beta_0],[\beta_1]) = \mathcal{M}_z([\beta_0],[\beta_1])/\mathbb{R}$. The issue is that for fixed $[\beta_1]$, the sum

$$\sum_z n_z([\beta_0],[\beta_1])[\beta_1]$$

could be infinite! To see why, suppose that z, z' represent two trajectories from $[\beta_0]$ to $[\beta_1]$ with $\mathcal{M}_z([\beta_0],[\beta_1])$ and $\mathcal{M}_{z'}([\beta_0],[\beta_1])$ one dimensional. Then $z^{-1}\circ z'$ represents a loop based at $[\beta_0]$ with $\mathrm{gr}_{z^{-1}\circ z'}([\beta_0],[\beta_0]) = 0$. From (21) we see that the topological energy of this loop must be $32\pi^2 l(4\alpha-1)$, which is a priori unbounded since $\alpha \neq 1/4$ and the monopole number l can in principle be any integer. Since $\mathcal{E}_{\mathfrak{p}}(z) - \mathcal{E}_{\mathfrak{p}}(z')$ differ (up to sign) by $32\pi^2 l(4\alpha-1)$, we see there is no way to guarantee the assumptions in Proposition (16).

The way to get out of this conundrum is via a **Novikov system**, but first we recall a few facts about the construction of homology groups using **local coefficients**. A good reference for the general construction is [15, Chapter 5]. We will use Γ to denote a local systems of coefficients. Also, if $[\beta] \in \mathcal{B}(Y,K,\alpha)$, then $o([\beta])$ will denote[6] the 2-element of orientations for $[\beta]$ [45, Section 3.6].

Suppose that to each $[\beta]$ we assign an abelian group $\Gamma_{[\beta]}$ and for each homotopy class of paths z from $[\beta_0]$ to $[\beta_1]$ there is an isomorphism Γ_z from $\Gamma_{[\beta_0]}$ to $\Gamma_{[\beta_1]}$ satisfying the usual composition law for paths.

Define the chain group $C_*(Y,K,\alpha;\Gamma)$ generated by irreducible critical points $[\beta]$ of $CS_{\mathfrak{p}}$ as

$$C_*(Y,K,\alpha;\Gamma) = \bigoplus_{[\beta]\in\mathfrak{C}} \mathbb{Z}o([\beta]) \otimes \Gamma_{[\beta]}$$

and the boundary map as

$$\partial = \sum_{([\alpha],[\beta],z)} \sum_{[\breve{A}]} \epsilon[\breve{A}] \otimes \Gamma_z \tag{22}$$

where $[A] \in \mathcal{M}_z([\beta_0],[\beta_1])$ belongs to a 1-dimensional space, $[\breve{A}]$ denotes the connection modulo the $\mathbb{R}$ action, and

$$\epsilon[\breve{A}] : o[\beta_0] \to o[\beta_1]$$

is an isomorphism obtained by comparing orientations. Written differently, we can think of the differential as

$$\partial(e) = \sum_{[\beta_1]\in\mathfrak{C}} \sum_z n_z([\beta_0],[\beta_1])\Gamma_z(e), \quad e \in \Gamma_{[\beta_0]}$$

which we will occasionally write as

$$\partial[\beta_0] = \sum_{[\beta_1]\in\mathfrak{C}} \sum_z n_z([\beta_0],[\beta_1])\Gamma_z[\beta_1]$$

After choosing trivializations for $o([\beta_0])$ and $o([\beta_1])$, the contribution for a given pair of critical points in the differential takes the form

$$\sum_z n_z \Gamma_z$$

where z runs through all relative homotopy classes satisfying the conditions

$$\mathrm{gr}_z([\beta_0],[\beta_1]) = 1$$

Now, define the **support**

$$\sup([\beta_0],[\beta_1]) = \{z \mid n_z \neq 0\}$$

By Proposition (16), for all C, the intersection

$$\sup([\beta_0],[\beta_1]) \cap \{z \mid \mathcal{E}_{\mathfrak{p}}(z) \leq C\}$$

[6]The notation Kronheimer and Mrowka use for the system of orientations is $\Lambda([\beta])$ but we will use Λ for the Novikov field so a sacrifice had to be made somewhere.

is finite. In the notation of definition 30.2.2 in [48], we will define a c-complete local system of coefficients which will allow us to make sense of infinite sums like $\sum_z n_z \varGamma_z$.

Definition 17. A subset

$$S \subset \pi_1([\beta_0], \mathcal{B}(Y, K, \alpha), [\beta_1])$$

is ***c*-finite** if the following two conditions hold:

i) For all C, the intersection

$$S \cap \{z \mid \mathcal{E}_{\mathfrak{p}}(z) \leq C\}$$

is finite.

ii) There exists d such that $|\mathrm{gr}_z([\beta_0], [\beta_1])| \leq d$ for all z in S.

Now suppose that $\varGamma$ is a local system of complete topological abelian groups on $\mathcal{B}(Y, K, \alpha)$. So $\varGamma([\beta])$ is a complete topological group, and the homomorphisms $\varGamma_z : \varGamma_{[\beta_0]} \to \varGamma_{[\beta_1]}$ are continuous. If $0 \in \varGamma[\beta]$ has a neighborhood basis (not necessarily countable) consisting of subgroups, then $\varGamma([\beta])$ is a complete filtered group. Let $\mathrm{hom}(\varGamma([\beta_0], \varGamma[\beta_1])$ denote the group of continuous homomorphisms, equipped with the compact-open topology. Then $\mathrm{hom}(\varGamma([\beta_0], \varGamma[\beta_1])$ becomes a topological group, and a neighborhood basis for 0 consists of the subgroups

$$\varOmega(N, V) = \{k : \varGamma([\beta_0]) \to \varGamma([\beta_1]) \mid k(N) \subset V\}$$

where N runs over compact subsets of $\varGamma([\beta_0])$ and V runs through open subgroups of $\varGamma([\beta_1])$. With this topology, a sufficient condition for a countable series $\sum_{k \in K} k$ in $\mathrm{hom}(\varGamma([\beta_0], \varGamma[\beta_1])$ to converge is that the terms converge to zero and that the terms are equicontinuous: for each open subgroup V in $\varGamma([\beta_1])$, there exists an open subgroup W in $\varGamma([\beta_0])$ such that $k(W) \subset V$ for all $k \in K$. When $\varGamma([\beta_0])$ contains a compact set which generates an open subgroup, equicontinuity is automatic. Equicontinous series allows to rearrange sums, which is captured in the definition of a c-complete system.

Definition 18. A local system of complete, filtered abelian groups $\varGamma$ is ***c*-complete** if it satisfies the following two conditions for every pair $[\beta_0]$, $[\beta_1]$:

i) for any c-finite set $S \subset \pi_1([\beta_0], \mathcal{B}(Y, K, \alpha), [\beta_1])$, the set $\{\varGamma_z \mid z \in S\} \subset \mathrm{hom}(\varGamma([\beta_0], \varGamma[\beta_1])$ is equicontinuous.

ii) for any c-finite set $S \subset \pi_1([\beta_0], \mathcal{B}(Y, K, \alpha), [\beta_1])$, the homomorphisms $\varGamma_z$ converge to zero in the compact-open topology as z runs through S.

Remark 19. i) The c-complete condition implies that a series like $\sum_z n_z \varGamma_z$ is convergent in the compact-open topology to a continuous limit.

ii) In the notation of [48], the "c" in c-complete represents the period class of a non-exact perturbation used to modify the Chern-Simons-Dirac functional. In our case, the holonomy perturbations are not changing the periods of CS, but we keep the c to use the same notation as Kronheimer and Mrowka. Moreover, since in our case the spectral flow around loops can be non-trivial, the system we define is not strongly c-complete, in the sense of [48, Definition 30.2.4].

Now we will exhibit a c-complete system suited for our purposes, based on [48, Section 30.2], [35, Section 4], [51, Section 2.2] , [77] and [78, Section 2.1]. As we will see, there are couple of natural candidates to use, but in order to minimize the number of auxiliary choices we have to make, we will stick with a particular local system which we will call the **universal Novikov field/local system** [78, Section 4]. Moreover, the universal Novikov field/local system has the advantage that the fibers $\varGamma_{[\beta]}$ assigned to each $[\beta]$ is independent of α (which does not happen for the other candidates), so it might be a more suited candidate if one wanted to compare the Floer homologies corresponding to different values of α (though Γ_z will depend on α which is to be expected as we will see in a moment).

First of all, if $\mathbb{F}$ is a ground field and $\Gamma \leq \mathbb{R}$ an additive subgroup, then the **Novikov field** $\Lambda^{\mathbb{F},\Lambda}$ associated to $\mathbb{F}, \Lambda$ is

$$\Lambda^{\mathbb{F},\Gamma} = \left\{ \sum_{r \in \Gamma} a_r T^r \mid a_r \in \mathbb{F} \text{ and } \#\{r \mid a_r \neq 0, r > C\} < \infty \text{ for all } C \in \mathbb{R} \right\}$$

In other words, we allow infinitely many terms in the negative direction. The example we will have in mind most of the time is $\mathbb{F} = \mathbb{Q}$ or $\mathbb{F} = \mathbb{C}$.

Now write $\pi_1(\mathcal{B}) = \pi_1(\mathcal{B}(Y, K, \alpha)) = \mathbb{Z} \oplus \mathbb{Z}$, and consider the universal covering $\tilde{\mathcal{B}}(Y, K, \alpha)$ of $\mathcal{B}(Y, K, \alpha)$. This is the space we obtain after taking the quotient of $\mathcal{C}(Y, K, \alpha)$ using only gauge transformations $g \in \mathcal{G}(Y, K)$ which satisfy $d(g) = (0, 0)$ (so they are in the connected component of the identity gauge transformation). The Chern-Simons function CS becomes real valued on $\tilde{\mathcal{B}}(Y, K, \alpha)$ so we can regard it as a map

$$CS : \tilde{\mathcal{B}}(Y, K, \alpha) \to \mathbb{R}$$

In this context $\tilde{\beta} \in \tilde{\mathcal{B}}(Y, K, \alpha)$ will denote a lift of $[\beta] \in \mathcal{B}(Y, K, \alpha)$. On $\pi_1(\mathcal{B}(Y, K, \alpha))$ we can also define the **period homomorphism**

$$\begin{aligned} \triangle_{CS} : \pi_1(\mathcal{B}(Y, K, \alpha)) &\to \mathbb{R} \\ g &\to 64\pi^2(k + 2\alpha l) \qquad\qquad d(g) = (k, l) \end{aligned}$$

as well as the **spectral flow** map

$$\begin{aligned} \mathrm{sf} : \pi_1(\mathcal{B}(Y, K, \alpha)) &\to \mathbb{Z} \\ z &\to \mathrm{sf}(z) \end{aligned}$$

whose kernel is the **annihilator**

$$\mathrm{Ann} = \ker \mathrm{sf} \subset \pi_1(\mathcal{B}(Y, K, \alpha))$$

We also define the additive subgroup of $\mathbb{R}$

$$I = \mathrm{im} \triangle_{CS} \subset \mathbb{R}$$

The different local systems we will now explain depend on how they are related to I, and each has their own set of advantages and disadvantages.

• **Minimal Novikov field/Local System:** here we work with Novikov field $\Lambda^{\mathbb{F},I'}$, where I' is defined as

$$I' = \triangle_{CS}(\mathrm{Ann})$$

Recalling the formula 20 for the grading, this means that we must evaluate $\triangle_{CS}$ on those loops for which

$$8k + 4l = 0$$

so in fact

$$I' = \{64\pi^2 k(1 - 4\alpha) \mid k \in \mathbb{Z}\}$$

The appealing feature of $\Lambda^{\mathbb{F},I'}$ is that when $\alpha = 1/4$ (the monotone case), I' collapses to $\{0\}$, in which case working over $\Lambda^{\mathbb{F},I'}$ is the same as working over $\mathbb{F}$, as should be the case in the monotone situation. The local system associated to $\Lambda^{\mathbb{F},I'}$ will assign to each configuration $[\beta]$ a copy of $\Lambda^{\mathbb{F},I'}$, i.e, $\Gamma_{[\beta]} = \Lambda^{\mathbb{F},I'}$. To specify what happens on paths, i.e, what the maps $\Gamma_z : \Gamma_{[\beta_0]} \to \Gamma_{[\beta_1]}$ should be, observe that since $\triangle_{CS} : \pi_1(\mathcal{B}(Y,K,\alpha)) \to \mathbb{R}$ is a homomorphism whose domain is a finitely generated abelian group and whose image is torsion free, there is an exact sequence

$$\ker \triangle_{CS} \rightarrowtail \pi_1(\mathcal{B}(Y,K,\alpha)) \twoheadrightarrow I$$

which splits, so $\pi_1(\mathcal{B}(Y,K,\alpha))$ can be identified with $\ker \triangle_{CS} \oplus I$, i.e,

$$\pi_1(\mathcal{B}(Y,K,\alpha)) \simeq \ker \triangle_{CS} \oplus I \tag{23}$$

By the same token, there is an exact sequence

$$\ker \mathrm{sf} \rightarrowtail \pi_1(\mathcal{B}(Y,K,\alpha)) \twoheadrightarrow 4\mathbb{Z}$$

which splits, so $\pi_1(\mathcal{B}(Y,K,\alpha))$ can be identified with $\mathrm{Ann} \oplus 4\mathbb{Z}$, i.e

$$\pi_1(\mathcal{B}(Y,K,\alpha)) \simeq \mathrm{Ann} \oplus 4\mathbb{Z} \tag{24}$$

This allows us to construction a projection

$$p : I \to I'$$

as follows. Regard I as a subset of $\pi_1(\mathcal{B}(Y,K,\alpha))$ under the first identification 23. Then using the second identification 24, project I onto Ann and let p be $\triangle_{CS}$ of this element. In other words

$$p(i) = \triangle_{CS}(\pi_{\mathrm{Ann}}(i))$$

Now, we can define Γ_z on **loops** as multiplication by $T^{p(-\mathcal{E}_{top}(z))}$. Fixing the map Γ_z on loops is enough to pin down the local system up to isomorphism. The "issue" with this option is that it is not clear to the author how the resulting Floer groups depend on the choice of projection p. In fact, the author was unable to write *explicitly* a projection p, so if one ever wanted to compute explicitly the resulting homology groups, it is not clear what to do in this circumstance.

• **Strongly c-complete Novikov field/Local System:** we call this intermediate case strongly c-complete in analogy with the examples given by Kronheimer and Mrowka in [48, Section 30.2]. In this case we use $\Lambda^{\mathbb{F},I}$ (so I instead of I') as the Novikov system and a local system which assigns to $[\beta]$ a copy of $\Lambda^{\mathbb{F},I}$, i.e, $\Gamma_{[\beta]} = \Lambda^{\mathbb{F},I}$. On loops Γ_z will act as multiplication by $T^{-\mathcal{E}_{top}(z)}$. Notice that this system has the advantage that we no longer need to choose a projection p. However, in order to specify actual groups (and not groups up to isomorphisms), further choices are needed, since knowing how Γ_z is defined on loops is not enough to specify a unique formula for how it should act on paths. More precisely, for a path $z : [\beta_0] \to [\beta_1]$ we would like $\Gamma_z : \Gamma_{[\beta_0]} \to \Gamma_{[\beta_1]}$ to be multiplication by $T^{-\mathcal{E}_{top}(z)}$. However, notice that for even a simple element inside $\Gamma_{[\beta_0]}$ like $T^{\triangle_{CS}(g)}$, for $g \in \pi_1([\beta_0], \mathcal{B}(Y,K,\alpha), [\beta_0])$, there is no reason why we must have that $\Gamma_z T^{\triangle_{CS}(g)} = T^{-\mathcal{E}_{top}(z)+\triangle_{CS}(g)}$ can be written as $T^{\triangle_{CS}(g')}$ for some $g' \in \pi_1([\beta_1], \mathcal{B}(Y,K,\alpha), [\beta_1])$. However, with additional choices we make sense of this formula. Namely, suppose that we choose a preferred lift $\tilde{\beta}$ for each $[\beta] \in \mathcal{B}(Y,K,\alpha)$. Then from formula (15) we have that $\triangle_{CS}(g) = 2\left(CS(\tilde{\beta}) - CS(g\tilde{\beta})\right)$, which means that an element inside $\Gamma_{[\beta_0]}$ can be rewritten as a formal power series

$$\sum_{r=\triangle_{CS}(g)} c_r T^{\triangle_{CS}(g)} = T^{2CS(\tilde{\beta}_0)} \sum_{r=\triangle_{CS}(g)} c_r T^{-2CS(g\tilde{\beta}_0)}$$

Then $\Gamma_z = \Gamma_{z,\tilde{\beta}_0,\tilde{\beta}_1}$ can be taken to be

$$\Gamma_{z,\tilde{\beta}_0,\tilde{\beta}_1}\left(\sum_{r=\triangle_{CS}(g)} c_r T^{\triangle_{CS}(g)}\right) \equiv T^{2CS(\tilde{\beta}_1)} \sum_{r=\triangle_{CS}(g)} c_r T^{-2CS(z\cdot g\cdot\tilde{\beta}_0)}$$

Notice that we added the "basepoints" in our notation to make it clear that an additional choice is needed. Here $z \cdot g \cdot \tilde{\beta}_0$ means the following: regard $g \cdot \tilde{\beta}_0$ as an element in $\tilde{\mathcal{B}}(Y, K, \alpha)$, projecting to $[\beta_0] \in \mathcal{B}(Y, K, \alpha)$, and use the path z to determine a (unique) element $z \cdot g \cdot \tilde{\beta}_0$, projecting to $[\beta_1] \in \mathcal{B}(Y, K, \alpha)$. Moreover, this formula uses implicitly the fact that if $r = \triangle_{CS}(g) = \triangle_{CS}(g')$, i.e, $CS(g'\tilde{\beta}) = CS(g\tilde{\beta})$ or equivalently $g' \circ g^{-1} \in \ker \triangle_{CS}$, then $CS(z \cdot g \cdot \tilde{\beta}_0) = CS(z \cdot g' \cdot \tilde{\beta}_0)$. In any case, these extra choices of basepoints will not make our lives any easier and do not gives us much of an advantage, so we will prefer to use the universal Novikov/ local system, which we will describe next.

• **Universal Novikov field/Local System**: as the name suggests, here we are working with $\Lambda^{\mathbb{F},\mathbb{R}}$. The local system assigns $[\beta]$ a copy of $\Lambda^{\mathbb{F},\mathbb{R}}$, i.e, $\Gamma_{[\beta]} = \Lambda^{\mathbb{F},\mathbb{R}}$. In order to define Γ_z for a path $z : [\beta_0] \to [\beta_1]$, we lift z to a path $\tilde{z} : \tilde{\beta}_0 \to \tilde{\beta}_1$ on the universal cover $\tilde{\mathcal{B}}(Y, K, \alpha)$ (where again CS is real valued), and define

$$\Gamma_z \equiv \text{multiplication by } T^{-2(CS(\tilde{\beta}_0) - CS(\tilde{\beta}_1))}$$

The factor of 2 in the exponent is chosen so that on loops it acts as multiplication by $T^{-\mathcal{E}_{top}(z)}$. Because of the gauge invariance of the perturbation $\mathfrak{p}$ we could also write this as

$$T^{-2(CS_{\mathfrak{p}}(\tilde{\beta}_0) - CS_{\mathfrak{p}}(\tilde{\beta}_1))} T^{-2(f_{\mathfrak{p}}([\beta_1]) - f_{\mathfrak{p}}([\beta_0]))}$$

Notice that the second term is independent of the lift of $[\beta_0], [\beta_1]$. In a way, it would have been more tempting to let Γ_z depend on the perturbation being used and define $\Gamma_{z,\mathfrak{p}}$ as multiplication by $T^{-2(CS_{\mathfrak{p}}(\tilde{\beta}_0) - CS_{\mathfrak{p}}(\tilde{\beta}_1))}$. This would give the advantage that for gradient flow-lines the exponent $-2(CS_{\mathfrak{p}}(\tilde{\beta}_0) - CS_{\mathfrak{p}}(\tilde{\beta}_1))$ would always be negative, hence making automatic the condition that the differential when we define it in the next section. However, once we analyze the case of cobordisms, it is more tricky to figure out what the right notion of perturbed topological energy is, given that we have to use many different kinds of holonomy perturbations, so this monotonicity property would not be automatic anyways. In other words, there is no way to avoid the fact that some control on the perturbations has to be imposed, and this is precisely what we will do in the next section (based on ideas from [13]).

4. Floer-Novikov Homology and its Relationship To the Knot Signature

Recall that for the construction of the usual instanton Floer homology groups on an integer homology sphere Y [20], the chain complex $CI(Y)$ is generated by (perturbed) irreducible flat connections and the proof that the differential ∂ on $CI(Y)$ squares to zero requires showing that no broken flow lines can factorize through the trivial flat connection θ, which cannot be eliminated through the use of perturbations.

Moreover, the proof that the homology $HI(Y)$ one obtains does not depend on the choice of metrics and perturbations, requires using the fact that in a generic one dimensional path of metrics and perturbations, a path of irreducible flat connections will stay away from the trivial connection θ.

To show these one uses the fact that the stabilizer of the trivial connection θ is positive dimensional and that the trivial connection is isolated from the irreducible (perturbed) flat connections. Even if there were no need for using local coefficients suitable analogues of these results are needed in our situation, which is what we will proceed to discuss next.

Recall that the critical points B of the unperturbed Chern-Simons functional CS on (Y, K, α) can be interpreted as flat connections on $Y \backslash K$, which modulo gauge will correspond to conjugacy classes of representations

$$\rho : \pi_1(Y \backslash K, y_0) \to SU(2)$$

Since $H_1(Y\backslash K;\mathbb{Z}) \simeq \mathbb{Z}[\mu_K]$ regardless of the pair (Y, K), the abelian representations of $\pi_1(Y\backslash K, y_0)$ are completely determined by their action on an oriented meridian μ_K of the knot K, which we already required to be conjugate to the matrix

$$\rho(\mu_K) \sim \begin{pmatrix} \exp(-2\pi i\alpha) & 0 \\ 0 & \exp(2\pi i\alpha) \end{pmatrix}$$

In other words, *for any fixed choice of α, there is only one reducible critical point of the unperturbed Chern-Simons functional CS,* which we denoted previously as $[\theta_\alpha]$.

In general, a representation $\rho : \pi_1(Y\backslash K, y_0) \to SU(2)$ will determine a local coefficient system $\mathfrak{g}_\rho$ on $Y\backslash K$, with fiber $\mathfrak{g} = \mathfrak{su}(2)$. In turn this gives rise to cohomology groups $H^i(Y\backslash K;\mathfrak{g}_\rho)$. If we identify ρ with a flat connection B, then its gauge orbit $[B]$ being isolated among the set of critical points $\mathfrak{C}$ of CS is equivalent to B being **non-degenerate,** that is, the kernel of the map

$$\ker : H^1(Y\backslash K;\mathfrak{g}_\rho) \to H^1(\mu_K;\mathfrak{g}_\rho) \tag{25}$$

is zero, where here μ_K is a collection of loops representing the meridians of all the components of K (which will be one since we are focusing on the case of a knot).

Here is an explanation of this condition as well of the notation. The map (25) is simply the one induced in cohomology by the inclusion of $\mu_K \hookrightarrow Y\backslash K$. When B is an *irreducible* flat connection, Lemma 3.13 in [45] shows that this condition is equivalent to the extended Hessian $\widehat{\text{Hess}}_B$ defined in (19) being invertible.

At the *reducible* flat connection θ_α, one can still use (25) as the criterion for being isolated, the only thing that changes is that this condition does not imply that the extended Hessian $\widehat{\text{Hess}}_{\theta_\alpha}$ is invertible, since at reducibles connections it will always have a non-trivial kernel [18, Section 2.5.4]. To better understand the requirement (25) at the reducible θ_α, notice that the Lie algebra $\mathfrak{g}_{\theta_\alpha} \simeq \mathfrak{su}(2) \simeq \mathfrak{so}(3)$ decomposes as

$$\mathfrak{g}_{\theta_\alpha} \simeq \mathbb{R} \oplus L_\alpha^{\otimes 2}$$

where $E = L_\alpha \oplus L_\alpha^{-1}$ is the decomposition of the $SU(2)$ bundle induced by θ_α. Hence

$$H^1\left(Y\backslash K;\mathfrak{g}_{\theta_\alpha}\right) \simeq H^1(Y\backslash K;\mathbb{R}) \oplus H^1(Y\backslash K; L_\alpha^{\otimes 2}) = \mathbb{R} \oplus H^1(Y\backslash K; L_\alpha^{\otimes 2})$$

One should think of the $\mathbb{R}$ summand as being the directions in the Zariski tangent space obtained from deforming the value of the holonomy α. Given that α is fixed for our problem and $H^1(Y\backslash K;\mathbb{R}) \to H^1(\mu_K;\mathfrak{g}_\rho)$ is an injection since $H^1(\mu_K;\mathfrak{g}_\rho) \simeq H^1(\mu_K;\mathbb{R})$ (see Lemma 63 in the Appendix), the condition (25) captures that we only need to worry about about the factor $H^1(Y\backslash K; L_\alpha^{\otimes 2})$. As we mentioned in the introduction, the condition (25) is only satisfied for certain values of α determined by the Alexander polynomial $\triangle_K(t)$ of K.

Lemma 20. *Suppose that K is a knot and that $\triangle_K(e^{-4\pi i\alpha}) \neq 0$. Then the reducible connection $[\theta_\alpha]$ is isolated from the irreducible flat connections.*

Proof. Again, it suffices to guarantee that $H^1(Y\backslash K; L_\alpha^{\otimes 2})$ vanishes. In the notation of the Appendix, the reducible connection $[\theta_\alpha]$ defines a local coefficient system with fiber $\mathbb{C}$ where the monodromy map $\hat{\rho}_\alpha : H_1(Y\backslash K;\mathbb{Z}) \to \mathbb{C}^*$ maps the meridian μ_K to $e^{-4\pi i\alpha}$. The vanishing of the cohomology $H^1(Y\backslash K; L_\alpha^{\otimes 2})$ now follows from Corollary 66 in the appendix.

□

For completeness sake, we will now give more details to Lemma 3.13 in [45], since our method of proof will generalize to the case of an embedded torus T inside a four manifold X with the homology of $S^1 \times S^3$.

Lemma 21. *Suppose that ρ is a flat connection corresponding to a critical point of CS and $\mathfrak{g}_\rho$ is the corresponding local system. Then the first (orbifold) cohomology $\check{H}^1(\check{Y};\mathfrak{g}_\rho)$ can be identified with* $\ker : H^1(Y\backslash K;\mathfrak{g}_\rho) \to H^1(\mu_K;\mathfrak{g}_\rho)$.

Proof. For the proof we will follow the suggestion of Lemma 3.13 in [45] and use the Mayer-Vietoris sequence for cohomology with local coefficients. More precisely, for $\epsilon > 0$ write $\check{Y}$ as

$$\check{Y} = (\check{Y}\backslash\check{\nu}_\epsilon(K)) \cup \check{\nu}(K)$$

where $\check{\nu}_\epsilon(K)$, $\check{\nu}(K)$ are tubular neighborhoods of K, and $\check{\nu}_\epsilon(K) \subset \check{\nu}(K)$. Then the first terms in the Mayer-Vietoris sequence for this decomposition read

$$\begin{aligned}
&\cdots \check{H}^0((\check{Y}\backslash\check{\nu}_\epsilon(K)) \cap \check{\nu}(K);\mathfrak{g}_\rho)\\
&\to \check{H}^1(\check{Y};\mathfrak{g}_\rho)\\
\to^{(\imath^*_{\check{Y}\backslash\check{\nu}_\epsilon(K)},\imath^*_{\check{\nu}(K)})}&\check{H}^1(\check{Y}\backslash\check{\nu}_\epsilon(K);\mathfrak{g}_\rho) \oplus \check{H}^1(\check{\nu}(K);\mathfrak{g}_\rho)\\
&\to \check{H}^1((\check{Y}\backslash\check{\nu}_\epsilon(K)) \cap \check{\nu}(K);\mathfrak{g}_\rho)\\
&\to \check{H}^2(\check{Y};\mathfrak{g}_\rho)\\
&\to \cdots
\end{aligned}$$

Here $\imath^*_{\check{Y}\backslash\check{\nu}_\epsilon(K)}$ and $\imath^*_{\check{\nu}(K)}$ denote the pullback in cohomology of the inclusions $\imath : \check{Y}\backslash\check{\nu}_\epsilon(K) \hookrightarrow \check{Y}$ and $\imath : \check{\nu}(K) \hookrightarrow \check{Y}$. Now using the usual properties of invariance of the cohomology groups under deformation retractions we can rewrite the previous sequence as

$$\begin{aligned}
&\cdots H^0(T_\epsilon;\mathfrak{g}_\rho)\\
&\to \check{H}^1(\check{Y};\mathfrak{g}_\rho)\\
\to^{(\imath^*_{Y\backslash K},\imath^*_{\check{\nu}(K)})}&H^1(Y\backslash K;\mathfrak{g}_\rho) \oplus \check{H}^1(\check{\nu}(K);\mathfrak{g}_\rho)\\
\to^{i^*_{T_\epsilon,\nu(K)} - i^*_{T_\epsilon,Y\backslash K}}&H^1(T_\epsilon;\mathfrak{g}_\rho)\\
&\to \cdots
\end{aligned}$$

where $T_\epsilon = \partial\check{\nu}_\epsilon(K)$. Notice that we have dropped the ˇ notation on the regions where the orbifold singularity is not present.

Observe first of all that since T_ϵ has abelian fundamental group then the restriction of $\mathfrak{g}_\rho$ to T_ϵ automatically becomes reducible and we can write

$$\mathfrak{g}_\rho \mid_{T_\epsilon} = \mathbb{R} \oplus L_\rho^{\otimes 2}$$

Now, because of the holonomy condition, for ϵ sufficiently small the local system defined by $L_\rho^{\otimes 2}$ is non-trivial, which means in particular that

$$H^\bullet(T_\epsilon; L_\rho^{\otimes 2}) \equiv 0$$

by Lemma (63) from the appendix. If we write μ_K and λ_K for the meridian and longitudes of K, then the Mayer-Vietoris sequence simplifies to

$$\begin{aligned}
&\cdots\mathbb{R}\\
&\to\check{H}^1(\check{Y};\mathfrak{g}_\rho) && (26)\\
&\to^{(\imath^*_{Y\setminus K},\imath^*_{\check{\nu}(K)})} H^1(Y\backslash K;\mathfrak{g}_\rho)\oplus\check{H}^1(\check{\nu}(K);\mathfrak{g}_\rho) && (27)\\
&\to^{i^*_{T_\epsilon,\nu(K)}-i^*_{T_\epsilon,Y\setminus K}}\mathbb{R}[\mu_K]\oplus\mathbb{R}[\lambda_K]\\
&\to\check{H}^2(\check{Y};\mathfrak{g}_\rho) && (28)\\
&\to\cdots
\end{aligned}$$

For computing $\check{H}^1(\check{\nu}(K);\mathfrak{g}_\rho)$ write

$$\check{\nu}(K)=S^1\times\check{D}^2$$

where we regard $\check{D}^2$ as an orbifold. In fact, we can think of $\check{D}^2$ as $D^2/\mathbb{Z}_\nu$, where $\mathbb{Z}_\nu$ denotes a cyclic action determining the cone angle $2\pi/\nu$. If $p:D^2\backslash\{0\}\to\check{D}^2\backslash\{0\}$ denotes the quotient map, then as long as the cone angle is sufficiently sharp and α is a rational value, the pullback of the flat connection ρ to $D^2\backslash\{0\}$ extends smoothly to a flat connection $p^*\rho$ on all of D^2, and in fact we can identify $\check{H}^1(\check{D}^2;\mathfrak{g}_\rho)$ with the equivariant cohomology $H^{1,\nu}(D^2;p^*\mathfrak{g}_\rho)$. Hence

$$\check{H}^1(\check{\nu}(K);\mathfrak{g}_\rho)=H^0(S^1;\mathfrak{g}_\rho)\otimes H^{1,\nu}(D^2;p^*\mathfrak{g}_\rho)\oplus H^1(S^1;\mathfrak{g}_\rho)\otimes H^{0,\nu}(D^2;p^*\mathfrak{g}_\rho)$$

Now, the factor $H^{1,\nu}(D^2;p^*\mathfrak{g}_\rho)$ will vanish since D^2 equivariantly retracts to the origin so that

$$H^{1,\nu}(D^2;p^*\mathfrak{g}_\rho)\simeq H^{1,\nu}(\{0\},p^*\mathfrak{g}_\rho)=0$$

Likewise, we have that

$$H^{0,\nu}(D^2;p^*\mathfrak{g}_\rho)\simeq H^{0,\nu}(\{0\};p^*\mathfrak{g}_\rho)\simeq\mathbb{R}$$

since only one factor of $\mathbb{R}^3\simeq p^*\mathfrak{g}_\rho\mid_{\{0\}\subset D^2}$ is preserved under the group action. To compute $H^1(S^1;\mathfrak{g}_\rho)$ we use again the fact that S^1 has abelian fundamental group which in particular means that $\mathfrak{g}_\rho$ reduces to the system

$$\mathfrak{g}_\rho\mid_{S^1}\simeq\mathbb{R}\oplus L_\rho^{\otimes 2}$$

In this case, $L_\rho^{\otimes 2}$ may or may not be the trivial system which means that

$$H^1(S^1;\mathfrak{g}_\rho)=\begin{cases}H^1(S^1;\mathbb{R}) & \text{if } L_\rho^{\otimes 2}\mid_{S^1}\ \text{non-trivial}\\ H^1(S^1;\mathbb{R})\otimes\mathbb{R}^3 & \text{if } L_\rho^{\otimes 2}\mid_{S^1}\ \text{trivial}\end{cases}$$

In any case that means that

$$\check{H}^1(\check{\nu}(K);\mathfrak{g}_\rho)\simeq H^1(S^1_{\lambda_K};\mathfrak{g}_\rho) \tag{29}$$

so the maps in the Mayer-Vietoris sequence 27 become

$$\begin{aligned}
\check{H}^1(\check{Y};\mathfrak{g}_\rho) &\to^{(\imath^*_{Y\setminus K},\imath^*_{\check{\nu}(K)})} && H^1(Y\backslash K;\mathfrak{g}_\rho)\oplus\check{H}^1(\check{\nu}(K);\mathfrak{g}_\rho)\\
\check{\omega} &\to && \left(\check{\omega}\mid_{Y\setminus K},\check{\omega}\mid_{\check{\nu}(K)}\right)
\end{aligned} \tag{30}$$

and

$$\begin{aligned}
H^1(Y\backslash K;\mathfrak{g}_\rho)\oplus\check{H}^1(\check{\nu}(K);\mathfrak{g}_\rho) &\to^{i^*_{T_\epsilon,\nu(K)}-i^*_{T_\epsilon,Y\setminus K}} && \mathbb{R}[\mu_K]\oplus\mathbb{R}[\lambda_K]\\
(\omega,\check{\varpi}) &\to && \left(\left\langle\omega\mid_{S^1_{\mu_K}},[\mu_K]\right\rangle,\left\langle(\omega-\check{\varpi})\mid_{S^1_{\lambda_K}},[\lambda_K]\right\rangle\right)
\end{aligned} \tag{31}$$

where the notation means that we are restricting the forms to the S^1 factors generated by the meridian and longitude respectively, and then using the pairing coming from the isomorphisms we established previously. In particular, since Mayer-Vietoris is an exact sequence this means that for any $\check{\omega} \in \check{H}^1(\check{Y};\mathfrak{g}_\rho)$ we have

$$\left(i^*_{T_\epsilon,\nu(K)} - i^*_{T_\epsilon,Y\backslash K}\right) \circ (\imath^*_{Y\backslash K}, \imath^*_{\check{\nu}(K)})\check{\omega} = 0$$

which according to the formulas 30 and 31 imply that

$$\left\langle \check{\omega} \mid_{S^1_{\mu_K}}, [\mu_K] \right\rangle = 0$$

In other words, there is a well defined map

$$\begin{aligned} \imath^* : \check{H}^1(\check{Y};\mathfrak{g}_\rho) \to & \quad \left\{\omega \in H^1(Y\backslash K;\mathfrak{g}_\rho) \mid \left\langle \omega \mid_{S^1_{\mu_K}}, [\mu_K] \right\rangle = 0\right\} \\ \check{\omega} \to & \quad \check{\omega} \mid_{Y\backslash K} \end{aligned}$$

or more succinctly $\imath^* : \check{H}^1(\check{Y};\mathfrak{g}_\rho) \to \ker : \left(H^1(Y\backslash K;\mathfrak{g}_\rho) \to H^1(m;\mathfrak{g}_\rho)\right)$ as we wrote in the statement of the lemma. We just need to show that this map is an isomorphism.

First we address the surjectivity of the map. Suppose that $\omega \in H^1(Y\backslash K;\mathfrak{g}_\rho)$ and that $\left\langle \omega \mid_{S^1_{\mu_K}}, [\mu_K] \right\rangle = 0$. Using the isomorphism 29 we can consider the element

$$\left(\omega, \omega \mid_{S^1_{\lambda_K}}\right) \in H^1(Y\backslash K;\mathfrak{g}_\rho) \oplus \check{H}^1(\check{\nu}(K);\mathfrak{g}_\rho)$$

Because $\left\langle \omega \mid_{S^1_{\mu_K}}, [\mu_K] \right\rangle = 0$ we have that $\left(\omega, \omega \mid_{S^1_{\lambda_K}}\right) \in \ker\left(i^*_{T_\epsilon,\nu(K)} - i^*_{T_\epsilon,Y\backslash K}\right)$ which means by exactness of the Mayer-Vietoris sequence that $\left(\omega, \omega \mid_{S^1_{\lambda_K}}\right) \in \mathrm{im}(\imath^*_{Y\backslash K}, \imath^*_{\check{\nu}(K)})$. In other words, there is $\check{\omega} \in \check{H}^1(\check{Y};\mathfrak{g}_\rho)$ such that $\imath^*(\check{\omega}) = \omega$, which is what we wanted to show.

For injectivity we simply observe that if $\imath^*(\check{\omega}) = [0]$ then $\omega = \check{\omega} \mid_{Y\backslash K}$ is exact, that is, $\omega = d_B\xi$ for some $\xi \in \Omega^0(Y\backslash K;\mathfrak{g}_\rho)$ and B is the connection representative of the flat connection ρ. Now we can compute the norm of $\check{\omega}$ in the following way, which is essentially the same computation as in [29, Proposition 2.10]

$$\begin{aligned} \|\check{\omega}\|^2_{\check{L}^2(\check{Y})} &= -\lim_{\epsilon\to 0}\int_{Y\backslash\check{\nu}_\epsilon(K)} \mathrm{tr}(*\omega \wedge \omega) \\ &= -\lim_{\epsilon\to 0}\int_{Y\backslash\check{\nu}_\epsilon(K)} \mathrm{tr}(*\omega \wedge d_B\xi) \\ &= \lim_{\epsilon\to 0}\left[\int_{Y\backslash\check{\nu}_\epsilon(K)} \mathrm{tr}(d_B(*\omega)\wedge\xi) - \int_{T_\epsilon}\mathrm{tr}(*\omega\wedge\xi)\right] \\ &= -\lim_{\epsilon\to 0}\int_{T_\epsilon}\mathrm{tr}(*\omega\wedge\xi) \\ &= 0 \end{aligned}$$

where we have used that $d_B(*\omega) = *d_B^*\omega = 0$ since $\check{\omega} \in \ker\check{\triangle}_B = \ker(\check{d}_B) \cap \ker(\check{d}_B^*)$. That $\lim_{\epsilon\to 0}\int_{T_\epsilon}\mathrm{tr}(*\omega\wedge\xi) = 0$ vanishes is essentially a restatement that on orbifolds one can use integration by parts [50, Section 2.1]. Alternatively, one can follow the approximation scheme in the proof of [41, Proposition 8.3]. Hence $\|\check{\omega}\|^2_{\check{L}^2(\check{Y})} = 0$ which also means that $\check{\omega} = 0$, which is what we needed to show for the proof of injectivity. $\square$

Using values of $\alpha \in \mathbb{Q} \cap (0, 1/2)$ which satisfy $\triangle_K(e^{-4\pi i\alpha}) \neq 0$, and with a local system which satisfies the properties discussed in the previous section, we can define the chain complex

$$(C_*(Y, K, \alpha; \Gamma, \mathfrak{p}), \partial)$$

where we are also indicating the dependence of the chain complex on the perturbation used to achieve transversality. Again, the local coefficient system we will have in mind is the universal Novikov field/local system described in the previous section, in order to minimize additional choices to get an actual Floer group, rather than a group up to isomorphism.

We will choose our perturbation $\mathfrak{p}$ as

$$\mathfrak{p} = \mathfrak{p}_{\text{crit}} + \mathfrak{p}_\partial$$

where $\mathfrak{p}_{\text{crit}}$ and $\mathfrak{p}_\partial$ mean the following. If $\mathfrak{p}_{\text{crit}}$ is a perturbation which makes the critical set $\mathfrak{C}_{\mathfrak{p}_{\text{crit}}}$ non-degenerate, then we can approximate $\mathfrak{p}_{\text{crit}}$ by a finite sum of holonomy perturbations. Since the non-degeneracy of the compact set $\mathfrak{C}_{\mathfrak{p}_{\text{crit}}}$ is an open condition, we can arrange that $f_{\mathfrak{p}_{\text{crit}}}$ is a finite sum by truncating $\mathfrak{p}_{\text{crit}}$ after finitely many terms. This guarantees that the support of the holonomy perturbations do not meet a neighborhood of the knot K. This argument appears in Proposition 3.1 of [47], although the setup in Kronheimer and Mrowka's paper is more complicated since they also need to guarantee certain finiteness condition for the moduli spaces of flow lines of dimension less or equal to 2. Then $\mathfrak{p}_\partial$ is the remaining perturbation needed to cut out the moduli spaces of trajectories transversely.

Because of this support condition, we can appeal to Herald's result [32, Theorem 0.1], which relates the signed count of elements in the (finite) set $\mathfrak{C}_{\mathfrak{p}_{\text{crit}}}$ to the Casson invariant and the knot signature of K.

Theorem 22. *Suppose that $\alpha \in \mathbb{Q} \cap (0, 1/2)$ satisfies $\triangle_K(e^{-4\pi i\alpha}) \neq 0$. Then the signed count of $\mathfrak{C}_{\mathfrak{p}_{crit}}$, which we denote $\#_s|\mathfrak{C}_{\mathfrak{p}_{crit}}|$, equals*

$$\#_s|\mathfrak{C}_{\mathfrak{p}_{crit}}| = 4\lambda_C(Y) + \frac{1}{2}\sigma_K(e^{-4\pi i\alpha}) \tag{32}$$

where $\lambda_C(Y)$ is the Casson invariant of Y and $\sigma_K(e^{-4\pi i\alpha})$ the Tristram-Levine knot signature of K evaluated at $e^{-4\pi i\alpha}$.

Moreover, as vector spaces over the Novikov field Λ, we have that

$$\chi_\Lambda(HI(Y, K, \alpha)) = 4\lambda_C(Y) + \frac{1}{2}\sigma_K(e^{-4\pi i\alpha})$$

Notice that the second part of previous statement implicitly assumed:

a) A choice for the orientation of the moduli spaces involved in the definition of $\mathfrak{C}_{\mathfrak{p}_{crit}}$ and $HI(Y, K, \alpha)$.

b) The claim that there is a well defined differential ∂ which squares to zero so that we can take $HI(Y, K, \alpha) = \ker \partial / \mathrm{im} \partial$.

c) A proof that $HI(Y, K, \alpha)$ is independent of the perturbations used.

We will start by discussing the orientation of the moduli spaces, which we have kept under the rug until this point. For our conventions we will follow closely the exposition in [71, Section 4.3], which in turn are based on those used in [18, 44, 45]. They also agree with the conventions used by Collin and Steer in their paper [11], which we used to pin down the signs in the formula (32).

The orientation set $o([\beta])$ of a critical point $\mathfrak{C}_{\mathfrak{p}}$ refers to the 2-element set of orientations of the real line $\det D_A$, where A is a connection on $\mathbb{R} \times Y$ such that $A \mid_{\{t\} \times Y}$ is (gauge) equivalent to θ_α for t sufficiently negative and (gauge) equivalent to β for t sufficiently large. Here D_A is the

Fredholm operator $-d_A^* \oplus d_A^+$, after suitable Sobolev completions have been introduced. In general any reference connection would work, but the advantage of using the reducible connection θ_α as one of the limiting connections is that it will automatically gives us an absolute $\mathbb{Z}/4\mathbb{Z}$ grading as we will explain momentarily.

In general, given two connections $\beta_1 \in \mathcal{C}(Y_1, K_1, \alpha)$ and $\beta_2 \in \mathcal{C}(Y_2, K_2, \alpha_2)$ and a cobordism

$$(W, \Sigma, \alpha) : \emptyset \to (-Y_1, -K_1, \alpha) \sqcup (Y_2, K_2, \alpha)$$

one can consider the operator (for $\epsilon > 0$ sufficiently small)

$$D'_A = -d_A^* \oplus d_A^+ : \check{L}^p_{m,\phi'_\epsilon}(\Lambda^1 \otimes \mathfrak{g}_P) \to \check{L}^p_{m-1,\phi'_\epsilon}((\Lambda^0 \oplus \Lambda^+) \otimes \mathfrak{g}_P))$$

where $A \in \mathcal{C}(W^*, \Sigma^*, \alpha)$ is a connection on the completion

$$\begin{aligned} W^* &= (\mathbb{R}^- \times Y_1) \cup W \cup (\mathbb{R}^+ \times Y_2) \\ \Sigma^* &= (\mathbb{R}^- \times K_1) \cup W \cup (\mathbb{R}^+ \times K_2) \end{aligned}$$

which is asymptotic to β_1 for t sufficiently negative and to β_2 for t sufficiently positive. Here $\check{L}^p_{m,\phi'_\epsilon} = e^{\phi'_\epsilon}\check{L}^p_m$ denotes a weighted Sobolev space, weighted by a real function $e^{\phi'_\epsilon}$, where ϕ'_ϵ is a non-positive smooth function equal to ϵt for t sufficiently negative on $\mathbb{R}^- \times Y_1$, to $-\epsilon t$ for t sufficiently large on $\mathbb{R}^+ \times Y_2$, and equal to 0 on W.

When β_1 is a perturbed instanton on $(Y_1, K_1, \alpha, \mathfrak{p}_1)$, with respect to the perturbation $\mathfrak{p}_1$, and similarly for β_2 with respect to the perturbation $\mathfrak{p}_2$ on $(Y_2, K_2, \alpha, \mathfrak{p}_2)$, then the moduli space $\mathcal{M}([\beta_1], (W, \Sigma), [\beta_2])$ of perturbed α-ASD instantons, which we will denote as $\mathcal{M}([\beta_1], \check{W}, [\beta_2])$ will decompose into connected components

$$\mathcal{M}([\beta_1], \check{W}, [\beta_2]) = \bigcup_z \mathcal{M}_z([\beta_1], \check{W}, [\beta_2])$$

and the dimension of $\mathcal{M}_z([\beta_1], \check{W}, [\beta_2])$ can be computed as $\text{ind} D'_A$, where A is an appropriate representative of an element $[A] \in \mathcal{M}_z([\beta_1], \check{W}, [\beta_2])$. For a composite cobordism, one has the additivity relation (assuming the critical points are non-degenerate) [71, eq. 4.3]

$$\begin{aligned} &\dim \mathcal{M}_{z' \circ z}([\beta_0], \check{W}' \circ \check{W}, [\beta_2]) \\ &= \dim \mathcal{M}_z([\beta_0], \check{W}, [\beta_1]) + \dim \text{stab}[\beta_1] + \dim \mathcal{M}_{z'}([\beta_1], \check{W}', [\beta_2]) \end{aligned} \tag{33}$$

When $[\beta_0], [\beta_1]$ are irreducible (perturbed) flat connections on (Y, K, α), we had defined in Section 3 a relative grading $\text{gr}([\beta_0], [\beta_1]) \in \mathbb{Z}/4\mathbb{Z}$ in terms of the spectral flow of the extended Hessian. Since the spectral flow of this operator can be interpreted as the index of the operator D'_A on the cylinder $\mathbb{R} \times Y$, this means that the relative grading can be interpreted as the dimension (mod-4) of the moduli space of flow-lines between $[\beta_0]$ and $[\beta_1]$:

$$\text{gr}([\beta_0], [\beta_1]) = \dim \mathcal{M}([\beta_0], [\beta_1]) \mod 4$$

where again we are looking at moduli spaces asymptotic to $[\beta_0]$ as $t \to -\infty$ and to $[\beta_1]$ as $t \to \infty$.

To make this an absolute grading, set

$$\text{gr}([\beta]) = -1 - \dim \mathcal{M}([\theta_\alpha], [\beta]) \mod 4 = \dim \mathcal{M}([\beta], [\theta_\alpha]) \mod 4 \tag{34}$$

and give grading 0 to the reducible connection $[\theta_\alpha]$. Notice that because $[\theta_\alpha]$ has a one-dimensional stabilizer according to (33) we have that

$$\begin{aligned}
&\mathrm{gr}([\beta_0]) - \mathrm{gr}([\beta_1]) \\
=&\dim \mathcal{M}([\theta_\alpha],[\beta_1]) - \dim \mathcal{M}([\theta_\alpha],[\beta_0]) \mod 4 \\
=&\dim \mathcal{M}([\theta_\alpha],[\beta_1]) + \dim \mathcal{M}([\beta_0],[\theta_\alpha]) + 1 \mod 4 \\
=&\dim \mathcal{M}([\beta_0],[\beta_1]) \mod 4 \\
=&\mathrm{gr}([\beta_0],[\beta_1])
\end{aligned}$$

With respect to this absolute grading, if $\mathfrak{C}_{\mathfrak{p},i}$ denotes the set of critical points associated to the perturbation $\mathfrak{p}$ whose absolute grading is $i \in \mathbb{Z}/4\mathbb{Z}$, then an expression like (32) means

$$\#_s|\mathfrak{C}_{\mathfrak{p}_{\mathrm{crit}}}| = \#|\mathfrak{C}_{\mathfrak{p}_{\mathrm{crit}},0}| + \#|\mathfrak{C}_{\mathfrak{p}_{\mathrm{crit}},2}| - \#|\mathfrak{C}_{\mathfrak{p}_{\mathrm{crit}},1}| - \#|\mathfrak{C}_{\mathfrak{p}_{\mathrm{crit}},3}|$$

Now we must turn to a discussion of the properties of the perturbations used in order to guarantee that we have a well defined differential and that the homology groups we obtain are independent of the perturbations used. First we need to understand what is at stake. Recall that we are assigning to each element $[\beta] \in \mathcal{B}(Y, K, \alpha)$ a vector space $\Gamma_{[\beta]}$ which is a copy of the universal Novikov field $\Lambda^{\mathbb{Q},\mathbb{R}}$. In other words, an element of $\Gamma_{[\beta]}$ is a formal power series

$$\sum_{r\in\mathbb{R}} a_r T^r \tag{35}$$

where $a_r \in \mathbb{Q}$, and $\#\{r \mid a_r \neq 0, r > C\} < \infty$ for all $C \in \mathbb{R}$. Therefore, the power series can extend indefinitely in the negative direction, but not the positive one. Now, the differential ∂ on the chain complex $C_*(Y, K, \alpha, \Gamma, \mathfrak{p})$ should be defined by the formula (for $[\beta_1] \in \mathfrak{C}(Y, K, \alpha, \mathfrak{p})$)

$$\partial[\beta_1] = \sum_{[\beta_2]\in\mathfrak{C}(Y,K,\alpha,\mathfrak{p})|\mathrm{gr}([\beta_1],[\beta_2])=1 \mod 4} \sum_z n_z([\beta_1],[\beta_2]) T^{-\mathcal{E}_{top}(z)}[\beta_2] \tag{36}$$

Here

$$\mathcal{E}_{top}(z) = \frac{1}{8\pi^2}\int_{\mathbb{R}\times\check{Y}} \mathrm{tr}(F_A \wedge F_A)$$

where $[A] \in \mathcal{M}_z([\beta_1],[\beta_2])$. In the absence of perturbations, i.e, $\mathfrak{p} = 0$ then $\mathcal{E}_{top}(z)$ would equal $\frac{1}{8\pi^2}\|F_A^-\|_{L^2} = 2(CS(\beta_1) - CS(\beta_2))$, since A would solve the α-ASD equation $F_A^+ = 0$. Therefore, the exponents in $T^{-\mathcal{E}_{top}(z)}$ would always be non-positive and do not accumulate on any finite subinterval, since they must always differ from each other by some integer combination of $64\pi^2$ and $128\pi^2\alpha$ (this follows from the formula (16) for the topological energy in the closed 4-manifold case). In other words, the candidate for the differential (36) does make sense as an element (35) of $\Lambda^{\mathbb{Q},\mathbb{R}}$.

In the presence of perturbations $\mathfrak{p}$, the inequality that is satisfied is

$$\mathcal{E}_{top}(z) + f_{\mathfrak{p}}([\beta_1]) - f_{\mathfrak{p}}([\beta_2]) \geq 0$$

Since $f_{\mathfrak{p}}$ is fully gauge equivariant as we mentioned when we introduced the holonomy perturbations, the lower bound for $\mathcal{E}_{top}(z)$

$$\mathcal{E}_{top}(z) \geq f_{\mathfrak{p}}([\beta_2]) - f_{\mathfrak{p}}([\beta_1]) \tag{37}$$

is independent of the specific moduli space $\mathcal{M}_z([\beta_1],[\beta_2])$ we are looking at, so the expression (36) continues to be well defined as an element of $\Lambda^{\mathbb{Q},\mathbb{R}}$.

After we know that ∂ makes sense, proving $\partial^2 = 0$ is in this setting is no different than the situation for monopole Floer homology with local coefficients [48, Section 30], except for two main differences:

(1) In the instanton setup, bubbling in general can play a role.
(2) We have excluded the reducible flat connection from our chain group, so we need to guarantee that the broken trajectories considered to show that $\partial^2 = 0$ do not include factorizations through this reducible connection.

Regarding the first point, this is not a problem for showing that $\partial^2 = 0$, since bubbles drop the dimension of the moduli spaces involve by at least 4 [45, Proposition 3.22], and our transversality assumptions for the moduli space of flow lines guarantee that negative dimensional moduli spaces are empty.

The second point involves analyzing the non-degeneracy of the reducible connection and some index formulas.

- Even after perturbations, there is still one reducible connection $[\theta_\alpha]$ up to gauge, the same one as the unperturbed case, since $[\theta_\alpha]$ is isolated from the irreducible connections so one can choose the holonomy perturbations in such a way that they vanish near $[\theta_\alpha]$. Notice that $[\theta_\alpha]$ is unobstructed since by Poincare duality $\check{H}^2(\check{Y};\mathfrak{g}_\alpha) \simeq \check{H}^1(\check{Y};\mathfrak{g}_\rho) = 0$.
- If $\mathcal{M}_z([\beta_0],[\beta_2])$ represents a d-dimensional moduli space, and there is a broken trajectory belonging to $\mathcal{M}_{z_1}([\beta_0],[\theta_\alpha]) \times \mathcal{M}_{z_2}([\theta_\alpha],[\beta_2])$ with dimensions d_1, d_2 respectively, the fact that $[\theta_\alpha]$ has one dimensional stabilizer implies that
$$\dim \mathcal{M}_z([\beta_0],[\beta_2]) = \dim \mathcal{M}_{z_1}([\beta_0],[\theta_\alpha]) + \dim \mathcal{M}_{z_2}([\theta_\alpha],[\beta_2]) + 1$$
so in particular the right hand side is bounded from below by 3, since each d_1, d_2 must be at least one dimensional. Hence moduli spaces admitting $\mathcal{M}_z([\beta_0],[\beta_2])$ such factorizations through the reducible connection $[\theta_\alpha]$ must be at least three dimensional, so they can be ignored for the definition of the differential. Notice that this is the same as what happens in the ordinary case of Instanton Floer homology for integer homology spheres, but in that case the bound for the dimension is 5, since the stabilizer of the trivial connection is three-dimensional.

This means that we can define the **Instanton Floer-Novikov homology for knots** $HI(Y,K,\alpha)$ as $\ker\partial/\mathrm{im}\partial$. Notice that our notation makes implicit that the homology we obtain is independent of the choice of perturbation $\pi \in \mathcal{P}$. To see why this is true, we adapt the proof of independence in [18, Section 5.3] to our situation, and discuss more generally the functoriality properties of these instanton Floer-Novikov groups.

Suppose that $(W,\Sigma) : (Y_1,K_1) \to (Y_2,K_2)$ is a concordance of the knots K_1, K_2. By this we mean that Y_1, Y_2 will be both be integer homology spheres and W homology cobordism, i.e $H_*(W;\mathbb{Z}) = H_*(Y_i;\mathbb{Z})$ for $i = 1,2$. Moreover, Σ will be an embedded annulus with $\partial\Sigma = -K_1 \sqcup K_2$.

Finally, suppose that the cobordism (W,Σ) is α-admissible in the sense of Definition 9. Recall that this means that for the unique reducible up to gauge $\theta_{W,\alpha}$ we have $H^1(W\backslash\Sigma; L^{\otimes 2}_{\theta_{W,\alpha}}) = 0$. Since $\theta_{Y_1,\alpha}$ and $\theta_{Y_2,\alpha}$ also satisfy $H^1(Y_i\backslash K_i; L^{\otimes 2}_{\theta_{Y_i,\alpha}}) = 0$ for $i = 1,2$, it is immediate that on the completion $H^1(W^*\backslash\Sigma^*; L^{\otimes 2}_{\theta_{W^*,\alpha}}) = 0$, and an straightforward adaptation of Lemma 21 will imply that for α-admissible cobordisms the reducible $\theta_{W,\alpha}$ is isolated (and non-degenerate as well).

We want to define a cobordism map
$$m_{(W,\Sigma)} : C_*(Y_1,K_1,\alpha,\Gamma_{\mathfrak{p}},\mathfrak{p}_1) \to C_*(Y_2,K_2,\alpha,\Gamma_{\mathfrak{p}},\mathfrak{p}_2)$$

via the formula

$$m_{(W,\Sigma)}[\beta_1] = \sum_{[\beta_2]} \sum_{z:[\beta_1]\to[\beta_2]} m_z([\beta_1], \check{W}, [\beta_2]) T^{-\mathcal{E}_{top}(z)}[\beta_2] \tag{38}$$

Here the sum is taking place over all homotopy classes z for which the moduli space $\mathcal{M}_z([\beta_1], \check{W}, [\beta_2])$ is zero dimensional, and the notation $\check{W}$ is emphasizing that we are regarding $\check{W}$ as an orbifold. Before showing that $m_z([\beta_1], \check{W}, [\beta_2])$ is well defined, let's discuss first how the cobordism maps should interact with the gradings $HI_i(Y, K, \alpha)$ of the Floer groups.

If $\mathrm{gr}([\beta_i])$ denotes the (absolute) $\mod 4$ grading of $[\beta_i]$ then [44, Proposition 4.4], [11, Section 4.3] shows that

$$\dim \mathcal{M}_z([\beta_1], \check{W}, [\beta_2]) = \mathrm{gr}([\beta_1]) - \mathrm{gr}([\beta_2]) - \frac{3}{2}(\chi(W) + \sigma(W)) - \chi(\Sigma) \mod 4 \tag{39}$$

In particular, given our assumptions on (W, Σ), we can see that the cobordism map $m_{(W,\Sigma)}$ is a sum over all elements $[\beta_2]$ whose relative grading is the same as $[\beta_1]$. What is left to see is why the formula for $m_{(W,\Sigma)}[\beta_1]$ defines an element in $\Lambda^{\mathbb{Q},\mathbb{R}}$.

This requires further discussion of how the ASD equation is perturbed on a cobordism. Here we follow [45, Section 3.8], [13, Section 2.2], [40, Section 3.1] and [59, Section 4]. On the cobordism W we choose a collar neighborhood of each boundary component, and if t denotes the coordinate (say in the collar $[0,1) \times Y_1$) , we choose a t -dependent holonomy perturbation $\mathfrak{p}_t$ equal to $\mathfrak{p}_1$ on the near $[0, 1/4) \times Y_1$. On $(3/4, 1) \times Y_1$ the perturbation $\mathfrak{p}_t$ vanishes and then we interpolate on $[1/4, 3/4] \times Y_1$ choosing an auxiliary perturbation $\tilde{\mathfrak{p}}_1 \in \mathcal{P}_{Y_1}$. The net effect is that on $[0,1) \times Y_1$ the perturbed equations take the form

$$F_A^+ + \beta_1(t) U_1(A) + \tilde{\beta}_1(t) \tilde{U}_1(A) = 0$$

where $\beta_1, \tilde{\beta}_1$ denote suitable cut-off functions and $U_1, \tilde{U}_1$ denote the perturbation terms associated to $\beta_1, \tilde{\beta}_1$. Similar remarks apply to Y_2. These were the perturbations used in [45, Section 3.8]. To deal with transversality issues involving flat connections on the cobordism, we also need interior holonomy perturbations, supported on a compact subset of $W \backslash \mathrm{nbd}(\Sigma \cup \partial W)$, which were introduced for closed 4-manifolds in [40], although analogous constructions appear for example in [17, 22].

These are constructed in a similar way to how cylinder functions were constructed on a 3 manifold. We sketch the construction given in [45, Section 3.8], which also appears in [59, Definition 4.2]. Namely, one chooses a closed ball $B \subset W \backslash \mathrm{nbd}(\partial W)$ and a finite collection of smooth submersions $q_i : S^1 \times B \to W \backslash \mathrm{nbd}(\Sigma \cup \partial W)$ so that $q_i(1, b) = b$ and $q_i(-, b)$ is an immersion for all $1 \leq i \leq n$ and $b \in B$. Choose also a self dual two form ω whose support is contained in B. Then for $\mathbf{q} = (q_1, \cdots, q_n)$ we have a section

$$V_{\mathbf{q},\omega} : \mathcal{C}(W, \Sigma, \alpha) \to \Omega^{2,+}(W; \mathfrak{g}_E)$$

given by

$$V_{\mathbf{q},\omega}(A)(x) = \mathrm{Hol}_{\mathbf{q}}(A) \otimes \omega(x)$$

Again, after introducing suitable completions one constructs a Banach space of secondary holonomy perturbations with a notion of L^1 convergence [59, p.51]. Therefore, on the cobordism with cylindrical ends W^* the perturbed α-ASD equations we must consider are of the form

$$F_A^+ + U_{\mathfrak{p}}(A) + V_{\boldsymbol{\omega}}(A) = 0 \tag{40}$$

where the term $U_{\mathfrak{p}}(A)$ is generic notation for the cylindrical holonomy perturbations and $V_{\boldsymbol{\omega}}(A)$ denotes the interior holonomy perturbations. The important property we need to know about $U_{\mathfrak{p}}$ and $V_{\boldsymbol{\omega}}$ is that their L^∞ norms are uniformly bounded: that is, there exists $K>0$ such that

- $\|U_{\mathfrak{p}}(A)\|_{L^\infty} \leq K\|\mathfrak{p}\|_{\mathcal{P}}$ (this is statement iii) in [45, Proposition 3.7].
- $\|V_{\boldsymbol{\omega}}(A)\|_{L^\infty} \leq K\|\boldsymbol{\omega}\|_{\tilde{\mathcal{P}}}$ (this is statement 2) in [59, Proposition 4.4], which first appeared in [40, Section 3.2]).

In fact, for transversality purposes, which is the reason why these perturbations were introduced in the first place, for any given $\epsilon>0$, we can assume that we chose perturbations $\mathfrak{p}$ and $\boldsymbol{\omega}$ satisfying $\|\mathfrak{p}\|_{\mathcal{P}}, \|\boldsymbol{\omega}\|_{\tilde{\mathcal{P}}} < \epsilon$, so in particular that will mean that

$$|f_{\mathfrak{p}}([\beta])| < \epsilon$$

for all $[\beta] \in \mathcal{B}(Y,K,\alpha)$. To bound $\mathcal{E}(z)$ from below, we follow the proof of [13, Proposition 2.15] and split W^* into three regions

$$\begin{aligned}
&\frac{1}{8\pi^2}\int_{W^*} \operatorname{tr}(F(A)\wedge F(A)) \\
=&\frac{1}{8\pi^2}\int_{\mathbb{R}^-\times \check{Y}_1} \operatorname{tr}(F(A)\wedge F(A)) + \frac{1}{8\pi^2}\int_{W} \operatorname{tr}(F(A)\wedge F(A)) + \frac{1}{8\pi^2}\int_{\mathbb{R}^+\times \check{Y}_2} \operatorname{tr}(F(A)\wedge F(A)) \\
\geq& \Big[f_{\mathfrak{p}}([A\mid_{\{0\}\times\check{Y}_1}]) - f_{\mathfrak{p}}([\beta_1])\Big] + \frac{1}{8\pi^2}\int_W \left(|F^-(A)|^2 - |F^+(A)|^2\right) + \Big[f_{\mathfrak{p}}([\beta_2]) - f_{\mathfrak{p}}([A\mid_{\{0\}\times\check{Y}_2}])\Big] \\
\geq& -2\epsilon + f_{\mathfrak{p}}([\beta_2]) - f_{\mathfrak{p}}([\beta_1]) - \frac{1}{8\pi^2}\int_W |U_{\mathfrak{p}}(A)+V_{\boldsymbol{\omega}}(A)|^2 \\
\geq& -2\epsilon + f_{\mathfrak{p}}([\beta_2]) - f_{\mathfrak{p}}([\beta_1]) - \frac{3\epsilon^2}{8\pi^2}
\end{aligned}$$

In these steps we used the inequality 37 to deal the first and third integrals while we used the equation 40 on the second integral. The specific bound is not that important, only knowing that it does not depend on the component of the moduli space that is being analyzed. Therefore $m_{(W,\Sigma)}[\beta_1]$ will define an element of $\Lambda^{\mathbb{Q},\mathbb{R}}$ once we know that the numbers $m_z([\beta_1],\check{W},[\beta_2])$ are well defined.

For that we follow [18, Proposition 5.9]. That is, we want to show that the 0-dimensional moduli spaces $\mathcal{M}_z([\beta_1],\check{W},[\beta_2])$ with topological energy $\mathcal{E}(z)$ are compact. This is because if we start with a sequence $[A_i]$ in $\mathcal{M}_z([\beta_1],\check{W},[\beta_2])$ (which is defined using a bundle $P([\beta_1],[\beta_2])$, then it would converge weakly to

- An ideal instanton $([A_\infty],x_\infty)$ on a bundle Q over W^*, asymptotic on each cylindrical end to $[\beta_1'],[\beta_2']$ respectively.
- A broken trajectory $([A_1],x_1)$ over $\mathbb{R}\times Y_1$ connection $[\beta_1]$ and $[\beta_1']$, and a broken trajectory $([A_2],x_2)$ over $\mathbb{R}\times Y_2$ connecting $[\beta_2']$ and $[\beta_2]$.

Additivity of the index says that (again because the critical points are non-degenerate)

$$\begin{aligned}
0 =& \operatorname{ind}P([\beta_1],[\beta_2]) \\
=& \operatorname{ind}[A_\infty] + \operatorname{ind}[A_1] + \operatorname{ind}[A_2] + \dim \check{H}^0(Y_1;\mathfrak{g}_{\beta_1'}) + \dim \check{H}^0(Y_2;\mathfrak{g}_{\beta_2'}) + 4(|x_\infty|+|x_1|+|x_2|)
\end{aligned} \tag{41}$$

- If $[A_\infty]$ is irreducible then by transversality $\operatorname{ind}[A_\infty]\geq 0$ and since $[A_1],[A_2]$ are both irreducible given that at least one of their limits is irreducible, then $\operatorname{ind}[A_1]\geq 0, \operatorname{ind}[A_2]\geq 0$ by transversality. Then the only way for the above equality to hold is if $[\beta_1]=[\beta_1']$, $[\beta_2]=[\beta_2']$ and there were no bubbles, i.e, $|x_\infty|=|x_1|=|x_2|=0$.

• If $[A_\infty]$ is reducible then our assumptions on homology imply that $H_1(W\backslash\Sigma;\mathbb{Z}) \simeq H_1(W^*\backslash\Sigma^*;\mathbb{Z}) \simeq \mathbb{Z}$ and because of the holonomy condition $[A_\infty]$ is determined to be the unique (up to gauge) reducible with S^1 stabilizer which is asymptotic to the reducibles $[\theta_{\alpha,Y_1}]$ and $[\theta_{\alpha,Y_2}]$ respectively. In this case $\mathrm{ind}[A_\infty]$ can be computed from the dimension formula 39, with the caveat that the formula as written only works assuming that the limits are irreducible connections. When the limits are reducible, one needs to take into account the stabilizer in the formula and one concludes that $\mathrm{ind}[A_\infty] = -1$. Since $\dim \check{H}^0(Y_1;\mathfrak{g}_{\theta_{\alpha,Y_1}}) = \check{H}^0(Y_2;\mathfrak{g}_{\theta_{\alpha,Y_2}}) = 1$ one finds in 41 that

$$-1 = \mathrm{ind}[A_1] + \mathrm{ind}[A_2] + 4(|x_\infty| + |x_1| + |x_2|) \geq 0$$

which is impossible. Therefore, $m_z([\beta_1], \check{W}, [\beta_2])$ is well defined.

Finally, to verify the chain property one needs to compute

$$\begin{aligned}
&\left(\partial_{Y_2} m_W + m_W \partial_1\right)[\beta_1] \\
=&\sum_{[\beta_2']}\sum_{z_2}\sum_{[\beta_2]}\sum_{z} m_z([\beta_1],\check{W},[\beta_2]) n_{z_2}([\beta_2],[\beta_2']) T^{-\mathcal{E}_{top}(z)} T^{-\mathcal{E}_{top}(z_2)}[\beta_2'] \\
+&\sum_{[\beta_2]}\sum_{w}\sum_{[\beta_1]}\sum_{z_1} n_{z_1}([\beta_1],[\beta_1']) m_w([\beta_1'],\check{W},[\beta_2]) T^{-\mathcal{E}_{top}(z_1)} T^{-\mathcal{E}_{top}(w)}[\beta_2]
\end{aligned}$$

and show that it vanishes. Again, the usual argument will still work as long as one remember that the topological energy is additive under concatenation of paths.

Similar arguments can be applied for showing that the composition law for α-admissible cobordisms holds, and in this way we have verified that $HI(Y,K,\alpha)$ is a topological invariant of the data (Y,K,α), together with a choice of cone parameter ν, which has been omitted from our notation.

5. Reduced Version

Now we will define a reduced version $HI_{red}(Y,K,\alpha)$ of the singular instanton Floer-Novikov homology for knots $HI(Y,K,\alpha)$ we just construced, following [21]. As we also mentioned in the introduction, the case of $\alpha = 1/4$, where local coefficients are not needed, was defined by Daemi and Scaduto earlier [14].

Our conventions will differ slightly from those of Frøyshov, since we are defining the homology version of the Floer groups, which means that some gradings of the groups differ, and in fact are closer to the ones used in [18, Section 3.3.2], [71, Section 9].The key difference between the reduced and unreduced versions of instanton Floer homologies is that the reduced version takes into account the flow-lines between the critical points and the reducible flat connection. Moreover, in the reduced version we can define a U-map, which in the situation of knots will arise from the μ-map evaluated at a point $x \in K$.

Now we want to define maps which take into account the interaction with the reducible connection $[\theta_\alpha]$. Consider a critical point $[\beta]$ with $\mathrm{gr}([\beta]) = 1$. From the grading formula (34), we can see that in principle there are non-empty moduli spaces $\mathcal{M}_1([\beta],[\theta_\alpha])$ asymptotic to $[\beta]$ as $t \to -\infty$ and to $[\theta_\alpha]$ as $t \to \infty$. After taking the quotient by the $\mathbb{R}$ action, we get a 0-dimensional moduli space $\check{\mathcal{M}}_1([\beta],[\theta_\alpha])$. We would like to define a map $\delta_1[\beta]$ obtained by counting the points in this 0-dimensional moduli space. However, because of non-monotonicity there can be a priori infinitely many components of this moduli space

$$\mathcal{M}_1([\beta],[\theta_\alpha]) = \bigcup_z \mathcal{M}_{1,z}([\beta],[\theta_\alpha])$$

Therefore, we define

$$\delta_1 : CI_1(Y, K, \alpha, \mathfrak{p}) \to \Lambda$$
$$[\beta] \to \sum_z \#\check{\mathcal{M}}_{1,z}([\beta], [\theta_\alpha]) T^{-\mathcal{E}_{top}(z)}$$

Recall that Λ is the Novikov field, while $CI_1(Y, K, \alpha, \mathfrak{p})$ denotes the chain complex of $\mathfrak{p}$-perturbed flat connections whose absolute grading is 1. The formula for $\delta_1[\beta]$ does give a well defined element in Λ for exactly the same reasons as the differential ∂ from the previous section being well defined. Just as in the case where K is absent, it is straightforward to see that δ_1 descends to a map in homology, that is:

Lemma 23. *The map δ_1 satisfies $\delta_1 \partial = 0$, so it induces a map in homology $\delta_1 : HI_1(Y, K, \alpha) \to \Lambda$.*

Likewise, suppose that $\mathrm{gr}([\beta]) = 2$, so that a priori there are non-empty moduli spaces $\mathcal{M}_1([\theta_\alpha], [\beta])$. Analogous to δ_1, define an element $\delta_2 \in CI_2(Y, K, \alpha, \mathfrak{p})$ by the formula

$$\delta_2 = \sum_{[\beta] \in CI_2} \sum_z \#\check{\mathcal{M}}_{1,z}([\theta_\alpha], [\beta]) T^{-\mathcal{E}_{top}(z)} [\beta]$$

As before, it is straightforward to check that:

Lemma 24. *The element δ_2, descends to an element in homology, i.e, $\partial \delta_2 = 0$ so that $\delta_2 \in HI_2(Y, K, \alpha)$.*

The next map to define is the μ -map, which we will denote μ_K, to emphasize the fact that it is not the ordinary μ-map. First we need to understand the homotopy type of the space of connections mod gauge, i.e, $\mathcal{B}(Y, K, \alpha)$, since for $x \in K$, $\mu_K(x)$ will be a degree 2 element in $H^*(\mathcal{B}(Y, K, \alpha); \mathbb{Q})$.

The main idea is to take advantage of the fact that singular connections have a stronger notion of framing than ordinary connections. We will follow the discussion in [64, Section 5] and [39, Section 4]. Recall that if G is a compact Lie group that acts on a topological space Z, then the homotopy quotient $Z//G$ is defined as $Z \times_G EG$, where EG is a contractible space with a free G action. The natural map $Z//G \to Z/G$ induces a map $H^*(Z/G, \mathbb{Z}) \to H^*(Z//G, \mathbb{Z})$, which is an isomorphism when G acts freely. Lemma 5.1 in [64] shows the following:

Lemma 25. *If $U(1)$ acts on Z and the stabilizer of every point in Z is $\{\pm 1\}$, then the pull-back map $H^*(Z/U(1), \mathbb{Q}) \to H^*(Z//U(1), \mathbb{Q})$ is an isomorphism.*

When V is a complex vector bundle over Z with a lift of the G action to V, we can also define $V//G = V \times_G EG$ and the G-equivariant Chern classes of V as $c_{i,G}(V) = c_i(V//G) \in H^{2i}(Z//G, \mathbb{Z})$. These are the pull-backs of the Chern classes on Z/G.

For our setup, when we are working with the pair (X, Σ), the bundle E decomposes near Σ as $E = L \oplus L^{-1}$. Moreover, we wrote an exact sequence 12 for the gauge group, which in particular implies that over Σ the $SU(2)$ gauge is broken to an $U(1)$ gauge. If $x \in \Sigma$ is a base-point and $\mathcal{G}_x$ the gauge transformations which act trivially on $E_x = L_x \oplus L_x^{-1}$, then we have the framed configuration space

$$\mathcal{B}^o(X, \Sigma, \alpha) = \mathcal{C}(X, \Sigma, \alpha)/\mathcal{G}_x(X, \Sigma)$$

Since $\mathcal{G}/\mathcal{G}_x \simeq U(1)$, a residual gauge group isomorphic to $U(1)$ acts on $\mathcal{B}^o(X, \Sigma, \alpha)$, and $\mathcal{B}^o(X, \Sigma, \alpha)/U(1) = \mathcal{B}(X, \Sigma, \alpha)$. Now we are ready to define the universal $SO(3)$ bundle and the corresponding μ map.

Definition 26. Define the **universal** $SO(3)$ **bundle**

$$\mathbb{E}^{ad} = \mathcal{C}(X, \Sigma, \alpha) \times_{\mathcal{G}_x} \mathfrak{g}_E \to (\mathcal{B}^*(X, \Sigma, \alpha) \times X)$$

Moreover, the $U(1)$ bundle

$$\mathbb{L}^o = \mathcal{C}(X, \Sigma, \alpha) \times_{\mathcal{G}_x} L \to \mathcal{B}^o(X, \Sigma, \alpha) \times \Sigma \tag{42}$$

descends to the **universal** $U(1)$ **bundle**

$$\mathbb{L}^{\otimes 2} \to (\mathcal{B}^*(X, \Sigma, \alpha) \times \Sigma)$$

Define for $\eta \in H_i(X; \mathbb{Q})$ and $\eta_\Sigma \in H_j(\Sigma; \mathbb{Q})$ the μ**-maps**

$$\mu(\eta) = -\frac{1}{4} p_1(\mathbb{E}^{ad})/\eta \in H^{4-i}(\mathcal{B}^*(X, \Sigma, \alpha); \mathbb{Q})$$
$$\mu_\Sigma(\eta_\Sigma) = -\frac{1}{2} e(\mathbb{L}^{\otimes 2})/\eta_\Sigma \in H^{2-j}(\mathcal{B}^*(X, \Sigma, \alpha); \mathbb{Q})$$

Remark 27. 1) We follow the sign conventions of Kronheimer for μ_Σ [39, Section 2.1].

2) Notice that since the homotopy type of $\mathcal{B}^*(X, \Sigma, \alpha)$ is independent of α, the μ-maps corresponding to different values of α can be identified with each other, which is why we do not indicate the value of α in our notation.

3) We can also view these as elements in $\mathcal{B}(X, \Sigma, \alpha)$, since the reducible connections form a stratum of infinite codimension in $\mathcal{B}(X, \Sigma, \alpha)$.

4) Notice that our construction singles out one of the line bundles in the decomposition $E = L \oplus L^{-1}$. In fact, had we used L^{-1} instead of L, then μ_Σ would differ only in sign.

Clearly a similar procedure can be used to define μ_K in the case of (Y, K): a cheap way to do this is to consider $X = [0, 1] \times Y$ and $\Sigma = [0, 1] \times K$. Therefore, for $x \in K$ we let

$$u_K(x) : CI_*(Y, K, \alpha, \mathfrak{p}) \to CI_{*-2}(Y, K, \alpha, \mathfrak{p})$$
$$[\beta_0] \to \sum_{[\beta_1] | \mathrm{gr}([\beta_0], [\beta_1]) = 2} \sum_z < u_K(x), \mathcal{M}_{2,z}([\beta_0], [\beta_1]) > T^{-\mathcal{E}_{top}(z)}[\beta_1]$$

Contrary to the case of δ_1, δ_2, $u_K(x)$ will not descend to a map between the Floer homology groups. To see why this is the case, observe that the maps

$$\partial u_K - u_K \partial$$

involve considering three dimensional moduli spaces, for which we had said factorizations through the reducibles can occur. More precisely, notice that for any $[\beta_0]$, we have [here we denote for convenience $< u_K(x), \mathcal{M}_{2,z}([\beta_0], [\beta_1]) >$ as U_z]

(43)

$$\begin{aligned}
&(\partial u_K - u_K \partial)[\beta_0] \\
= &\sum_{[\beta_2] | \mathrm{gr}([\beta_2], [\beta_1]) = 1} \sum_{z_{12}} \sum_{[\beta_1] | \mathrm{gr}([\beta_0], [\beta_1]) = 2} \sum_{z_{01}} U_{z_{01}}([\beta_0], [\beta_1]) n_{z_{12}}([\beta_1], [\beta_2]) T^{-\mathcal{E}_{top}(z_{01})} T^{-\mathcal{E}_{top}(z_{12})} [\beta_2] \\
&- \sum_{[\beta_2] | \mathrm{gr}([\beta_2], [\beta_1]) = 2} \sum_{w_{12}} \sum_{[\beta_1] | \mathrm{gr}([\beta_1], [\beta_0]) = 1} \sum_{w_{01}} n_{w_{01}}([\beta_0], [\beta_1]) U_{w_{12}}([\beta_1], [\beta_2]) T^{-\mathcal{E}_{top}(w_{01})} T^{-\mathcal{E}_{top}(w_{12})} [\beta_2]
\end{aligned}$$

The typical argument would look at a three dimensional moduli space $\mathcal{M}_3([\beta_0],[\beta_2])$ and consider the possible ends of this moduli space. Some of the ends correspond to the terms in 43, but when $\mathrm{gr}[\beta_0]=1$, a priori it is also possible to have factorizations of the form

$$\mathcal{M}_1([\beta_0],[\theta_\alpha])\times\mathcal{M}_1([\theta_\alpha],[\beta_2])$$

which needs to be accounted for. In fact, we have the analogue of [21, Theorem 4], [24, Proposition 8] and [18, Lemma 7.6].

Lemma 28. *The map u_K satisfies the relation*

$$\partial u_K - u_K\partial - \frac{1}{2}\delta_1\otimes\delta_2 = 0 \tag{44}$$

Proof. Of the references cited above, the closest to our argument is in fact [24, Proposition 8], since the monopole case also uses a universal $U(1)$ bundle to define the corresponding u-map. Therefore we obtain the same formula as the one Frøyshov writes for the monopole case, except for the difference in conventions for the constants in front of the μ-maps.

In fact, the only place where one needs to be careful with the previous argument is that any of the proofs quoted above use the holonomy of a connection A along the path $\mathbb{R}\times\{x\}\subset\mathbb{R}\times Y$.

More precisely, consider a 3-dimensional moduli space $\mathcal{M}_{3,z}([\beta_0],[\beta_2])$, and choose a representative $A\in\mathcal{M}_{3,z}([\beta_0],[\beta_1])$ whose "centre of mass" is 0, i.e,

$$\int_{\mathbb{R}\times Y} t|F_A|^2 = 0$$

If $\mathrm{ad}\beta_0,\mathrm{ad}\beta_1$ are the corresponding (perturbed) flat $SO(3)$ bundles corresponding to $[\beta_0],[\beta_1]$, then we can choose a base point $y\in Y$, which is *close* to $x\in K$, without being equal to it. Using a normal neighborhood $\nu(K)$ of K, we may assume that y belongs to a normal disk to K, centered at x, for which y has polar coordinates (r,θ). Since we are away from the knot, there is no controversy as to what we mean by

$$h_A(r,\theta)=\mathrm{hol}_A\left(\mathbb{R}\times\{y\}\right)$$

That is, the holonomy of A along the path $\mathbb{R}\times\{y\}$, where y has coordinates (r,θ). Comparing the frames for the fibres of the bundles $\mathrm{ad}\beta_0,\mathrm{ad}\beta_1$, we get an element in $SO(3)$, i.e, $h_A(r,\theta)\in SO(3)$. Now, for fixed θ, as r decreases the decomposition $E=L\oplus L^{-1}$, $\mathrm{ad}E=\mathbb{R}\oplus L^{\otimes 2}$, becomes asymptotically parallel with respect to A. Therefore, we obtain an element

$$h_A(\theta)=\lim_{r\to 0}\mathrm{hol}_A\left(\mathbb{R}\times\{y\}\right)\in U(1)$$

which is obtained by comparing the frames for the fibres of the $U(1)$ bundles $\widetilde{\mathrm{ad}\beta_0},\widetilde{\mathrm{ad}\beta_1}$, over the $U(1)$ line bundle $\mathbb{L}^{\otimes 2}\to\Sigma$ [compare with the description of the universal bundle 42]. The existence of this limit follows for example from [72], or one can also use the fact that we are working with orbifold connections, as Kronheimer and Mrowka do in [49, Section 3.1].

However, we still need to analyze what happens as we vary the angle at which we approach the point x. It is not difficult to see that as we vary the angle by a full revolution, i.e, $\theta\to\theta+2\pi$, then the holonomy picks out $h_A(\theta)$ picks out the asymptotic holonomy factor $e^{-4\pi i\alpha}$, in other words

$$h_A(\theta+2\pi)=e^{-4\pi i\alpha}h_A(\theta)$$

So in the case of rational holonomy, we can take

$$\begin{cases} h_A(x)\equiv[h_A(\theta)]^q & \alpha=\frac{2p+1}{2q}\\ h_A(x)\equiv[h_A(\theta)]^{2q+1} & \alpha=\frac{2p}{2q+1}\end{cases}$$

as our desired holonomy map. For example, the case of $\alpha = \frac{1}{4} = \frac{1}{2q}$ implies that we should square the limiting holonomy maps, which is exactly what Kronheimer and Mrowka do in [49, Section 3.1]. The difference with their construction is mainly stylistically, since they pass first to a local cover of a neighborhood of x, where the pull-backs of the connections extend smoothly.

Once we know how to take the holonomy along a point on the knot $x \in K$, the proof follows in exactly the same way as in [24, Proposition 8], where the coefficient of $\delta_1 \otimes \delta_2$ is the Euler number of the rank 1 Hermitian vector bundle over $S^2 = D^2 \cup_{S^1} D^2$ whose "clutching map" $S^1 \to U(1)$ has degree 1. $\square$

Now we interpret the equation 44 to find out how the reduced Floer groups should be defined.

- Case when $[\beta] \in HI_1(Y,K,\alpha)$ and $\delta_1[\beta] = 0$: then $\partial u_K[\beta] - u_K\partial[\beta] = 0$, which means that u_K descends to a map
$$u_K : \ker \delta_1 \subset HI_1(Y,K,\alpha) \to HI_{-1}(Y,K,\alpha) = HI_3(Y,K,\alpha)$$
- Case when $[\beta_0] \in HI_0(Y,K,\alpha)$: abusing notation write a representative of $[\beta_0]$ as $[\beta]+\partial[\beta_1]$, where $\partial[\beta] = 0$ and $[\beta_1] \in HI_1(Y,K,\alpha)$. Then
$$u_K[\beta_0] = u_K[\beta] + u_K\partial[\beta_1] = u_K[\beta] + \partial u_K[\beta_1] - \frac{1}{2}\delta_1([\beta_1])\delta_2$$
Therefore, we must identify elements which differ by an element on the "ray" $\Lambda\delta_2$, in other words, we get a map
$$u_K : HI_0(Y,K,\alpha) \to \mathrm{coker}(\delta_2) = HI_2(Y,K,\alpha)/(\Lambda\delta_2)$$
- On the other summands $HI_2(Y,K,\alpha)$ and $HI(Y,K,\alpha)$, the u_K map is actually well defined without additional considerations so we get maps
$$\begin{cases} u_K : HI_2(Y,K,\alpha) \to HI_0(Y,K,\alpha) \\ u_K : HI_3(Y,K,\alpha) \to HI_1(Y,K,\alpha) \end{cases}$$

The definition the reduced Floer homology groups is now identical to [21, Definition 1], since as long as we work with α-admissible cobordism it is straightforward to adapt section 3 of [21] (which analyzes the behavior of δ_1, δ_2, u_K under cobordisms) to obtain:

Theorem 29. *Let $K \subset Y$ be a knot and $\alpha \in \mathbb{Q} \cap (0,1/2)$ be such that $\triangle_K(e^{-4\pi i\alpha}) \neq 0$. The **reduced Instanton Floer homology groups** $HI_i^{red}(Y,K,\alpha)$*

$$\begin{aligned} HI_0^{red}(Y,K,\alpha) &= & HI_0(Y,K,\alpha)/(\textstyle\sum im(u_K^{2l+1}\delta_2)) \\ HI_1^{red}(Y,K,\alpha) &= & \cap_{l\geq 0}\ker(\delta_1 u_K^{2l}) \subset HI_1(Y,K,\alpha) \\ HI_2^{red}(Y,K,\alpha) &= & HI_2(Y,K,\alpha)/(\textstyle\sum im(u_K^{2l}\delta_2)) \\ HI_3^{red}(Y,K,\alpha) &= & \cap_{l\geq 0}\ker(\delta_1 u_K^{2l+1}) \subset HI_3(Y,K,\alpha) \end{aligned} \tag{45}$$

are topological invariants of the data (Y,K,α), together with the choice of cone parameter ν used to define the groups.

*Moreover, the **Frøyshov knot invariants***

$$h(Y,K,\alpha) = \chi_\Lambda(HI^{red}(Y,K,\alpha)) - \chi_\Lambda(HI(Y,K,\alpha)) \in \mathbb{Z} \tag{46}$$

where χ_Λ denotes the Euler characteristic with respect to the Novikov field Λ, are also invariants of (Y,K,α) (and the cone parameter).

Remark 30. The usual h-invariants have a factor $\frac{1}{2}$ in front of the difference in Euler characteristics. We choose not to include this factor because in our case the Euler characteristics of the groups may be odd, since these groups need not be 2-periodic as opposed to the 4-periodic instanton Floer homology groups on Y, and we prefer to obtain an integer rather than a half-integer.

6. Singular Orbifold Furuta-Ohta and Tori Signature

In this section we define an analogue of the Furuta-Ohta invariant $\lambda_{FO}(X)$ [28] to the case of an embedded torus T inside X satisfying certain conditions.

In other words, we want to define an invariant $\lambda_{FO}(X, T, \alpha)$, which in the best case scenario can be interpreted as a signed count of irreducible representations $\pi_1(X\backslash T) \to SU(2)$ (modulo conjugacy) satisfying a certain holonomy condition determined by the parameter α. As usual, perturbations of the flatness equation will be needed, so the interpretation of $\lambda_{FO}(X, T, \alpha)$ as a count of flat connections is slightly more complicated. In any case, our construction will be cooked up in such a way that when we take $X = S^1 \times Y$ and $T = S^1 \times K$ then $\lambda_{FO}(X, T, \alpha)$ agrees with $2\lambda_{CH}(Y, K, \alpha)$.

First we want to explain why we consider only torus complements $X\backslash T$, and not more general surface complements $X\backslash \Sigma$. Moreover, we will discuss what conditions are needed to make the definition of $\lambda_{FO}(X, T, \alpha)$ work.

Recall from section 2 that $SU(2)$ bundles $E \to X$ are characterized by two topological invariants, the instanton number k and the monopole number l. The moduli space of α-ASD connections $\mathcal{M}(X, \Sigma, k, l, \alpha)$ satisfying the asymptotic condition 9 has the expected dimension [41, Eq 1.6]

$$\dim \mathcal{M}(X, \Sigma, k, l, \alpha) = 8k + 4l - 3(b_2^+ - b^1 + 1) - (2g - 2) \tag{47}$$

while the formula for the topological energy is [41, Eq. 1.7]

$$\mathcal{E}(X, \Sigma, k, l, \alpha) = \frac{1}{8\pi^2} \int_{\check{X}} \operatorname{tr}(F_A \wedge F_A) = k + 2\alpha l - \alpha^2 \Sigma \cdot \Sigma \tag{48}$$

Since $H_*(X; \mathbb{Z}) \simeq H_*(S^1 \times S^3; \mathbb{Z})$ these formulas simplify to

$$\begin{aligned} \dim \mathcal{M}(X, \Sigma, k, l, \alpha) &= 8k + 4l - (2g - 2) \\ \mathcal{E}(X, \Sigma, k, l, \alpha) &= k + 2\alpha l \end{aligned} \tag{49}$$

Notice that α-flat connections (i.e, flat connections satisfying 9) are equivalent to energy zero α-ASD instantons (i.e, ASD instantons satisfying 9). In particular, this means that α-flat connections only exist a priori on bundles whose monopole and instanton numbers are related as

$$k + 2\alpha l = 0$$

Clearly $k = l = 0$ is always a solution of this equation. Since k, l must always be integers, for irrational values of α, $k = l = 0$ is the only solution. In fact, the next lemma shows that this continue to hold regardless of the value of α.

Lemma 31. *Suppose that E is an $SU(2)$ bundle over (X, Σ) with instanton and monopole numbers (k, l). If E supports an α-flat connection then $(k, l) = (0, 0)$ and thus E is the trivial bundle over X.*

Proof. As discussed before, any α-flat connection can only exist on a bundle E for which

$$k + 2\alpha l = 0$$

If α is irrational we already explained that $k = l = 0$ is automatic.

For α rational, it suffices to show that l vanishes, since the previous equation will force k to vanish and we will be done. To understand why l vanishes, we must use the fact that l can be computed as [39, Eq 17]

$$l = \lambda + \alpha \Sigma \cdot \Sigma \tag{50}$$

Here we use the fact the orbifold connection has a locally well defined restriction as an abelian connection on Σ, so the curvature decomposes as

$$F_A = \begin{pmatrix} \omega & 0 \\ 0 & -\omega \end{pmatrix}$$

when regarded as a 2-form on Σ. The quantity λ can then be computed as

$$\lambda = \frac{i}{2\pi} \int_{\Sigma} (-\omega)$$

In the case of a flat connection it is clear that $\omega = 0$ thus $\lambda = 0$. Since $\Sigma \cdot \Sigma = 0$ equation 50 now implies that $l = 0$, as desired. ☐

This means that for counting α-flat connections we can simply concentrate on the case $k = l = 0$, for which the expected dimension of the moduli space is 49

$$\dim \mathcal{M}(X, \Sigma, 0, 0, \alpha) = -2(g-1)$$

So it is now clear that the expected dimension of the moduli space of α-flat connections is zero dimensional if and only if $g = 1$, i.e, Σ must be an embedded torus T. In this case the energy and dimension formulas become

$$\begin{aligned} \dim \mathcal{M}(X, T, k, l, \alpha) &= 8k + 4l \\ \mathcal{E}(X, T, k, l, \alpha) &= k + 2\alpha l \end{aligned} \tag{51}$$

Our next objective will be to analyze which further hypothesis on T must be made in order to have a well defined invariant $\lambda_{FO}(X, T, \alpha)$.

But first, we also analyze what are the possible α-ASD instantons which are reducible on (X, Σ).

Lemma 32. *Suppose that E is an $SU(2)$ bundle over (X, Σ) with instanton and monopole numbers (k, l). If E supports an α-ASD connection which is reducible then $(k, l) = (0, 0)$ and thus E is the trivial bundle over X.*

Proof. We follow the remarks after Proposition 5.9 in [41], more specifically the remark "iii) Transversality". For an α-reducible ASD connection the bundle E must first of all split globally as $E = L \oplus L^{-1}$, where L is a complex line bundle which admits an α-ASD connection. This in turn can be represented by a smooth, harmonic, anti-self-dual 2-form ω whose cohomology class represents $c_1(L) + \alpha[\Sigma]$. Again, because $b_2(X) = 0$ this means in fact that ω represents the class 0, i.e, it must be an α-flat twisted connection. But we already know from Lemma 31 that this forces k and l to be both zero so we are done. ☐

Back to the case where $\Sigma = T$, we need to discuss under which conditions we can expect to define a count of α-representations $\pi_1(X \backslash T) \to SU(2)$. As in the case of a knot K inside a homology sphere Y, we have to guarantee that the α-reducible representations are isolated from the α-irreducible representations. To analyze the α-reducible representations we need to consider the first homology of the torus complement, i.e, $H_1(X \backslash T; \mathbb{Z})$. Now, $H_1(X \backslash T; \mathbb{Z})$ will be sensitive on the embedding of the torus, for example, T could be null-homologous or not. In fact, we are interested in the case where T is homologically indistinguishable from the product situation $S^1 \times K \subset S^1 \times Y$, so we will make the following assumption.

Assumption 33. *T will be an embedded torus inside X such that the natural map $H_1(T;\mathbb{Z}) \twoheadrightarrow H_1(X;\mathbb{Z})$ is a surjection.*

More precisely, if we take our torus T to be the image of an embedding $\imath_T : S^1 \times S^1 \hookrightarrow X$, then we will assume that $\vartheta = (\imath_T)_*(S^1 \times \{pt\})$ is a generator of $H_1(X;\mathbb{Z})$. A tubular neighborhood of T looks like $T \times D^2$, and if we use polar coordinates (r,θ) for the second factor, then

$$H_1(X\backslash T;\mathbb{Z}) \simeq \mathbb{Z}[\vartheta] \oplus \mathbb{Z}[\mu_T]$$

where μ_T is a "meridian" for the torus T, and can be represented for example as $(\imath_{T\times D^2})_*(\{pt\} \times S^1 \times (\epsilon,0))$, where ϵ is sufficiently small.

If ρ is an α flat connection then we want to understand the Zariski tangent space $\check{H}^1(\check{X};\mathfrak{g}_\rho)$. Just as in Lemma 21 we have:

Lemma 34. *Suppose $\alpha \in \mathbb{Q} \cap (0,1/2)$ and that ρ is an α-flat connection on the orbifold $\check{X}$. Then the first (orbifold) cohomology $\check{H}^1(\check{X};\mathfrak{g}_\rho)$ can be identified with* $\ker : H^1(X\backslash T;\mathfrak{g}_\rho) \to H^1(\mu_T;\mathfrak{g}_\rho)$.

Proof. The argument is completely analogous. Namely, after applying a Mayer-Vietoris decomposition to

$$\check{X} = (\check{X}\backslash\check{\nu}_\epsilon(T)) \cup \check{\nu}(T)$$

we end up with the analogue of equations 30 and 31

$$\check{H}^1(\check{X};\mathfrak{g}_\rho) \to^{(i^*_{X\backslash T}, i^*_{\nu(T)})} H^1(X\backslash T;\mathfrak{g}_\rho) \oplus (\mathbb{R}[\lambda_T] \oplus \mathbb{R}[\vartheta]) \to^{i^*_{T_\epsilon,\nu(K)} - i^*_{T_\epsilon, Y\backslash K}} \mathbb{R}[\mu_T] \oplus \mathbb{R}[\lambda_T] \oplus \mathbb{R}[\vartheta]$$

The composition being 0 now says that $\check{\omega} \to \left\langle \omega \mid_{S^1_{\mu_T}}, [\mu_T] \right\rangle$ vanishes, so there is a map $\check{H}^1(\check{X};\mathfrak{g}_\rho) \to \ker(H^1(X\backslash T;\mathfrak{g}_\rho) \to H^1(\mu;\mathfrak{g}_\rho))$. The surjectivity and injectivity of this map are proven in exactly the same way as before. □

In particular, for an α-reducible representation ρ we have $\mathfrak{g}_\rho = \mathbb{R} \oplus L^{\otimes 2}$

$$\begin{aligned}
&\check{H}^1(\check{X};\mathfrak{g}_{\rho_\alpha}) \\
\simeq &\ker\left(H^1(X\backslash T;\mathbb{R}) \oplus H^1(X\backslash T;L^{\otimes 2}) \to H^1(\mu_T;\mathbb{R}) \oplus H^1(\mu_T;L^{\otimes 2})\right) \\
\simeq &\ker\left(\mathbb{R}[\vartheta] \oplus \mathbb{R}[\mu_T] \oplus H^1(X\backslash T;L^{\otimes 2}) \to \mathbb{R}[\vartheta]\right) \\
\simeq &\mathbb{R}[\mu_T] \oplus H^1(X\backslash T;L^{\otimes 2})
\end{aligned}$$

So at an α-reducible representation ρ, $\check{H}^1(\check{X};\mathfrak{g}_{\rho_\alpha})$ is at least one dimensional, and for ρ to be isolated from the irreducible representations it suffices to assume that $H^1(X\backslash T;L^{\otimes 2})$ vanishes. Using Corollary 65 in the Appendix, we can characterize this in terms of the Alexander polynomial of the torus complement $X\backslash T$.

Corollary 35. *Let $\alpha \in \mathbb{Q} \cap (0,1/2)$ and T be an embedded oriented torus such that $H_1(T;\mathbb{Z}) \twoheadrightarrow H_1(X;\mathbb{Z})$. Then $\check{H}^1(\check{X};\mathfrak{g}_\rho)$ is one dimensional for every α-reducible representation ρ if and only if $H^1(X\backslash T;L_\rho^{\otimes 2}) = 0$, where $\mathfrak{g}_\rho = \mathbb{R} \oplus L_\rho^{\otimes 2}$. Equivalently, for every α-reducible representation we have $\triangle_{X\backslash T}(\hat{\rho}) \neq 0$, where $\hat{\rho} \in \widehat{\pi_1(X\backslash T)}$ is the character determined by the local system $H^1(X\backslash T;L^{\otimes 2})$.*

Remark 36. For the examples of embedded tori we will analyze, we find it easier to verify directly that $H^1(X\backslash T;L^{\otimes 2}) = 0$ vanishes, rather than using the condition on the Alexander polynomial, since our constructions will arise from some operation on a knot inside an integer homology sphere.

It would be interesting to study an example of an embedded torus where it is easier to verify the condition on the Alexander polynomial directly. It would probably need to be more four dimensional in nature.

Notice that whenever the previous condition is satisfied then the obstruction space (i.e, the second cohomology group $\check{\mathcal{H}}^2(\check{X}; A_\rho)$ of the deformation complex defined by ρ as a solution of the α-ASD equations) must vanish. This is because the Euler characteristic of the deformation complex is (minus) the virtual dimension of the moduli space, which is zero in our situation, so

$$\dim \check{\mathcal{H}}^0(\check{X}; A_\rho) - \dim \check{\mathcal{H}}^1(\check{X}; A_\rho) + \dim \check{\mathcal{H}}^2(\check{X}; A_\rho) = 0$$

The first factor is 1 because that is the dimension of the stabilizer of A_ρ and the one in the middle is also 1 by assumption so that forces $\check{\mathcal{H}}^2(\check{X}; A_\rho)$ to vanish.

Now we are finally ready to give a definition $\lambda_{FO}(X, T, \alpha)$.

Definition 37. Suppose that $\alpha \in \mathbb{Q} \cap (0, 1/2)$ and T be an oriented embedded torus such that $H_1(T; \mathbb{Z}) \twoheadrightarrow H_1(X; \mathbb{Z})$. Suppose moreover that for every α-reducible representation ρ we have $H^1(X \backslash T; L^{\otimes 2}) = 0$, where $\mathfrak{g}_\rho = \mathbb{R} \oplus L^{\otimes 2}$. Choose a homology orientation for (X, T), that is, an orientation of $H^1(X; \mathbb{Z})$, which in turn is determined by the orientation of T. Given this homology orientation, there is an orientation of the moduli spaces $\mathcal{M}(X, T, k, l, \alpha)$ [43, Section 2.i)]. We define the **singular Furuta-Ohta invariant** $\lambda_{FO}(X, T, \alpha)$ as follows.

Choose the trivial $SU(2)$ bundle $E \to X$ corresponding to the instanton and monopole numbers $k = l = 0$. Then, after perturbations if necessary, the irreducible α-ASD connections $\mathcal{M}^*(X, T, 0, 0, \alpha)$ will form a 0-dimensional compact moduli space. We define $\lambda_{FO}(X, T, \alpha) \in \mathbb{Z}$ as the signed count of elements inside $\mathcal{M}^*(X, T, 0, 0, \alpha)$.

Remark 38. The perturbations we have in mind for the statement of the previous theorem are exactly the same as the interior holonomy perturbations that were needed to define the cobordism maps for the Floer groups $HI(Y, K, \alpha)$.

Despite the fact that $\lambda_{FO}(X, T, \alpha)$ is morally defined as a count of flat connections, it is important to notice that it is the equation $F_A^+ = 0$ which is perturbed, not the flatness equation $F_A = 0$. In other words, $\lambda_{FO}(X, T, \alpha)$ is better interpreted as a degree 0 Donaldson invariant.

This raises the question of whether this is the only degree 0 Donaldson invariant which can be defined for the embedded torus T. In fact, it is possible to define additional invariants $D_0(X, T, \alpha, k)$ where $k \in \mathbb{Z}$ is an integer, and provided $\alpha \neq 1/4$. Interestingly enough, **for $k \neq 0$, the invariants** *$D_0(X, T, \alpha, k)$ can be defined for any embedded torus, independent of whether it is null-homologous or not.* To see why this is the case, we need to go back to the formulas 51 for the energy and dimension of the moduli spaces

$$\dim \mathcal{M}(X, T, k, l, \alpha) = 8k + 4l$$
$$\mathcal{E}(X, T, k, l, \alpha) = k + 2\alpha l$$

Notice that $\mathcal{M}(X, T, k, l, \alpha)$ is zero-dimensional whenever

$$l = -2k$$

The corresponding energy of this moduli space is

$$\mathcal{E}(X, T, k, -2k, \alpha) = k(1 - 4\alpha)$$

In particular, when $\alpha = 1/4$, the energy of $\mathcal{M}(X, T, k, -2k, 1/4)$ is zero, which means it can only consist of α-flat connections. But we already know from Lemma 31 that this can only happen when

$k = 0$, which means that

$$k \neq 0 \implies \mathcal{M}(X, T, k, -2k, 1/4) = \emptyset$$

However, when $\alpha \neq 1/4$, the moduli spaces $\mathcal{M}(X, T, k, -2k, \alpha)$ are a priori non-empty, at least provided that $\mathcal{E}(X, T, k, -2k, \alpha) \geq 0$. A more important question is whether they are compact.

Since they are already 0 dimensional, the only way for $\mathcal{M}(X, T, k, -2k, \alpha)$ to be non-compact is if a sequence of α-ASD connections $[A_i]$ inside $\mathcal{M}(X, T, k, -2k, \alpha)$ bubbles off and converges weakly to an α-ASD connection $[A_\infty]$ on some moduli space $\mathcal{M}(X, T, k', l', \alpha)$ of *negative* dimension. In fact, since bubbles drop dimensions by 4, $\dim \mathcal{M}(X, T, k', l', \alpha) \leq -4$. Fortunately, by Lemma 32, $\mathcal{M}(X, T, k', l', \alpha)$ can admit no reducible α-ASD connections, so for generic perturbations there is no risk in assuming that $\mathcal{M}(X, T, k', l', \alpha)$ is empty (this is explained in great detail in [40, Sections 3 and 5]).

Therefore, for generic perturbations $\mathcal{M}(X, T, k, -2k, \alpha)$ is in fact compact, and a count of signed points in $\mathcal{M}(X, T, k, -2k, \alpha)$ will be independent of the perturbation chosen, because a path of perturbations will generically miss non-empty negative dimensional moduli spaces. Notice that for $k \neq 0$, $\alpha \neq 1/4$, $\mathcal{M}(X, T, k, -2k, \alpha)$ has no reducibles to begin with, which in particular means that the count of signed points inside $\mathcal{M}(X, T, k, -2k, \alpha)$ can be made regardless of whether T is null-homologous or not.

These observations allows us to define the additional invariants $D_0(X, T, k, \alpha)$ we promised earlier.

Definition 39. Suppose that $\alpha \in \mathbb{Q} \cap (0, 1/2)$ and $k \in \mathbb{Z} \backslash \{0\}$ is a non-zero integer such that $k(1 - 4\alpha) \geq 0$. Let T be a oriented embedded torus inside X (null-homologous or not). After choosing an orientation of $H^1(X; \mathbb{R})$, define $D_0(X, T, k, \alpha)$ as the signed count of points inside the moduli space $\mathcal{M}(X, T, k, -2k, \alpha) = \mathcal{M}^*(X, T, k, -2k, \alpha)$. When $\alpha = 1/4$, set $D_0(X, T, k, 1/4) = 0$.

Remark 40. Notice that the choice of homology orientation is no longer determined in a canonical way by an orientation of T, if we allow T to be null-homologous.

Also, it is not all clear what is the geometric meaning of the invariants $D_0(X, T, \alpha, k)$. It is possible they may not contain any interesting topological information about the torus T. For example, if we could solve the issue of the implicit dependence of the invariants on the cone angle being used, and moreover if we succeeded in defining them for irrational values of α as well, then one could try to use a deformation argument to show that for $k \neq 0$, $D_0(X, T, \alpha, k)$ must vanish, since in this case $D_0(X, T, \alpha, k)$ could be compared to $D_0(X, T, 1/4, k)$, which we already know vanishes.

Likewise, we can construct an analogue of the invariants $D_0(X, T, \alpha, k)$ for the case of an embedded sphere $S^2 \hookrightarrow X$. Namely, the expected dimension of the moduli spaces $\mathcal{M}(X, S^2, k, -2k, \alpha)$ is now 2, as can be seen from the formula 49. As long as $\alpha \neq 1/4$ and $k \neq 0$, some of these moduli spaces could be non-empty, and they will be compact and free of reducibles by a similar argument. Hence, by pairing them with $\mu_{S^2}(x)$ for $x \in S^2$, we can define a degree-two Donaldson invariant $D_2(X, S^2, k, \alpha)$. However, the fact that these invariants cannot be defined when $\alpha = 1/4$ (which is the best value α could take from many points of view), suggests to us these invariants $D_2(X, S^2, k, \alpha)$ will probably end up giving no interesting topological information.

We finish this section by analyzing the action of $H^1(X; \mathbb{Z}/2)$ on the moduli spaces $\mathcal{M}(X, T, k, -2k, \alpha)$, as was promised in the introduction. We follow [68, Section 4.6] and [70, Section 3] in order to describe this action.

First of all, $H^1(X; \mathbb{Z}_2) = \hom(\pi_1(X); \mathbb{Z}_2)$ parametrizes isomorphism classes of complex line bundles (with connection) χ with holonomy $\{\pm 1\}$ (along the loop ϑ in our case). Since χ lifts to an integral homology class, the bundle L_χ is trivial and thus for any (k,l), the bundles $E(k,l)$ and $E(k,l)\otimes L_\chi$ are isomorphic. Thus the action of $H^1(X;\mathbb{Z}_2)$ on $\mathcal{M}(X,T,k,-2k,\alpha)$ can be regarded as the one which sends a connection $[A]$ to $[A\otimes\chi]$.

In general this action may or may not be free. We only care about the freeness of the action on the irreducible part of the moduli space $\mathcal{M}^*(X,T,k,-2k,\alpha)$ (again, when $k\neq 0$, this coincides with the entire moduli space $\mathcal{M}(X,T,k,-2k,\alpha)$), which we will show in the next lemma.

Here we only need to analyze the action on the unperturbed moduli spaces, since the idea is that once we know the action is free in the unperturbed case, one can find perturbations that are $H^1(X;\mathbb{Z}_2)$ equivariant and still guarantee transversality for the moduli spaces [68, section 4.6].

Lemma 41. *Suppose that the (unperturbed) moduli space $\mathcal{M}^*(X,T,k,-2k,\alpha)$ is non-empty. Then $H^1(X;\mathbb{Z}/2)$ acts freely on $\mathcal{M}^*(X,T,k,-2k,\alpha)$.*

Proof. Let $[A]\in\mathcal{M}^*(X,T,k,-2k,\alpha)$ be an irreducible α-ASD connection. The connection A induces a connection A^{ad} on the adjoint bundle $E^{ad}(k,-2k)$ of $E(k,-2k)$. On the adjoint bundle it makes sense to consider the gauge group $\mathcal{G}_{SO(3)}(X,T)$ of all $SO(3)$ gauge transformations (as opposed to the $SU(2)$ gauge transformations which are the ones we have been working with). As in the non-singular case (i.e, when T is not present), it is still the case that [44, Section 5.1]

$$\mathcal{G}_{SO(3)}(X,T)/(\mathcal{G}(X,T)/\{\pm 1\})\simeq H^1(X;\mathbb{Z}/2)$$

Therefore, the action of $H^1(X;\mathbb{Z}/2)$ is free on $[A]$ if and only if the stabilizer of A^{ad} with respect to the full gauge group $\mathcal{G}_{SO}(X,T)$ is trivial. In general, since A is irreducible with respect to $\mathcal{G}(X,T)$, $\mathrm{stab}_{SO(3)}A^{ad}$ can only be one of three possibilities: 1, $\mathbb{Z}_2$ or the Klein-4 group V_4. Therefore we must rule out that $\mathbb{Z}_2$ and V_4 can arises as potential stabilizers.

The case of V_4 is easy: a connection A with stabilizer V_4 must be flat [70, Section 4], and thus cannot belong to $\mathcal{M}(X,T,k,-2k,\alpha)$ for $k\neq 0$, since these a priori do not support any flat connections.

In the case that $k=0$ but $\alpha\neq 1/4$, we just need to use the fact that every α-representation ρ such that ρ^{ad} has stabilizer V_4 also has holonomy V_4, hence ρ^{ad} will correspond in general to representations of $\pi_1(X\backslash T)$ with image into $V_4\subset SO(3)$ ([70, Section 4], [47, Examples 2.9]). Given that every element of V_4 has order 2, the only value of α compatible with V_4 representations corresponds to $\alpha=1/4$. Notice that ρ will then have image contained in the quaternionic subgroup $Q_8=\{\pm 1,\pm\mathbf{i},\pm\mathbf{j},\pm\mathbf{k}\}$ when we identify $SU(2)$ with the unit quaternions.

For the case of $k=0$ and $\alpha=1/4$, we need to use the fact that the existence of Klein-4 representations is a homological phenomenon. Namely, they exist whenever one can find three nontrivial real line bundles which are *distinct* [70, Section 3] on the manifold. In our case, the manifold to consider is $X\backslash\mathrm{nbd}(T)$, where $\mathrm{nbd}(T)$ is an open neighborhood of the torus T. Since $H^1(X\backslash\mathrm{nbd}(T);\mathbb{Z}_2)\simeq\mathbb{Z}_2\oplus\mathbb{Z}_2$, one can find three distinct nontrivial real line bundles which determines a Klein-4 representation. Call these (real) line bundles $\epsilon_1,\epsilon_2,\epsilon_3=\epsilon_1+\epsilon_2$, where ϵ_1 generates the first factor of $H^1(X\backslash\mathrm{nbd}(T);\mathbb{Z}_2)$ while ϵ_2 the second factor. It is easy to see that in this case

$$\begin{aligned}
&w_2(E^{ad}\mid_{X\backslash\mathrm{nbd}(T)})\\
=&w_2\left(\epsilon_1\oplus\epsilon_2\oplus\epsilon_3\right)\\
=&w_1(\epsilon_1)w_1(\epsilon_2)+w_1(\epsilon_1)w_1(\epsilon_3)+w_1(\epsilon_2)w_1(\epsilon_3)\\
=&w_1(\epsilon_1)w_1(\epsilon_2)+[w_1(\epsilon_1)]^2+w_1(\epsilon_1)w_1(\epsilon_2)+w_1(\epsilon_2)w_1(\epsilon_1)+[w_1(\epsilon_2)]^2\\
=&w_1(\epsilon_2)w_1(\epsilon_1)\neq 0
\end{aligned}$$

In other words, the Klein 4 representation we found exists on a bundle with non-trivial w_2. However, the bundle $E^{ad}(0,0)$ over X has vanishing w_2 (since X is an integral homology $S^1 \times S^3$, or alternatively, because $E(0,0)$ was the trivial bundle to begin with), which means that its restriction to the torus complement should have vanishing w_2 as well by naturality of the Stiefel-Whitney classes. Therefore, $\mathcal{M}^*(X,T,0,0,1/4)$ is also free of connections with stabilizer V_4 (with respect to $\mathcal{G}_{SO(3)}(X,T)$).

The case of $\mathbb{Z}_2$ stabilizer corresponds to the so-called twisted reducibles [42, Section 2 i)]: these are those connections which preserve a splitting $\mathfrak{g}_E = \lambda \oplus P$, where now λ is a non-orientable real line bundle and P is a non-orientable real two plane bundle with orientation bundle isomorphic to λ. Now we can use the fact that our connections have a prescribed model near the surface T: as explained before Lemma 2.22 in [42], if A were a twisted reducible, then P would have to coincide with $\pm L^{\otimes 2}$ in a tubular neighborhood of T, and therefore λ must be trivial on T.

Now, because $H_1(T;\mathbb{Z}_2)$ maps onto $H_1(X;\mathbb{Z}_2)$, then λ will be trivial on all of X, which means that it is orientable, thus we obtain a contraction. Notice that the paragraph we refer to from [42] starts by stating that A must be a non-flat connection. However, this condition is not used in this argument, rather it was assumed by Kronheimer and Mrowka because they were interested in obtaining a generic metrics theorem in the presence of twisted reducibles, and in general this cannot be achieved whenever there are flat connections. $\square$

Remark 42. Since for $k \neq 0$, $\alpha \neq 1/4$, the moduli spaces $\mathcal{M}(X,T,k,-2k,\alpha)$ are free of reducible α-ASD connections, free of α-flat connections and free of twisted reducible connections, one can also obtain transversality for these moduli spaces using the generic metrics theorem, thanks to [42, Lemma 2.17]. Hence, one could avoid using holonomy perturbations for defining the invariants $D_0(X,T,k,\alpha)$ for $k \neq 0$.

It is also interesting to understand why one gets stuck if one tries to adapt the argument Ruberman and Saveliev give in [68, Proposition 4.1] for the freeness of the action of $H^1(X;\mathbb{Z}/2)$ on the moduli space needed to define $\lambda_{FO}(X)$.

Notice that if we had found Klein 4 representations in one of our moduli spaces the corresponding $SU(2)$ representation would have had image inside Q_8. More generally, we have the so called binary dihedral representations [33], whose definition we now recall.

Definition 43. A representation $\rho : \pi_1 M \to SU(2)$ is called a **binary dihedral representation** if the image of ρ is contained inside $S^1_i \cup S^1_j$. Here we identified $SU(2)$ with the unit sphere S^3 inside $\mathbb{R}^4$ (equivalently the unit quaternions), and set

$$S^1_i = \{a + b\mathbf{i} \mid a, b \in \mathbb{R}, a^2 + b^2 = 1\}$$
$$S^1_j = \{c\mathbf{j} + d\mathbf{k} \mid c, d \in \mathbb{R}, c^2 + d^2 = 1\}$$

We will denote by BD the subset of representations (modulo conjugacy) whose image is binary dihedral. When we want to consider binary dihedral representations with the correct holonomy condition, we will write BD_α.

Remark 44. i) The set $S^1_i \subset S^1_i \cup S^1_j$ is a subgroup of index 2 and hence normal.

ii) Binary dihedral representations can also be defined as representations $\rho : \pi_1 M \to SU(2)$ for which there exists $u \in SU(2)$ satisfying $\chi(h)\rho(h) = u\rho(h)u^{-1}$ for some cocycle χ and all $h \in SU(2)$, i.e, a homomorphism $\chi : \pi_1 M \to \mathbb{Z}/2$ (see remarks after proof of Proposition 3.3 in [10]).

The reason why we mention the more general case of a binary dihedral representation, instead of just the representations with image Q_8, is that if one tries to adapt the argument Ruberman and

Saveliev give in [68, Proposition 4.1] for the freeness of the action of $H^1(X;\mathbb{Z}/2)$ on the moduli space needed to define $\lambda_{FO}(X)$ (assuming as well that the cyclic cover of X has the homology of S^3), the best one can find without relying on the behavior of the connections near the surface isthat the action could fail to be free at those representations which are binary dihedral, i.e, on BD_α.

To see why this is the case, notice that $H^1(X\backslash T;\mathbb{Z}_2)\simeq\mathbb{Z}_2\oplus\mathbb{Z}_2$, where one factor just corresponds to the pullback of the elements of $H^1(X;\mathbb{Z}_2)$ under the inclusion map $\imath : X\backslash T\hookrightarrow X$. Therefore, we can regard χ as a homomorphism $\pi_1(X\backslash T)\to\mathbb{Z}_2$ and thus the action of χ on an α-representation $\rho:\pi_1(X\backslash T)\to SU(2)$ is simply given by

$$\chi(\rho)(h)=\chi(h)\rho(h),\quad h\in\pi_1(X\backslash T) \tag{52}$$

The conjugacy class of ρ will be fixed under this action if there exists an element $u\in SU(2)$ such that

$$\chi(\rho)(h)=u\rho(h)u^{-1} \tag{53}$$

for all $h\in\pi_1(X\backslash T)$. Since $\chi(h)\in\mathbb{Z}_2=Z(SU(2))$ we also have

$$\begin{aligned}&\rho(h)\\ =&\chi(h)^2\rho(h)\\ =&\chi(h)\left[u\rho(h)u^{-1}\right]\\ =&u\chi(h)\rho(h)u^{-1}\\ =&u^2\rho(h)(u^{-1})^2\end{aligned}$$

which implies that $u\in\mathrm{stab}\rho$. Since ρ is irreducible (with respect to $SU(2)$ gauge transformations), we have that $u^2=\pm1$.

If $u^2=1$ then $u=\pm1$ and 52 and 53 would imply that for all $h\in\pi_1(X\backslash T)$

$$\chi(h)\rho(h)=\chi(\rho)(h)=u\rho(h)u^{-1}=\rho(h)$$

Since we are assuming χ is non-trivial, there must exists at least one $h\in\pi_1(X\backslash T)$ such that $\chi(h)=-1$ and so the previous equation would imply that

$$-\rho(h)=\rho(h)\in SU(2)$$

which is clearly impossible since $0\notin SU(2)$. Now, if $u^2=-1$, we can assume up to conjugacy that $u=\mathbf{i}$, thus for all $h\in\pi_1(X\backslash T)$

$$\chi(h)\rho(h)=\mathbf{i}\rho(h)\mathbf{i}^{-1} \tag{54}$$

which means that ρ is a binary dihedral representation. After this it is no longer possible to adapt the argument of Ruberman and Saveliev. Notice that we made no special assumptions on the value of α. However, it is interesting that in the case of knot complements, irreducible binary dihedral representations can only occur when $\alpha=1/4$. The following lemma, explains why this is the case. We are very thankful to Nikolai Saveliev for providing to us a proof of it.

Lemma 45. *Let K be a knot inside an integer homology sphere Y. Every irreducible binary dihedral representation $\rho:\pi_1(Y\backslash K)\to S^1_i\cup S^1_j$ maps the meridian of K to a trace-free matrix, that is, an $\rho\in BD_{1/4}$.*

Proof. Let $G = \pi_1(Y\backslash K)$ be the knot group and the commutator subgroup, i.e, $G' = [G, G]$. If $G \to \mathbb{Z}$ is the abelianization homomorphism, which sends the meridian of the knot to a generator of $\mathbb{Z}$, we have the exact sequence

$$1 \to G' \to G \to \mathbb{Z} \to 1$$

which in fact *splits*. Any binary dihedral representation $\rho : \pi_1(Y\backslash K) \to S^1_i \cup S^1_j$ sends G' to the commutator subgroup of $S^1_i \cup S^1_j$, which happens to be $U(1)$. Since ρ is assumed irreducible, and the sequence of groups splits, if ρ sends both G' and the meridian to the $U(1)$ factor, it must send the entire group G to $U(1)$, which is a contradiction. Therefore, the meridian must be sent to a matrix inside S^1_j, all of which have zero trace. □

Based on this, we make the following conjecture:

Conjecture 46. *Let $\rho : \pi_1(X\backslash T) \to S^1_i \cup S^1_j$ be an irreducible α-representation, where $T \subset X$ is an embedded torus such that $H_1(T;\mathbb{Z})$ surjects onto $H_1(X;\mathbb{Z})$. Then $\alpha = 1/4$.*

Before discussing some examples and properties of $\lambda_{FO}(X, T, \alpha, k)$, we will prove the splitting formula for $\lambda_{FO}(X, T, \alpha, k)$, which was one of our main motivations for defining the invariant.

7. The Splitting Formula

Our first step for finding the splitting formula is understanding how α-admissibility for self-concordances $(W, \Sigma) : (Y, K) \to (Y, K)$ is related to the reducible representations being isolated from the irreducible representations in the case of (X, Σ). The next lemma says that in fact both notions are equivalent.

Lemma 47. *Suppose that $(W, \Sigma) : (Y, K) \to (Y, K)$ is a self-concordance of a knot and we choose a parameter α for which $\triangle_K(e^{-4\pi i\alpha}) \neq 0$. Let (X, T) be the closed 4-manifold obtained by closing up (W, Σ). Then $\lambda_{FO}(X, T, \alpha)$ is well defined if and only if the cobordism (W, Σ) is α-admissible.*

Proof. We can think of (X, T) as being obtained from (W, Σ) after attaching a tube $(I \times Y, I \times K)$ to the boundary of (W, Σ), where I is some interval. That is,

$$(X, T) = (W, \Sigma) \cup (I \times Y, I \times K)$$

Then we want to apply Mayer-Vietoris to this decomposition of (X, T), where we enlarged I a little bit so that the overlap of (W, Σ) with $(I \times Y, I \times K)$ is the disjoint union

$$(W, \Sigma) \cap (I \times Y, I \times K) = (I_1 \times Y, I_1 \times K) \sqcup (I_2 \times Y, I_2 \times K)$$

where I_1, I_2 are two small subintervals. Recall that on the orbifold $\check{W}$ there is only one α-flat reducible $\theta_{W,\alpha}$, while on Y we have the reducible θ_α. For any α-flat reducible A_ρ on $\check{X}$, it must restrict to $\theta_{W,\alpha}$ and θ_α on $\check{W}$ and $\check{Y}$ respectively, which means that exact sequence for the (orbifold) cohomology groups reads

$$\begin{aligned}
0 \to & \check{H}^0(\check{X};\mathfrak{g}_\rho) \\
\to & \check{H}^0(\check{W};\mathfrak{g}_{\theta_{W,\alpha}}) \oplus \check{H}^0(I\times\check{Y};\mathfrak{g}_{\theta_\alpha}) \\
\to & \check{H}^0(\left(I_1 \sqcup I_2 \times \check{Y}\right);\mathfrak{g}_{\theta_\alpha}) \\
\to & \check{H}^1(\check{X};\mathfrak{g}_\rho) \\
\to & \check{H}^1(\check{W};\mathfrak{g}_{\theta_{W,\alpha}}) \oplus \check{H}^1(I\times\check{Y};\mathfrak{g}_{\theta_\alpha}) \\
\to & \check{H}^1((I_1 \sqcup I_2)\times\check{Y});\mathfrak{g}_{\theta_\alpha}) \\
\to & \cdots
\end{aligned}$$

Since $\triangle_K(e^{-4\pi i\alpha}) \neq 0$, we have that $\check{H}^1(\check{Y};\mathfrak{g}_{\theta_\alpha}) = 0$ so we can simplify the previous exact sequence into

$$\begin{aligned}
0 \to & \mathbb{R} \\
\to & \mathbb{R}\oplus\mathbb{R} \\
\to & \mathbb{R}\oplus\mathbb{R} \\
\to & \check{H}^1(\check{X};\mathfrak{g}_\rho) \\
\to & \check{H}^1(\check{W};\mathfrak{g}_{\theta_{W,\alpha}}) \\
\to & 0 \\
\to & \cdots
\end{aligned}$$

From this we can conclude that the alternating sum of the dimensions of these vector spaces is zero, which means that

$$\dim \check{H}^1(\check{X};\mathfrak{g}_\rho) = 1 + \dim \check{H}^1(\check{W};\mathfrak{g}_{\theta_{W,\alpha}})$$

Thus, if (W,Σ) is α-admissible (i.e, $\dim \check{H}^1(\check{W};\mathfrak{g}_{\theta_{W,\alpha}}) = 0$) then $\check{H}^1(\check{X};\mathfrak{g}_\rho)$ vanishes for all α-reducible representations ρ, and conversely, if $\lambda_{FO}(X,T,\alpha)$ can be defined (i.e, $\dim \check{H}^1(\check{X};\mathfrak{g}_\rho) = 1$) then $\dim \check{H}^1(\check{W};\mathfrak{g}_{\theta_{W,\alpha}})$ must vanish which is the condition for the cobordism to be α-admissible. □

We are finally ready to state the splitting formula.

Theorem 48. *(Splitting Theorem) Suppose that $(W,\Sigma) : (Y,K) \to (Y,K)$ is a self-concordance of a knot and we choose a parameter $\alpha \in \mathbb{Q}\cap(0,1/2)$ for which $\triangle_K(e^{-4\pi i\alpha}) \neq 0$. Let (X,T) be the closed 4-manifold obtained by closing up (W,Σ). Suppose that $\lambda_{FO}(X,T,\alpha)$ is well defined, or equivalently, that (W,Σ) is α-admissible. Then we have the* ***splitting formula***

$$\sum_k D_0(X,T,\alpha,k)T^{-\mathcal{E}_{top}(X,T,k,-2k,\alpha)} = 2\boldsymbol{Lef}(W \mid HI(Y,K,\alpha)) = 2\boldsymbol{Lef}(W \mid HI^{red}(Y,K,\alpha)) - 2h(Y,K,\alpha) \tag{55}$$

Proof. (first equality of Theorem 48) The argument is standard and analogous to the one given in [24, Section 11], [23, Section 11.1] and [53, Section 9]. In fact, since we already analyzed the action of $H^1(X;\mathbb{Z}/2)$ on the moduli spaces, as we mentioned in the introduction, it can be regarded as a consequence of Proposition 5.5 in [44] and the remarks after it.

Namely, the idea is to compare the moduli spaces $\mathcal{M}(X,T,k,-2k,\alpha)$ $(k \in \mathbb{Z})$ with the 0-dimensional moduli spaces $\mathcal{M}_0([\beta],W,[\beta])$, i.e, those which are asymptotic to $[\beta]$ as $t \to \pm\infty$. Notice that because of the failure of monotonicity, specifying the dimension is not enough (when

$\alpha \neq 1/4$), i.e, $\mathcal{M}_0([\beta], W, [\beta])$ has a priori different connected components which are indexed by the energy

$$\mathcal{M}_0([\beta], W, [\beta]) = \bigcup_{k \in \mathbb{Z}} \mathcal{M}_{0,E(k)}([\beta], W, [\beta])$$

Now, we can focus on one of the individual moduli spaces $\mathcal{M}(X, T, k, -2k, \alpha)$ and introduce a parameter R which keeps track of the length of the cylinder in the usual stretching the neck argument, so that we are consider the manifold $\check{X}(R)$ where a cylinder of length $2R$ $[-R, R] \times \check{Y}$ has been introduced along $\check{Y}$.

The gluing argument says that for R sufficiently large, $\mathcal{M}(X, T, k, -2k, \alpha)$ can be identified with $\bigcup_{[\beta] \in \mathfrak{C}^*(Y,K,\alpha)} \mathcal{M}_{0,E(k)}([\beta], W, [\beta])$. Notice that $\mathcal{M}_{0,E(k)}([\theta_\alpha], W, [\theta_\alpha])$ does not enter into our identification because it has negative dimension.

However, this correspondence is two to one, since when we are closing the bundle over W^* to produce the bundle over X, there are two ways to do this since the stabilizer of $[\beta]$ is $\mathbb{Z}_2 = Z(SU(2))$, given that we are dealing with an irreducible connection [45, Remark p.893]. This explains the factor of 2 in the statement of the theorem. □

Now we will proof the second part of the Splitting Theorem. The proof is an adaptation word by word of the one Anvari gives in [2], but we will redo most of the arguments since the reader may be less familiar with this proof (relative to the first part of the splitting formula).

Proof. (second equality of Theorem 48) We start by recalling the behavior of the maps

$$\begin{aligned} \delta_{1,n} =& \delta_1 u_K^n : HI_{1+2n}(Y, K, \alpha) \to \varLambda \\ \delta_{2,n} =& u_K^n \delta_2 : \varLambda \to HI_{2-2n}(Y, K, \alpha) \end{aligned}$$

which appear implicitly in our definition of the reduced Floer groups 45. In the case of a self-concordance $(W, \varSigma)$, the induced cobordism map $m_{\check{W}} : HI_*(Y, K, \alpha) \to HI_*(Y, K, \alpha)$ acts on the $\delta_{1,n}$ and $\delta_{2,n}$ as follows [21, Theorem 7]: there are integers a_{ij}, b_{ij} such that

$$\begin{aligned} \delta_{1,n} m_{\check{W}} =& \delta_{1,n} + \sum_{i=0}^{n-1} a_{in} \delta_{1,n} \\ m_{\check{W}} \delta_{2,n} =& \delta_{2,n} + \sum_{i=0}^{n-1} b_{in} \delta_{2,n} \end{aligned} \tag{56}$$

Since $m_{\check{W}}$ preserves gradings we can see that $a_{in} = 0$ and $b_{in} = 0$ whenever i, n have opposite parity. We will also use the fact (which follows from Lemma 44) that either δ_1 or δ_2 must vanish. Finally, we also need the fact that for a commutative diagram of exact sequences of finite dimensional vector spaces over an arbitrary field $\varLambda$

$$\begin{array}{cccc} 0 \to & V_0 \to & V_1 \to & V_2 \\ & \downarrow_\alpha & \downarrow_\beta & \downarrow_\gamma \\ 0 \to & W_0 \to & W_1 \to & W_2 \end{array}$$

we have

$$\operatorname{tr}(\beta) = \operatorname{tr}(\alpha) + \operatorname{tr}(\gamma) \tag{57}$$

Likewise, for an exact sequence of the form

$$\begin{array}{cccc} V_0 \to & V_1 \to & V_2 \to & 0 \\ \downarrow_\alpha & \downarrow_\beta & \downarrow_\gamma & \\ W_0 \to & W_1 \to & W_2 \to & 0 \end{array}$$

we again have

$$\mathrm{tr}(\beta) = \mathrm{tr}(\alpha) + \mathrm{tr}(\gamma)$$

To show that

$$h(Y,K,\alpha) = \mathrm{Lef}(W \mid HI^{red}(Y,K,\alpha)) - \mathrm{Lef}(W \mid HI(Y,K,\alpha)) \tag{58}$$

we make cases based on the vanishing of δ_1, δ_2.

Case $\delta_2 = 0$: In this situation the reduced Floer groups 45 simplify to

$$\begin{cases} HI_1^{red}(Y,K,\alpha) = \cap_l \ker(\delta_{1,2l}) \\ HI_3^{red}(Y,K,\alpha) = \cap_l \ker(\delta_{1,2l+1}) \\ HI_0^{red}(Y,K,\alpha) = HI_0(Y,K,\alpha) \\ HI_2^{red}(Y,K,\alpha) = HI_2(Y,K,\alpha) \end{cases}$$

Therefore the difference in Lefschetz numbers simplifies to

$$\begin{aligned} &\mathrm{Lef}(W \mid HI^{red}(Y,K,\alpha)) - \mathrm{Lef}(W \mid HI(Y,K,\alpha)) \\ =&\mathrm{Tr}(W \mid HI_1(Y,K,\alpha) \oplus HI_3(Y,K,\alpha)) - \mathrm{Tr}(W \mid HI_1^{red}(Y,K,\alpha) \oplus HI_3^{red}(Y,K,\alpha)) \\ =&\mathrm{Tr}(W \mid HI_1(Y,K,\alpha)) - \mathrm{Tr}(W \mid HI_1^{red}(Y,K,\alpha)) + \mathrm{Tr}(W \mid HI_3(Y,K,\alpha)) - \mathrm{Tr}(W \mid HI_3^{red}(Y,K,\alpha)) \end{aligned}$$

In this case $h(Y,K,\alpha)$ 46 simplifies to

$$\begin{aligned} &\chi_\Lambda(HI^{red}(Y,K,\alpha)) - \chi_\Lambda(HI(Y,K,\alpha)) \\ =& -\dim_\Lambda HI_1^{red}(Y,K,\alpha) - \dim_\Lambda HI_3^{red}(Y,K,\alpha) + \dim_\Lambda HI_1(Y,K,\alpha) + \dim_\Lambda HI_3(Y,K,\alpha) \\ =& \dim_\Lambda HI_1(Y,K,\alpha) - \dim_\Lambda HI_1^{red}(Y,K,\alpha) + \dim_\Lambda HI_3(Y,K,\alpha) - \dim_\Lambda HI_3^{red}(Y,K,\alpha) \\ =& \dim_\Lambda(HI_1(Y,K,\alpha)/\cap_l \ker(\delta_{1,2l})) + \dim_\Lambda(HI_3(Y,K,\alpha)/\cap_l \ker(\delta_{1,2l+1})) \end{aligned}$$

So it clearly suffices to show that

$$\begin{aligned} \dim_\Lambda(HI_1(Y,K,\alpha)/\cap_l \ker(\delta_{1,2l})) &= \mathrm{Tr}(W \mid HI_1(Y,K,\alpha)) - \mathrm{Tr}(W \mid HI_1^{red}(Y,K,\alpha)) \\ \dim_\Lambda(HI_3(Y,K,\alpha)/\cap_l \ker(\delta_{1,2l+1})) &= \mathrm{Tr}(W \mid HI_3(Y,K,\alpha)) - \mathrm{Tr}(W \mid HI_3^{red}(Y,K,\alpha)) \end{aligned}$$

to obtain our result. The argument is completely analogous in both cases, so let us do verify the second identity. The result will be obtained from induction on the sequence of subspaces

$$Z_{3,k} = \cap_{l=0}^k \delta_{1,2l+1}$$

From 56 we can see that for $k = 0$ we have $\delta_1 m_{\check{W}} = \delta_1$, which in other words means that there is an exact sequence

$$\begin{array}{cccc} 0 \to & Z_{3,0} \to & HI_3 \to^{\delta_1} & \Lambda \\ & \downarrow_{m_{3,0}} & \downarrow_{m_{\check{W}}} & \downarrow_{\mathrm{id}} \\ 0 \to & Z_{3,0} \to & HI_3 \to_{\delta_1} & \Lambda \end{array}$$

Here $m_{3,0}$ denotes the restriction of the cobordism map to $Z_{3,0}$. The additivity of the trace formula 57 now says that

$$\mathrm{tr}(m_{\breve{W}}) = \mathrm{tr}(m_{3,0}) + \mathrm{tr}(\mathrm{id}) = \mathrm{tr}(m_{3,0}) + 1 = \mathrm{tr}(m_{3,0}) + \dim_{\Lambda}(HI_3(Y,K,\alpha)/Z_{3,0})$$

provided that $\delta_1 \neq 0$ (which in any case is the only interesting situation since we already assumed that $\delta_2 = 0$). An induction argument on k [2, p. 7] based on the identities 56 now says that

$$\mathrm{tr}(m_{\breve{W}}) = \mathrm{tr}(m_{3,k}) + \dim(HI_3(Y,K,\alpha)/Z_{3,k})$$

Since the sequence of the $Z_{3,k}$ stabilizer once k is large enough then we find that

$$\mathrm{tr}(m_{\breve{W}} \mid HI_3(Y,K,\alpha)) - \mathrm{tr}(m^{red}_{\breve{W}} \mid HI^{red}_3(Y,K,\alpha)) = \dim(HI_3(Y,K,\alpha)/Z_{3,k})$$

Again, the case for $HI_1(Y,K,\alpha)$ is completely analogous so 58 has been verified in this situation.

Case $\delta_1 = 0$**:** In this situation the reduced Floer groups 45 simplify to

$$\begin{cases} HI^{red}_0(Y,K,\alpha) = HI_0(Y,K,\alpha)/\left(\sum \mathrm{im}(\delta_{2,2l+1}\right) \\ HI^{red}_2(Y,K,\alpha) = HI_2(Y,K,\alpha)/\left(\sum \mathrm{im}(\delta_{2,2l})\right) \\ HI^{red}_1(Y,K,\alpha) = HI_1(Y,K,\alpha) \\ HI^{red}_3(Y,K,\alpha) = HI_3(Y,K,\alpha) \end{cases}$$

In this case the difference in Lefschetz numbers simplify to

$$\begin{aligned} &\mathrm{Lef}(W \mid HI^{red}(Y,K,\alpha)) - \mathrm{Lef}(W \mid HI(Y,K,\alpha)) \\ =&\mathrm{Tr}(W \mid HI^{red}_0(Y,K,\alpha)) - \mathrm{Tr}(W \mid HI_0(Y,K,\alpha)) + \mathrm{Tr}(W \mid HI^{red}_2(Y,K,\alpha)) - \mathrm{Tr}(W \mid HI_2(Y,K,\alpha)) \end{aligned}$$

while $h(Y,K,\alpha)$ 46 simplifies to

$$\begin{aligned} &\chi_{\Lambda}(HI^{red}(Y,K,\alpha)) - \chi_{\Lambda}(HI(Y,K,\alpha)) \\ =& -\dim_{\Lambda}(\sum \mathrm{im}(\delta_{2,2l+1})) - \dim_{\Lambda}(\sum \mathrm{im}(\delta_{2,2l})) \end{aligned}$$

Once again it suffices to show that

$$\begin{aligned} -\dim_{\Lambda}(\sum \mathrm{im}(\delta_{2,2l+1})) &= \mathrm{Tr}(W \mid HI^{red}_0(Y,K,\alpha)) - \mathrm{Tr}(W \mid HI_0(Y,K,\alpha)) \\ -\dim_{\Lambda}(\sum \mathrm{im}(\delta_{2,2l})) &= \mathrm{Tr}(W \mid HI^{red}_2(Y,K,\alpha)) - \mathrm{Tr}(W \mid HI_2(Y,K,\alpha)) \end{aligned} \tag{59}$$

Both cases are proven in exactly the same way so we will do the second one. Define, the sequence of subspaces

$$B_{2,k} = \sum_{l=0}^{k} \mathrm{im}(\delta_{2,2l})$$

For k sufficiently large the sequence $B_{2,k}$ stabilizes so the following diagram is exact and commutes

$$\begin{array}{cccc} B_{2,k} \to & HI_2 \to & HI^{red}_2 \to & 0 \\ \downarrow_{m_{2,k}} & \downarrow_{m_{\breve{W}}} & \downarrow_{m^{red}_{\breve{W}}} & \\ B_{2,k} \to & HI_2 \to & HI^{red}_2 \to & 0 \end{array}$$

which implies by additivity of the trace that

$$\mathrm{tr}(m_{\breve{W}}) = \mathrm{tr}(m^{red}_{\breve{W}}) + \mathrm{tr}(m_{2,k})$$

so it suffices to show that (for k sufficiently large)$\mathrm{tr}(m_{2,k}) = \dim_{\Lambda} B_{2,k} = \dim_{\Lambda}(\sum \mathrm{im}(\delta_{2,2l}))$ to conclude the result.

For $k=0$ the relations 56 say that $m_{\check{W}}\delta_2=\delta_2$ gives us the diagram

$$\begin{array}{cccc}
\Lambda \to^{\delta_2} & HI_2(Y,K,\alpha)\to & HI_2(Y,K,\alpha)/B_{2,0}\to & 0\\
\downarrow_{\mathrm{id}} & \downarrow_{m_{\check{W}}} & \downarrow_{\bar{m}_{2,0}} & \downarrow\\
\Lambda \to^{\delta_2} & HI_2(Y,K,\alpha)\to & HI_2(Y,K,\alpha)/B_{2,0}\to_{\delta_1} & 0
\end{array}$$

which gives us the relation
(60)
$$\mathrm{tr}(m_{\check{W}})=1+\mathrm{tr}(\bar{m}_{2,0})=\dim_\Lambda HI_2(Y,K,\alpha)-\dim_\Lambda(HI_2(Y,K,\alpha)/B_{2,0})+\mathrm{tr}(\bar{m}_{2,0})=\dim_\Lambda B_{2,0}+\mathrm{tr}(\bar{m}_{2,0})$$

provided $\delta_2\neq 0$ (which is the only interesting case anyway). Also, using induction [2, p. 9], one concludes that for k sufficiently large

$$\mathrm{tr}(\bar{m}_{2,0})=\dim_\Lambda(B_{2,k}/B_{2,0})+\mathrm{tr}(m^{red}_{\check{W}}) \tag{61}$$

Combining equations 60 and 61 we find that

$$\begin{aligned}
&\mathrm{tr}(m^{red}_{\check{W}}\mid HI^{red}_2(Y,K,\alpha))-\mathrm{tr}(m_{\check{W}}\mid HI_2(Y,K,\alpha))\\
=&\mathrm{tr}(\bar{m}_{2,0})-\dim_\Lambda(B_{2,k}/B_{2,0})-\mathrm{tr}(m_{\check{W}})\\
=&\mathrm{tr}(m_{\check{W}})-\dim_\Lambda B_{2,0}-\dim_\Lambda(B_{2,k}/B_{2,0})-\mathrm{tr}(m_{\check{W}})\\
=&-\dim_\Lambda(B_{2,k})\\
=&-\dim_\Lambda\Big(\sum\mathrm{im}(\delta_{2,2l})\Big)
\end{aligned}$$

and this is precisely what we needed to show 59. ☐

Now we will show that under the previous circumstances (X,T) can be assigned an h-invariant.

Theorem 49. *Suppose that (X,T) can be written as the closed-up version of two different self-concordances $(W,\Sigma):(Y,K)\to(Y,K)$ and $(W',\Sigma'):(Y',K')\to(Y',K')$ which satisfy $\triangle_K(e^{-4\pi i\alpha})\neq 0$ and $\triangle_{K'}(e^{-4\pi i\alpha})\neq 0$ for some $\alpha\in\mathbb{Q}\cap(0,1/2)$. Suppose $\lambda_{FO}(X,T,\alpha)$ can be defined, or equivalently, either of the concordances (and hence the other) is α-admissible. Then*

$$\mathit{Lef}(W\mid HI^{red}(Y,K,\alpha))=\mathit{Lef}(W\mid HI^{red}(Y',K',\alpha))$$

and thus we can use the splitting formula 55 to define $h(X,T,\alpha)$ as $h(Y,K,\alpha)=h(Y',K',\alpha)$.

Proof. The proof is indistinguishable from the one Frøyshov gives for monopole h-invariant in [24, Section 13]. First Frøyshov proves Lemma 10 in [24], which in our case translates to the following statement: let A, B be $r\times r$ matrices with coefficients in the universal Novikov field $\Lambda^{\mathbb{C},\mathbb{R}}$, and let m be a natural number such that $\mathrm{tr}(A^n)=\mathrm{tr}(B^n)$ for all natural numbers n satisfying $m\leq n<2r+m$. Then A and B have the characteristic polynomial, and in particular $\mathrm{tr}(A)=\mathrm{tr}(B)$.

Frøyshov states this Lemma for matrices with coefficients over the complex field $\mathbb{C}$, but a quick inspection of the proof reveals that the only property used about $\mathbb{C}$ is that it is algebraically closed, so that the characteristic polynomial of A (or B) has roots, which are the eigenvalues of A (or B). That $\Lambda^{\mathbb{C},\mathbb{R}}$ is algebraically closed is proven in [27, Lemma A.1]. Once we know this lemma holds, the argument Frøyshov gives is the same, just replace every occurrence of $X,W,Y,X_\infty,X_{j,\infty},W_{j,n}$,etc in his proof with their orbifold versions $\check{X},\check{W},\check{Y},\check{X}_\infty,\check{X}_{j,\infty},\check{W}_{j,n}$. ☐

8. Some examples and Properties

Product Case.

As a corollary of the splitting theorem 48 we can verify our basic desiderata for $\lambda_{FO}(X,T,\alpha)$.

Corollary 50. *Suppose that (Y,K) is such that $\triangle_K(e^{-4\pi i\alpha}) \neq 0$ for $\alpha \in \mathbb{Q} \cap (0,1/2)$. Then $\lambda_{FO}(X,T,\alpha)$ can be defined on $(X,T)=(S^1\times Y, S^1\times K)$.*

Moreover, $\lambda_{FO}(X,T,\alpha) = 2\lambda_{CLH}(Y,K,\alpha)$ and the additional degree zero Donaldson invariants $D_0(X,T,\alpha,k)$ vanish for $k\neq 0$.

Proof. In this case the self-concordance is $(W,\Sigma)=([0,1]\times Y,[0,1]\times K)$ which will clearly be α-admissible.

According to the splitting formula 55 we have for $(X,T)=(S^1\times Y, S^1\times K)$

$$\sum_k D_0(X,T,\alpha,k)T^{-\mathcal{E}(X,T,k,-2k,\alpha)} = 2\mathrm{Lef}(Id \mid HI(Y,K,\alpha)) = 2\chi_\Lambda(HI(Y,K,\alpha)) = 2\lambda_{CH}(Y,K,\alpha)$$

from which we will conclude that

$$\begin{cases} \lambda_{FO}(S^1\times Y, S^1\times K,\alpha) = 2\lambda_{CH}(Y,K,\alpha) & \\ D_0(X,T,\alpha,k)=0 & k\neq 0 \end{cases}$$

□

Remark 51. In particular, notice that one cannot use the invariants $D_0(X,T,\alpha,k)$ (for $k\neq 0$) to obtain new invariants for knots, as one might have suspected all along.

Flip symmetry.

Now we briefly discuss the proof of Theorem 13 from the introduction, i.e, the flip symmetry $\mathcal{F}$ obtained by changing the holonomy parameter from α to $\frac{1}{2}-\alpha$. This is defined similar to the action of $H^1(X;\mathbb{Z}/2)$, in the sense that there is a natural map $\mathcal{F}:\mathcal{M}(X,T,k,l,\alpha)\to\mathcal{M}(X,T,k+l,-l,\frac{1}{2}-\alpha)$ obtained by tensoring the bundle $E(k,l)$ with a line bundle χ whose holonomy on the small circles linking T is -1. The new values for k and l after the flip symmetry is performed were computed in [41, Section 2, iv)].

Moreover, in [43, Appendix 1, ii)] they discuss the effect of $\mathcal{F}$ on the orientation of the moduli spaces. In particular, $\mathcal{F}$ preserves or reverses orientation according to the parity of $\frac{1}{4}T\cdot T-(g-1)$, which in our case vanishes, which means that $\mathcal{F}$ is orientation preserving.

Recalling that $D_0(X,T,k,\alpha)$ was computed using $\mathcal{M}(X,T,k,-2k,\alpha)$, this means that $D_0(X,T,k,\alpha) = D_0\left(X,T,-k,\frac{1}{2}-\alpha\right)$ as was claimed in Theorem 13.

The effect on the Floer homologies $\mathcal{F}: HI(Y,K,\alpha)\to HI(Y,K,\frac{1}{2}-\alpha)$ can be analyzed in a similar way. The only thing to worry about is about the effect of $\mathcal{F}$ on the gradings of a critical point before and after the flip has been performed. Clearly $\mathcal{F}([\theta_\alpha]) = \mathcal{F}([\theta_{1/2-\alpha}])$ and more generally $\mathcal{F}(\mathcal{M}([\beta],[\theta_\alpha]) = \mathcal{M}([\mathcal{F}\beta],[\theta_{1/2-\alpha}])$ which from the formula for the absolute grading 13 will imply that $\mathrm{gr}([\mathcal{F}\beta]) = \mathrm{gr}([\beta])$, in other words, $\mathcal{F}$ will be grading preserving, in particular the Euler characteristic is preserved under the flip operation, which could also be checked directly from the formula for $\lambda_{CLH}(Y,K,\alpha)$.

The effect of $\mathcal{F}: HI^{red}(Y,K,\alpha)\to HI^{red}(Y,K,\frac{1}{2}-\alpha)$ can be analyzed in a similar way. The only thing to be aware of is that under $\mathcal{F}$ the u-map $\mu_K(x)$ changes sign, i.e, $\mathcal{F}(\mu_K(x)) = -\mu_K(x)$, as is discussed in [39, Section 4.2]. However, changing the sign of the u map still is compatible with the definition of the reduced Floer groups 45 so the Euler characteristic of $HI^{red}(Y,K,\alpha)$ is preserved under flip symmetries. From the formula for the Frøyshov knot invariants 46 it is immediate to conclude that $h(Y,K,\alpha) = h(Y,K,\frac{1}{2}-\alpha)$, which was the first statement of Theorem 13.

Duality and the figure eight knot. It is well known that in general it is very difficult to compute the instanton Floer homology groups of some three manifold whenever there are non-trivial differentials between the critical points. Moreover, in general one can expect that computing the singular Floer groups $HI(Y,K,\alpha)$ will be even harder, considering that the metric has a conical singularity and using local coefficients requires keeping track of additional data.

Therefore, we will illustrate indirect approaches for trying to compute the groups $HI(Y,K,\alpha)$, using the pairing and duality properties Floer homologies are equipped with. This example first appeared in [7] for the case of $\alpha = 1/4$, but as will become clear the argument is entirely formal so it can be readily adapted to other values of α as long as the same properties about the character varieties hold true.

In order to do this we need to discuss how the Floer groups $HI(Y,K,\alpha)$ are related to those of $HI(-Y,-K,\alpha)$, where $(-Y,-K)$ denotes the pair (Y,K) with the opposite orientation on both factors. When there is no knot present, the way to understand the Floer homology $HI(-Y)$ in terms of the Floer homology $HI(Y)$ is standard, what we need to do discuss is that happens in the presence of the knot K as well as the effect on the local systems $\mathit{\Gamma}_{[\beta]}$. We will follow the discussion in [48, Sections 22.5 and 32.1], [71, Section 7.4].

The Chern-Simons functionals $CS_{\check{Y}}$ and $CS_{-\check{Y}}$ on the orbifolds $\check{Y}$ and $-\check{Y}$ are related as

$$CS_{\check{Y}} = -CS_{-\check{Y}}$$

Therefore the critical points of the functionals can be identified in a natural way with each other. The perturbation on $-Y$ can be taken to be $-\mathfrak{p}$, while the vector field $\mathrm{grad} CS_{-\check{Y}}$ is the negative of $\mathrm{grad} CS_{\check{Y}}$. In other words, if $\gamma(t)$ is a trajectory on $\check{Y}$ of $CS_{\check{Y}}$ then $\gamma(-t)$ is a trajectory on $-\check{Y}$ of $CS_{-\check{Y}}$. The coefficient system on $-\check{Y}$ can be described as a coefficient system on $\mathcal{B}(\check{Y},\alpha) = \mathcal{B}(Y,K,\alpha)$, by saying that the fiber at each point is still $\mathit{\Lambda}^{\mathbb{Q},\mathbb{R}}$, but now along paths z from $[\beta_0]$ to $[\beta_1]$ one multiplies by $T^{+\mathcal{E}_{top}(z)}$ instead of $T^{-\mathcal{E}_{top}(z)}$.

To obtain the grading formula, consider the cylinders $\mathbb{R}\times\check{Y}$ and $\mathbb{R}\times -\check{Y}$. Let $[\beta]\in\mathfrak{C}(Y,K,\alpha)$ be a critical point and $[\bar{\beta}]$ the corresponding class in $\mathfrak{C}(-Y,-K,\alpha)$. Recall that 34

$$\mathrm{gr}([\beta]) = -1 - \dim\mathcal{M}([\theta_\alpha],[\beta]) \mod 4 = \dim\mathcal{M}([\beta],[\theta_\alpha]) \mod 4$$

A flow line on the moduli space $\mathcal{M}([\bar{\beta}],[\bar{\theta}_\alpha])$ can be identified with a flow line of the moduli space $\mathcal{M}([\theta_\alpha],[\beta])$, since the time direction has been reversed, which means

$$\mathrm{gr}([\bar{\beta}]) = \mathcal{M}([\bar{\beta}],[\bar{\theta}_\alpha]) \mod 4 = \dim\mathcal{M}([\theta_\alpha],[\beta]) \mod 4 = -1-\mathrm{gr}([\beta])$$

Therefore we have found ([11, Proposition 4.3]):

Theorem 52. *For each $i\in\mathbb{Z}/4\mathbb{Z}$, there is an isomorphism*

$$HI_i(-Y,-K,\alpha)\simeq HI_{-i-1}(Y,K,\alpha) \tag{62}$$

In order to extract some useful consequences from the above isomorphism we need to say some words about the mirror of a knot. We will use the terminology of [55, Definitions 2.3.2, 2.3.4]

Definition 53. Given a knot (Y,K), the **mirror image or the obverse** is the knot mK which corresponds to the data $(-Y,K)$, the **reverse** knot is rK which corresponds to the data $rK = (M,-K)$, and the **inverse knot** $-K$ is the knot rmK which corresponds to the data $(-Y,-K)$.

For the case of a 3 manifold Y admitting an orientation preserving diffeomorphism $\imath : -Y \to Y$, then $\imath(mK)$ and $\imath(-K)$ can be regarded as knots in Y. In the case of S^3 every orientation preserving diffeomorphism is isotopic to the identity, so the choice of $\imath$ does not matter.

A knot K in S^3 is **positive amphicheiral** if K and mK are isotopic, K is **negative amphicheiral** if K and $-K$ are isotopic, and **reversible** if K and rK are isotopic. A knot K is called **amphicheiral** if it is either positive or negative amphicheiral.

The example we want to consider is the Figure 8 knot $K = 4_1$, which is both positive and negative amphicheiral [55, Exercise 2.3.5]. In this case the duality isomorphism 62 implies that for all values of α such that $\triangle_{4_1}(e^{-4\pi i\alpha}) \neq 0$ we have the isomorphism

$$HI_i(S^3, 4_1, \alpha) \simeq HI_{-i-1}(S^3, 4_1, \alpha)$$

or in other words

$$\begin{cases} HI_0(S^3, 4_1, \alpha) \simeq HI_3(S^3, 4_1, \alpha) \\ HI_1(S^3, 4_1, \alpha) \simeq HI_2(S^3, 4_1, \alpha) \end{cases} \tag{63}$$

Now, the Alexander polynomial and character varieties of the figure 8 knot is well understood. We follow the summary given in [31, Section 5.2] for details on the character variety of $S^3\backslash 4_1$.

The figure 8 has symmetrized Alexander polynomial $\triangle_{4_1}(t) = 3 - t - t^{-1}$. In particular, from this description it is clear that the polynomial $\triangle_{4_1}$ has no roots on the unit circle, so in particular any value $\mathbb{Q} \cap (0, 1/2)$ is fair game for defining the Floer homologies. This knot is a particular case of a 2-bridge knot, which [38, 6, 65] analyzed. In particular, $\mathcal{R}(S^3, 4_1)$ consists of an arc of abelian representations and a disjoint circle of nonabelian representations. Therefore, for fixed α, $\mathcal{R}^*(S^3, 4_1, \alpha)$ will consists of either 0, 2 or 1 points, the latter corresponding to the values of α where the number of points change from 0 to 2. For the value $\alpha = 1/4$ there are exactly two irreducible flat connections.

The additional thing we must understand is whether or not these connections satisfy the non-degeneracy condition stated in Lemma 21. In our case this means that $\ker H^1(S^3\backslash 4_1; \mathfrak{g}_\rho) \to H^1(\mu_{4_1}; \mathfrak{g}_\rho)$ must vanish at each of the irreducible connections. As discussed in [80, Section 7], this is verified by Hedden, Herald and Kirk [31] for all 2-bridge knots (for $\alpha = 1/4$), which includes 4_1. Therefore, the instanton Floer homology of $HI(S^3, 4_1, 1/4)$ can be computed using two critical points (perturbations may obviously be needed in order to cut out the flow lines transversely).

If all the differentials between these two critical points vanished (either because they are supported in the same grading or in grading difference at least two), then because of the duality properties 63 the total rank of the Floer groups would be 4, which is impossible! Therefore, we conclude that the critical points must have relative grading one and moreover there are non-trivial flow lines. Again, this example was known to Collin but we believe it could be interesting to find other instances were similar ideas could be used. In particular, the reduced Floer homology of 4_1 must also vanish for $\alpha = 1/4$ which means that

Theorem 54. *The figure eight knot 4_1 has vanishing instanton Floer homology and reduced Instanton Floer homology for $\alpha = 1/4$, i.e, $HI(S^3, 4_1, 1/4) \equiv 0 \equiv HI(S^3, 4_1, 1/4)$. As a consequence, the knot h-invariant of the figure eight knot vanishes, i.e, $h(S^3, 4_1, 1/4) = 0$.*

Some examples of tori inside mapping tori.

These examples can be considered as the orbifold version of [67]. Before writing a general statement, let's consider a toy model. Suppose that we have a knot K' inside an integer homology sphere Y' and we choose holonomy $\alpha' = \frac{1}{15}$ along the K'.

Moreover, assume that after taking the 3-fold branched cover along K' we obtain a 3 manifold Y which is still an integer homology sphere . Notice that Y comes with a natural $\mathbb{Z}_3$ action τ, whose fixed point set is a knot K. Now choose holonomy $\alpha = \frac{1}{5}$ along the knot K. Then the mapping torus

$$X_\tau = ([0,1] \times Y)/(\{0\} \times Y \sim^\tau \{1\} \times Y)$$

of (Y, τ) will be a homology $S^1 \times S^3$ with a natural torus T_τ obtained as the mapping torus of $K \subset Y$. In this situation the analogue of [67, Proposition 3.1] will tell us that there is a two to one correspondence between α-representations on (X_τ, T_τ) and α-representations on (Y, K) which are τ-equivariant.

Clearly, every α'- representation of (Y', K') will pullback to a α-representation on (Y, K) which is τ -equivariant. So the only question is whether this exhausts all the possibilities for being a τ-equivariant representation. In fact, it does not! Suppose we had chosen holonomy $\tilde{\alpha}' = \frac{6}{15}$ along K'. Then the pull-back of an $\tilde{\alpha}'$ representation of (Y', K') will have holonomy $6/5$ along K upstairs. Recall that we are using the normalization for the holonomy to be between 0 and $1/2$, and since

$$\frac{6}{5} = \frac{1}{5} + 2 \cdot \frac{1}{2}$$

this means that after performing two half-twists, holonomy $6/5$ is equivalent to holonomy $1/5$. In general all the allowable values of holonomy on (Y', K') which give rise to holonomy α upstairs will be of the form

$$\frac{\alpha}{p} + \frac{n}{p}$$

where n is an any integer such that

$$0 < \frac{\alpha}{p} + \frac{n}{p} < \frac{1}{2}$$

which clearly means that there are only finitely many values n can take. In fact, since n must be an integer and $\alpha < \frac{1}{2}$ the only possibilities are

$$n = 0, 1, \cdots, \left\lfloor \frac{p}{2} - \alpha \right\rfloor = \left\lfloor \frac{p}{2} \right\rfloor$$

where $\lfloor\rfloor$ denotes the floor function. In our toy model $p = 3$ and $\alpha = 1/5$ so $n = 0, 1$ are the only possibilities, which means that $\alpha'_0 = \frac{1}{15}$ and $\alpha'_1 = \frac{6}{15} = \frac{3}{5}$ are the only holonomy values that have the desired properties.

Our claim is that in this case

$$\begin{aligned}
&\lambda_{FO}(X_\tau, T_\tau, \alpha) \\
=&2\lambda_{CLH}(Y', K', \alpha'_0) + 2\lambda_{CLH}(Y', K', \alpha'_1) \\
=&16\lambda_C(Y') + \sigma_{K'}(e^{-4\pi i \alpha'_0}) + \sigma_{K'}(e^{-4\pi i \alpha'_1})
\end{aligned}$$

Besides the correspondence between the different representation spaces, the other important thing we need to check is how to compare the orientations between the different moduli spaces, and how to identify the representations in case perturbations are needed. We will address each of these issues in stages, but first we state the main result.

Theorem 55. *Let (Y', K') be a pair of an oriented knot inside an integer homology sphere and take $\alpha' = \frac{r}{pq}$, where p, q, r are all odd integers, relatively prime and such that $0 < \alpha' < \frac{1}{2}$. Suppose moreover that $0 < \frac{r}{q} < \frac{1}{2}$ and assume that the p-fold branched cover along K' is an integer homology* *sphere Y. Let K denote the fixed point set of the $\mathbb{Z}_p$ action τ on Y, which will be another oriented knot. For $\alpha=\frac{r}{q}$ consider the $l=\left\lfloor \frac{p}{2}\right\rfloor+1$ integers*

$$n_0=0, n_1=1,\cdots,n_{l-1}=\left\lfloor \frac{p}{2}\right\rfloor$$

and the l holonomy values

$$\alpha_0=\alpha',\alpha'_1=\alpha'+\frac{1}{p},\cdots,\alpha'_{l-1}=\alpha'+\frac{l-1}{p}$$

Assume furthermore that for each $j=0,\cdots,l-1$, $\triangle_{K'}(e^{-4\pi i\alpha'_j})\neq 0$ and that the reducible representation θ_α is τ non-degenerate, i.e, $\check{H}^{1,\tau}(\check{Y};\mathfrak{g}_{\theta_\alpha})=0$. Denote (X_τ,T_τ) the τ-mapping torus of (Y,K). Then $\lambda_{FO}(X_\tau,T_\tau,\alpha)$ is well defined and moreover

$$\begin{aligned}&\lambda_{FO}\left(X_\tau,T_\tau,\alpha\right)\\ =&\sum_{j=0}^{\left\lfloor \frac{p}{2}\right\rfloor}2\lambda_{CLH}\left(Y',K',\alpha'_j\right)\\ =&8\left(\left\lfloor \frac{p}{2}\right\rfloor+1\right)\lambda_C(Y')+\sum_{j=0}^{\left\lfloor \frac{p}{2}\right\rfloor}\sigma_{K'}(e^{-4\pi i\alpha'_j})\end{aligned}$$

Remark 56. i) There are certainly infinitely many examples that satisfy our assumptions. For example, the Brieskorn spheres $\Sigma(p,a,b)$ may be realized as the p-fold branched covering of S^3 along a torus knot $T(a,b)$ [9]. A way to guarantee θ_α is τ isolated as an α representation of $\pi_1(\Sigma(p,a,b)\backslash K)$ is for it to be isolated in the ordinary sense, i.e, $\check{H}^1(\check{\Sigma}(p,a,b);\mathfrak{g}_{\theta_\alpha})=0$. Since the Alexander polynomial of both K' and K have at most a finite number of roots on the unit circle, there are infinitely many numbers of the form $\frac{r}{pq}$ which will serve our purposes.

ii) A statement involving one of the integers p,q being even would be slightly more complicated, since the center of $SU(2)$ is $\mathbb{Z}_2$ and there can be a lifting issue when one tries to lift an action of $\mathbb{Z}_p$ for p even on a 3-manifold to an $SU(2)$ bundle [3].

To keep the proof manageable we will break it into several pieces: first we will discuss the identification at the level of the critical sets, then we will address how the Zariski tangent spaces are related, third we will discuss how to relate the critical sets after perturbations have been introduced into the picture, and finally we will discuss how to relate the orientations of the moduli spaces. Our steps should be regarded as the orbifold version of the corresponding statements in [67].

First we need to review some brief facts about the orbifold fundamental group, as well as some aspects about equivariant gauge theory. Our main sources are [3, 9, 10, 67].

At this point it is a matter of preference whether one wants to think we are working over $Y\backslash K$ or the orbifold $\check{Y}$ for analyzing the action, so we will change perspectives whenever it is more convenient.

Over the manifold $Y\backslash K$ we have an action $\mathbb{Z}_p$ and a principal $SU(2)$ bundle $P\to Y\backslash K$. Let τ denote the generator of $\mathbb{Z}_p$. When one is trying to understand the action of a group on some principal bundle, one needs to make the assumption that

$$\tau^*(P)\simeq P \tag{64}$$

Usually on a closed-4 manifold, this means verifying that the characteristic numbers of the bundle are preserved ($c_2(P)$ in the $SU(2)$ case for example). On any 3-manifold the $SU(2)$ bundles are necessarily trivial, so one may think that 64 is automatically guaranteed. However, we now need to take into account that we are working with connections with a prescribed singularity along the knot, so we need to check that the action of τ preserves the model connection. Conversely, from the orbifold perspective, this is the same as checking that the isotropy data of the bundle is preserved [11].

Recall that using a tubular neighborhood for the knot K, the model connection 10 in a trivialization could be understood given by the matrix valued 1-form

$$i\begin{pmatrix} \alpha & 0 \\ 0 & -\alpha \end{pmatrix} d\theta$$

where (r,θ) are polar coordinates. We are choosing the action $\mathbb{Z}_p$ in such a way that it becomes an isometry and orientation preserving, and from the local model of a branched cover it is not difficult to see that $\mathbb{Z}_p$ acts on the coordinates as

$$\tau \cdot (r,\theta) = (r, \theta + 2\pi/p)$$

Clearly this will preserve $d\theta$ and thus the local model of the connection. In other words, we have verified the singular (orbifold) analogue of the condition 64. If $\mathcal{G}(Y,K,\alpha)$ is the usual gauge group and $\mathscr{G}(Y,K,\alpha)$ the group of bundle automorphisms of P covering an element of $\mathbb{Z}_p$, then we have an exact sequence

$$1 \to \mathcal{G}(Y,K,\alpha) \to \mathscr{G}(Y,K,\alpha) \to \mathbb{Z}_p \to 1 \tag{65}$$

There is an action of $\mathscr{G}(Y,K,\alpha)$ by pullbacks on the space of connections $\mathcal{C}(Y,K,\alpha)$. Let $\tilde{\tau} : P \to P$ denote a lift of τ. By 65, for any two lifts $\tilde{\tau}_1, \tilde{\tau}_2$ of τ, there is a gauge transformation $g \in \mathcal{G}(Y,K,\alpha)$ such that

$$\tilde{\tau}_2 = \tilde{\tau}_1 \cdot g$$

Thus there is a well defined action τ^* on $\mathcal{B}^*(Y,K,\alpha)$. We will denote the fixed point set of τ^* by $\mathcal{B}^\tau(Y,K,\alpha)$. Let $[B] \in \mathcal{B}^\tau(Y,K,\alpha)$. Then we can find a representative B and a lift $\tilde{\tau} \in \mathscr{G}(Y,K,\alpha)$ such that

$$\tilde{\tau}^* B = B$$

If there were another such $\tilde{\tau}'$ then $\tilde{\tau}' \circ \tilde{\tau}^{-1}$ would be an element of $\mathcal{G}(Y,K,\alpha)$ fixing B, hence it would belong to the stabilizer of B, which since we assume was irreducible must be $\mathbb{Z}/2$. In other words

$$\tilde{\tau}' = \pm \tilde{\tau}$$

which means $\tilde{\tau}$ is well defined up to a sign. Moreover, $(\tilde{\tau})^p$ is an element of $\mathcal{G}(Y,K,\alpha)$ fixing B, so by the same token

$$(\tilde{\tau})^p = \pm 1$$

Therefore we can write

$$\mathcal{B}^\tau(Y,K,\alpha) = \sqcup_{[\tilde{\tau}]} \mathcal{B}^{\tilde{\tau}}(Y,K,\alpha) \tag{66}$$

where the disjoint union is over the lifts such that $\tilde{\tau}^p = \pm 1$ and the equivalence relation is $\tilde{\tau}_1 \sim \tilde{\tau}_2$ if and only if $\tilde{\tau}_2 = \pm g \cdot \tilde{\tau}_1 \cdot g^{-1}$ for some gauge transformation $g \in \mathcal{G}(Y,K,\alpha)$.

Each $\mathcal{B}^{\tilde{\tau}}(Y,K,\alpha)$ can be described as follows: for a fixed lift $\tilde{\tau}$, let $\mathcal{C}^{\tilde{\tau}}(Y,K,\alpha)$ denote the irreducible connections B such that $\tilde{\tau}^* B = B$. Define $\mathcal{G}^{\tilde{\tau}}(Y,K,\alpha) = \{g \in \mathcal{G}(Y,K,\alpha) \mid g\tilde{\tau} = \pm \tilde{\tau} g\}$. Then $\mathcal{B}^{\tilde{\tau}}(Y,K,\alpha) = \mathcal{C}^{\tilde{\tau}}(Y,K,\alpha)/\mathcal{G}^{\tilde{\tau}}(Y,K,\alpha)$.

We are after the analogue of [67, Proposition 2.1], which will be useful for the discussion of the spectral flow calculations. Namely, any lift $\tilde{\tau} : P \to P$ can be written in the base-fiber coordinates as

$$\tilde{\tau}(y, f) = (\tau(y), \sigma(y)f)$$

where $\sigma : Y \backslash K \to SU(2)$. Notice that when we regard it as an orbifold, then the automorphism σ must be S^1 valued along K, because the automorphisms of P, i.e $\mathcal{G}(Y, K, \alpha)$, consists of maps $Y \to SU(2)$ which restrict to $S^1 \subset SU(2)$ along K.

Therefore, we will say that the lift $\tilde{\tau}$ is **constant** if there exists $u \in S^1 \subset SU(2)$ such that $\sigma(y) = u$ for all $y \in \check{Y}$. Notice for $y \in \mathrm{Fix}(\tau)$ we must have that

$$\tilde{\tau}^p(y, f) = (y, \sigma^p(y)f) = (y, \pm f)$$

Since $\sigma \mid_K$ is circle valued, for fixed $y \in K$, $\sigma(y)$ must be one of the p-th roots of ± 1. Clearly this is a discrete set, and given that we can assume that the gauge transformations are continuous, this automatically says that $\sigma \mid_K$ is constant.

Let $u_j = \pm \begin{pmatrix} e^{-2\pi i j/p} & 0 \\ 0 & e^{2\pi i j/p} \end{pmatrix}$, for $j = 0, \cdots, \left\lfloor \frac{p}{2} \right\rfloor$ and consider the constant lift

$$\tilde{\tau}_u(y, f) = (\tau(y), uf)$$

The orbifold bundles $P^{\mathrm{ad}}/\tilde{\tau}$ and $P^{\mathrm{ad}}/\tilde{\tau}_u$ have the same holonomy along K', which is $2\alpha' + \frac{2j}{p}$ (modulo some twists to normalize it so that it belongs to the interval $(0, 1/2)$ again), therefore they are isomorphic. Take any isomorphism and pull it back to an (equivariant) gauge transformation $g^{ad} \in \mathcal{G}_{SO(3)}(Y, K, \alpha)$. Then as $SO(3)$ orbifold bundles we have that

$$\mathcal{B}^{\tilde{\tau}}_{ad}(Y, K, \alpha) = \mathcal{B}^{u}_{ad}(Y, K, \alpha)$$

Since $H^1(Y; \mathbb{Z}/2) = 0$ there is no obstruction to lifting g^{ad} to a gauge transformation $g \in \mathcal{G}(Y, K, \alpha)$, in fact there are two choices for such a lift. Since $\mathcal{G}^{\tilde{\tau}}(Y, K, \alpha)$ incorporates the ambiguity of the lift already, we have as well that

$$\mathcal{B}^{\tilde{\tau}}(Y, K, \alpha) = \mathcal{B}^{u}(Y, K, \alpha)$$

We will start now discussing the relation between equivariant α- representations on (Y, K, α) and the different α'_j -representations on (Y', K', α').

Lemma 57. *Assume the hypothesis of Theorem 55. Then there is a bijective correspondence between $\bigcup_{j=0}^{l-1} \mathcal{R}(Y', K', \alpha')$ and $\mathcal{R}^{\tau}(Y, K, \alpha)$. Likewise, there is a two to one correspondence between $\mathcal{R}^*(X_\tau, T_\tau, \alpha)$ and $\mathcal{R}^{\tau}(Y, K, \alpha)$.*

Proof. From covering space theory, the covering

$$Y \backslash K \to Y' \backslash K'$$

induces a homotopy exact sequence

$$1 \to \pi_1(Y \backslash K) \to \pi_1(Y' \backslash K') \to \mathbb{Z}_p \to 1 \tag{67}$$

Recall that the orbifold fundamental group can be defined in terms of the knot complement as

$$\check{\pi}_1 (Y, K, q) = \pi_1(Y \backslash K) / \langle \mu_K^q \rangle \tag{68}$$

Moreover, τ induces an action τ_* on $\pi_1(Y \backslash K)$ which we denote as $\tau \cdot h$ for $h \in \pi_1(Y \backslash K)$. The induced action on $\mathcal{R}(Y \backslash K)$ is given by [9, Section 2]

$$\tau^*(\rho)(h) = \rho(\tau \cdot h)$$

This action descends to an action on the orbifold group and we denote the fixed point as $\check{\pi}_1^{\tau}(Y, K, r/q)$. From the exact sequence we obtain a *split* exact sequence

$$1 \to \check{\pi}_1(Y, K, q) \to \check{\pi}_1(Y', K', pq) \to \mathbb{Z}_p \to 1 \tag{69}$$

Notice that every α'_l -representation of (Y', K') (for any $l = 0, \cdots, \lfloor \frac{p}{2} \rfloor$) can be regarded as an element of $\check{\pi}_1(Y', K', pq)$.

What we need to check is that the pull-back of any α'_l *irreducible* representation of (Y', K') continues to be an α- irreducible representation of (Y, K), and conversely, an τ-equivariant α-irreducible representation of (Y, K) pushes down to an α'_l-irreducible representation of (Y', K'), for some α'_l. The second statement will follow from the first one, so let's focus on the former. Since the sequence 69 splits, we can write $\check{\pi}_1(Y', K', pq)$ as a semi-direct product of $\check{\pi}_1(Y, K, q)$ and $\mathbb{Z}_p$. As discussed in [4, Section 7.4], the representations of a semi-direct product, like $\check{\pi}_1(Y', K', pq) \simeq \check{\pi}_1(Y, K, q) \rtimes \mathbb{Z}_p$ are determined in terms of the representations of $\check{\pi}_1(Y, K, q)$ and $\mathbb{Z}_p$.

Let $\rho' : \check{\pi}_1(Y, K, q) \to SU(2)$ be an α'_l representation and denote by ρ the induced α representation on $\check{\pi}_1(Y, K, q)$. If t is a generator of $\mathbb{Z}_p$ (with unit 1) and e is the identity of $\pi_1(Y \backslash K)$, then $\rho(h) = \rho'(h, 1)$ for $h \in \pi_1(Y \backslash K)$ and $u = \rho'(t) = \rho'(e, t) \in SU(2)$ determine the induced representations from the semidirect product 69. Notice that

$$\begin{aligned}
&\rho'((h, t^m)) \\
=&\rho'((e, t^m) \cdot (h, 1)) \\
=&\rho'((e, t^m))\rho'(h, 1) \\
=&u^m \rho(h)
\end{aligned}$$

The fact that ρ must be τ equivariant implies that [36, Proposition 7.7]

$$\tau^* \rho = u \rho u^{-1}$$

Now, if we assume that ρ is abelian, then $\tau^* \rho$ is completely specified by how τ acts on $H_1(Y \backslash K; \mathbb{Z})$. Since τ is a diffeomorphism of odd order, it is not difficult to check that $\tau_* = \text{id}$ on $H_1(Y \backslash K; \mathbb{Z})$. Hence u and ρ commute. Therefore we have

$$\begin{aligned}
&\rho'((h_1, t^{m_1}) \cdot (h_2, t^{m_2})) \\
=&u^{m_1} \rho(h_1) u^{m_2} \rho(h_2) \\
=&u^{m_2} \rho(h_2) u^{m_1} \rho(h_1) \\
=&\rho'((h_2, t^{m_2}) \cdot (h_1, t^{m_1}))
\end{aligned}$$

This means that ρ' vanishes on commutators hence it must be an abelian representation, giving rise to a contradiction.

Therefore, irreducibility is preserved under pull back of connections and push-down of connections. Once we know this, that $\bigcup_{j=0}^{l-1} \mathcal{R}^*(Y', K', \alpha')$ and $\mathcal{R}^{*,\tau}(Y, K, \alpha)$ are in bijection is immediate.

The two to one correspondence between $\mathcal{R}^*(X_\tau, T_\tau, \alpha)$ and $\mathcal{R}^{*,\tau}(Y, K, \alpha)$ is similar and follows the proof of [67, Proposition 2.1]. First of all, we have a splitting exact sequence

$$1 \to \check{\pi}_1(Y, K, q) \to \check{\pi}_1(X_\tau, T_\tau, q) \to \mathbb{Z} \to 0$$

The same argument we just gave applies to show that the pullback of an irreducible α-representation $\rho_{\check{X}}$ of (X_τ, T_τ) gives rise to an irreducible $\rho_{\check{Y}}$ α-representation of (Y, K) (or one can also think about this in terms of unique continuation coming from restricting the corresponding α -flat connection on (X_τ, T_τ) to a slice (Y,K)). Notice, that the pull back representation is equivariant, so we can write as before

$$\tau^*\rho_{\check{Y}} = u\rho_{\check{Y}}u^{-1}$$

where $u = \rho_{\check{X}}(e,t)$ and $\rho_{\check{Y}} = \rho_{\check{X}}(h,1)$. The correspondence is two to one because replacing u by $-u$ gives rise to a new representation of (X_τ, T_τ) which induces the same representation $\rho_{\check{Y}}$. This is in fact coming from the action of $H^1(X_\tau; \mathbb{Z}/2)$, and the fact that the new representation is a new one is granted by the freeness of this action, which is a true because of Lemma 41. That every equivariant representation on (Y,K) induces one on (X_τ, T_τ) is also clear ([68, Theorem 6.1] and [67, Proposition 3.1]). □

Now we will discuss the non-degeneracy condition. First we will show that under the assumptions of Theorem 55 that $\lambda_{FO}(X_\tau, T_\tau, \alpha)$ is well defined. Likewise, we will show that the Zariski tangent spaces of the α-representations on (X_τ, T_τ) can be identified with the equivariant Zariski tangent spaces of the α-representations of (Y, K, α).

Lemma 58. *Suppose that the conditions of Theorem 55 hold. Then $\lambda_{FO}(X_\tau, T_\tau, \alpha)$ is well defined. Moreover, the unperturbed moduli space $\mathcal{M}^*(X, T, 0, 0, \alpha)$ use to compute $\lambda_{FO}(X_\tau, T_\tau, \alpha)$ is non-degenerate if and only if $\mathcal{R}^{*,\tau}(Y, K, \alpha)$ is non-degenerate.*

Proof. Our proof is essentially the same as the one in [67, Proposition 3.3]. Let $\rho_{\check{X}}$ denote an α-representation on (X_τ, T_τ). By restriction to a slice it induces an α representation $\rho_{\check{Y}}$ on (Y,K). To study the non-degeneracy of the representation we use Lemma 34. Therefore, we want to compute $H^1(X\backslash T; \mathfrak{g}_{\rho_{\check{X}}})$.

From the fibration $X\backslash T \to S^1$ with fiber $Y\backslash K$ we have by Leray-Serre a spectral sequence whose E_2^{pq} page is

$$H^p(S^1, H^q(Y\backslash K; \mathfrak{g}_{\rho_{\check{Y}}}))$$

This spectral sequence collapses for all $p \geq 2$ so

$$H^1(X\backslash T; \mathfrak{g}_{\rho_{\check{X}}}) = H^0(S^1, H^1(Y\backslash K; \mathfrak{g}_{\rho_{\check{Y}}})) \oplus H^1(S^1, H^0(Y\backslash K; \mathfrak{g}_{\rho_{\check{Y}}})) \tag{70}$$

Now we analyze this decomposition in two cases:

• Case when $\rho_{\check{X}}$ is a reducible α-flat connection: in this situation $\mathfrak{g}_{\rho_{\check{X}}} \simeq \mathbb{R} \oplus L^{\otimes 2}_{\rho_{\check{X}}}$ and we just need to analyze the $L^{\otimes 2}_{\rho_{\check{X}}}$ part of the decomposition 70. Since $\rho_{\check{X}}$ induces the reducible connection ρ_{θ_α} on $Y\backslash K$ 70 becomes

$$H^1(X\backslash T; L^{\otimes 2}_{\rho_{\check{X}}}) = H^0(S^1, H^1(Y\backslash K; L^{\otimes 2}_{\theta_\alpha})) \oplus H^1(S^1, H^0(Y\backslash K; L^{\otimes 2}_{\theta_\alpha}))$$

Observe that $H^0(Y\backslash K; L^{\otimes 2}_{\theta_\alpha})$ vanishes since $\dim H^0(Y\backslash K; \mathbb{R} \oplus L^{\otimes 2}_{\theta_\alpha}) = \dim \mathrm{stab}\theta_\alpha = 1$ and we already know that $\dim H^0(Y\backslash K; \mathbb{R}) = 1$. Therefore $H^1(S^1, H^0(Y\backslash K; L^{\otimes 2}_{\theta_\alpha}))$ vanishes.

To compute $H^0(S^1, H^1(Y\backslash K; L^{\otimes 2}_{\theta_\alpha}))$ notice that the generator of $\pi_1(S^1)$ acts on $H^1(Y\backslash K; L^{\otimes 2}_{\theta_\alpha})$ as $\tau^* : H^1(Y\backslash K; L^{\otimes 2}_{\theta_\alpha}) \to H^1(Y\backslash K; L^{\otimes 2}_{\theta_\alpha})$, thus $H^0(S^1, H^1(Y\backslash K; L^{\otimes 2}_{\theta_\alpha}))$ is the fixed point set of τ^*, which is the equivariant cohomology $H^{1,\tau}(Y\backslash K; L^{\otimes 2}_{\theta_\alpha})$. This term must vanish by the assumption on Theorem 55 that $\check{H}^{1,\tau}(\check{Y}; \mathfrak{g}_{\theta_\alpha}) = 0$.

In conclusion, $H^1(X\backslash T; L^{\otimes 2}_{\rho_{\check{X}}})$ vanishes for all α-reducible representations $\rho_{\check{X}}$ which is the condition needed for $\lambda_{FO}(X_\tau, T_\tau, \alpha)$ to be well defined.

• Case when $\rho_{\check{X}}$ is an irreducible α-flat connection: since the restriction $\rho_{\check{Y}}$ is an irreducible α-flat connection then $H^0(Y\backslash K; \mathfrak{g}_{\rho_{\check{Y}}})$ will vanish which means that the second term in 70 will vanish as well.

As in the previous case, $H^0(S^1, H^1(Y\backslash K; \mathfrak{g}_{\rho_{\tilde{Y}}}))$ can be identified with $H^{1,\tau}(Y\backslash K; \mathfrak{g}_{\rho_{\tilde{Y}}}))$. Also, that the meridian μ_T restricts in a natural way to the meridian of μ_K, and any local system restricted to either μ_T or μ_K always reduces (since the loops have abelian fundamental group so any flat connection becomes reducible). Because of the holonomy condition, it cannot be the trivial local system $\simeq \mathbb{R}^3$ in either case which means that

$$\mathbb{R} \simeq H^1(\mu_T; \mathfrak{g}_\rho) \simeq H^1(\mu_K; \mathfrak{g}_{\tilde{\rho}_Y}) \simeq H^{1,\tau}(\mu_K; \mathfrak{g}_{\tilde{\rho}_Y})$$

where the last isomorphism follows from the fact that τ restricts to the identity on the knot. Therefore, ker : $H^1(X\backslash T; \mathfrak{g}_\rho) \to H^1(\mu_T; \mathfrak{g}_\rho)$ will vanish if and only if ker : $H^{1,\tau}(Y\backslash K; \mathfrak{g}_{\rho_{\tilde{Y}}}) \to H^1(\mu_K; \mathfrak{g}_{\tilde{\rho}_Y})$ vanishes, which means that $\mathcal{M}^*(X, T, 0, 0, \alpha)$ is non-degenerate if and only if $\mathcal{R}^{*,\tau}(Y, K, \alpha)$ is non-degenerate. $\square$

The case where perturbations are needed follow essentially the arguments in [67]. Namely, for computing $\lambda_{FO}(X_\tau, T_\tau, \alpha)$ as well as the different $\lambda_{CLH}(Y', K', \alpha')$ we can always use (finitely many) holonomy perturbations whose stays away from the singularity (i.e, the torus or knot). This was already discussed before the proof of Theorem 22, where we used this condition to identify the Euler characteristic of our Floer groups with the Casson Lin Herald invariant. Since the support of the holonomies do not meet the singularity, we can still find equivariant perturbations $\mathfrak{p}^\tau$ to achieve transversality for $\mathcal{R}^{*,\tau}(Y, K, \alpha, \mathfrak{p}^\tau)$ as in [67, Section 5.1]. Since the action is free away from the knot K, these equivariant holonomy perturbations can be pushed down to the quotient (Y', K') in such a way that they guarantee transversality for the different moduli spaces $\bigcup_{j=0}^{l-1} \mathcal{R}(Y', K', \alpha', \mathfrak{p}^\tau)$, as was done in [10, Section 3.8].

That the perturbations needed to achieve transversality for $\mathcal{R}^{*,\tau}(Y, K, \alpha)$ and $\mathcal{R}^*(X_\tau, T_\tau, \alpha)$ can be chosen in a consistent matter corresponds precisely to [67, Section 5.3]. Moreover, that the correspondence between the perturbed versions of $\mathcal{R}^{*,\tau}(Y, K, \alpha)$ and $\mathcal{R}^*(X_\tau, T_\tau, \alpha)$ continues to be 2 to 1 is proven in exactly the same way as [67, Proposition 5.3].

Finally, that the orientations of the moduli spaces for $\bigcup_{j=0}^{l-1} \mathcal{R}(Y', K', \alpha', \mathfrak{p}^\tau)$, $\mathcal{R}^{*,\tau}(Y, K, \alpha)$ and $\mathcal{R}^*(X_\tau, T_\tau, \alpha)$ can be chosen in a consistent way is a consequence of the analysis in [67, Section 3.5], where we just need to use the existence of a constant lift $\tilde{\tau}$, which we already know exist from our previous discussion when we defined the spaces $\mathcal{B}^{\tilde{\tau}}(Y, K, \alpha)$.

Some tori inside circle bundles over homology $S^1 \times S^2$.

Now we will work out an orbifold version of the examples discussed in [68, Section 8]. There Ruberman and Saveliev analyzed a family of homologies $S^1 \times S^3$ for which $\lambda_{FO}(X)$ can be computed, and in fact vanishes identically. The four manifolds X they considered arise as circle bundles $\pi : X \to Y_0$ over a 3 manifold with the integral homology of $S^1 \times S^2$ (i.e, a homology handle).

The manifolds Y_0 can be obtained from doing 0 surgery on a knot K in an integral homology sphere Y, and in order to guarantee that $\lambda_{FO}(X)$ is well defined, it is assumed that $\triangle_K(t) \equiv 1$ and moreover that the Euler class $e \in H^2(Y; \mathbb{Z}) = \mathbb{Z}$ of the bundle satisfies $e = 1$. In order to be able to compute $\lambda_{FO}(X)$, they furthermore assume that $\pi_2(Y) = 0$ so that the homotopy exact sequence of the S^1 bundle $\pi : X \to Y_0$ allows them to consider $\pi_1 X$ as a central extension of $\pi_1 Y_0$ by the integers

$$1 \to \mathbb{Z} \to \pi_1 X \to^{\pi_*} \pi_1 Y_0 \to 1$$

From here one can identify $\mathcal{M}^*(X, SO(3))$ with $\mathcal{R}^*(Y, SO(3))$ as well as the corresponding Zariski tangent spaces.

The natural tori inside X that can be used for trying to compute $\lambda_{FO}(X, T, \alpha)$ are related to the **Longitudinal Floer Homology** Kronheimer and Mrowka define in [45, Section 4.4]. First of all, observe that Y_0 contains a natural knot K_0, which is the core of the solid torus used in the surgery. Furthermore, K_0 represents a primitive element in the first homology of Y_0, so in particular $H_1(K_0;\mathbb{Z})$ generates $H_1(Y_0;\mathbb{Z})$.

Therefore, when we look at the inverse image of K_0 under the map $\pi : X \to Y_0$ it is clear that we obtain an embedded torus T satisfying the condition that $H_1(T;\mathbb{Z})$ surjects onto $H_1(X;\mathbb{Z})$.

Example 59. Consider the unknot $K = \circ \subset S_3$. After doing 0 -surgery on K we obtain $Y_0 = S^1 \times S^2$. A natural S^1 bundle over Y_0 satisfying the our requirements is $\pi : S^1 \times S^3 \to S^1 \times S^2$ where we are using the Hopf fibration $S^3 \to S^2$ on the second factor and the trivial projection on the first factor. Notice that in this case K_0 can be identified with $S^1 \times \{pt\}$ and therefore the natural torus T is $T = S^1 \times S^1$, where the second factor is the standard circle as well (an unknot). It is clear in this case that for any α we have $\lambda_{FO}(X, T, \alpha) = 0$, since the fundamental group of $X \backslash T$ is abelian.

In fact, our expectation is that the previous example is the norm in the following sense:

Conjecture 60. *Let $\pi : X \to Y_0$ as before and consider the torus $T = \pi^{-1}(K_0)$. Then whenever $\lambda_{FO}(X, T, \alpha)$ can be defined it will vanish.*

To give insight into why we are making this conjecture, we need to explain a bit more how $\lambda_{FO}(X)$ was shown to vanish by Ruberman and Saveliev and how this computation would be modified in the case of $\lambda_{FO}(X, T, \alpha)$.

As we mentioned before, Ruberman and Saveliev identified $\mathcal{M}^*(X, SO(3))$ with $\mathcal{R}^*(Y, SO(3))$ in an orientation preserving way. The action of $H^1(X;\mathbb{Z}_2)$ continues to be free in this situation, so in particular $\lambda_{FO}(X)$ can be computed as one half the signed count of elements in $\mathcal{R}^*(Y, SO(3))$ (after perturbations if needed). This count ends up being the same as $\triangle''_K(1)$, which is zero in this case.

The interesting feature of this calculation is that $\triangle''_K(1)$ is the Euler characteristic of the Instanton Floer homology Floer defined for a homology $S^1 \times S^2$. Therefore, we expect that $\lambda_{FO}(X, T, \alpha)$ is related to the Euler characteristic of an orbifold version of Instanton Floer homology on homology handles. But this is precisely the Longitudinal Floer homology $HIL(Y_0, K_0)$ Kronheimer and Mrowka defined in [45]! Here the connections are allowed to have a singularity along the knot K_0, where they used holonomy $\alpha = 1/4$ in order to avoid the compactness issues due to non-monotonicity.

If one is willing to use local coefficients, there is no difficulty in obtaining a version $HIL(Y_0, K_0, \alpha)$ which is always well defined, since there are no reducible α-flat connections to worry about in this situation (because we are now using the non-trivial $SO(3)$ bundle over Y_0). They conjectured that $\chi(HIL(Y_0, K_0))$ should be $2\triangle''_K(1)$ in general, which in our situation would imply that $\chi(HIL(Y_0, K_0))$ vanishes.

For the other values of α there is a brief discussion in [8, Section 4.3], where it is promised that a definition of what we are defining as $HIL(Y_0, K_0, \alpha)$ would be constructed eventually. In any case, based on the Property 3 Collin states in the survey we conjecture the following:

Conjecture 61. *For $\alpha \in \mathbb{Q} \cap (0, 1/2)$ the Euler characteristic of $HIL(Y_0, K_0, \alpha)$ over the Novikov ring Λ equals $2\triangle''_K(1)$. In particular, for the tori we are considering arising as $\pi^{-1}(K_0)$ we would have that $\lambda_{FO}(X, T, \alpha)$ vanishes in all of these cases.*

Remark 62. Besides the fact that conjecture 61 might be used for computing $\lambda_{FO}(X, T, \alpha)$ for this family of examples, it could be of independent interest to compute the Euler characteristic of

$HIL(Y_0, K_0, \alpha)$. In fact, we plan to verify this in the future, adapting the ideas of Floer as described in [5].

9. Appendix: (Co)homology with Local Coefficients and the Alexander Polynomial

In this appendix we recall the definition of (co)homology with local coefficients, some useful ways to compute it in simple cases, and its relation to the Alexander polynomial of a manifold.

There is an algebraic and a geometric way to think of homology with local coefficients [15, Chapter 5], [30, Section 3.H].

From the algebraic point of view, if X is a path-connected space and $\tilde{X}$ its universal cover, there is a natural action of $\pi = \pi_1(X)$ on the group $C_n(\tilde{X})$ of singular n-chains in $\tilde{X}$: $\gamma \in \pi$ acts on a singular simplex $\triangle^n$ by sending it to the composition $\triangle^n \to \tilde{X} \to^{\gamma} \tilde{X}$. Thus $C_n(\tilde{X})$ becomes a module over the group ring $\mathbb{Z}[\pi]$.

In general, if A is a $\mathbb{Z}[\pi]$ module, then we can define the twisted chain complex

$$\left(C_n(\tilde{X}) \otimes_{\mathbb{Z}[\pi]} A, \partial \otimes \mathbf{1}\right)$$

and the **homology groups with local coefficients** $H_n(X; A)$ are obtained as the homology groups of the previous chain complex.

An useful (in fact, equivalent) way to think of a (left) $\mathbb{Z}[\pi]$ module A is via a homomorphism (representation)

$$\rho : \pi \to \mathrm{Aut}_{\mathbb{Z}}(A)$$

where A is regarded as an abelian group. The $\mathbb{Z}[\pi]$ module structure is obtained by taking the action

$$\left(\sum_{\gamma\in\pi} m_\gamma \gamma\right) \cdot a = \sum_{\gamma\in\pi} m_\gamma \rho(\gamma)(a)$$

Using this interpretation sometimes $H_n(X; A)$ is written as $H_n(X; A_\rho)$.

For computing the **cohomology groups with local coefficients** $H^n(X; A)$ one uses instead with the co-chain complex

$$C^n(X; A) = \hom_{\mathbb{Z}[\pi]}(C_n(\tilde{X}), A)$$

and proceed as before.

Sometimes it is useful to have ways of computing $H_n(X; A)$ that do not involve the universal cover directly [30, Example 3H.2]. Suppose that π' is the kernel of the homomorphism $\rho : \pi \to \mathrm{Aut}_{\mathbb{Z}}(A)$. Let $X' \to X$ be the cover corresponding to the normal subgroup π' of π, then

$$C_n(\tilde{X}) \otimes_{\mathbb{Z}[\pi]} A \simeq C_n(X') \otimes_{\mathbb{Z}[\pi/\pi']} A$$

so we can compute $H_n(X; A_\rho)$ using X' instead of $\tilde{X}$.

The other way to compute homology with local coefficients is in terms of bundles (which in fact is more in line with the gauge theory interpretation).

Namely, let $p : E \to X$ be a system of local coefficients with fiber a discrete abelian group A and structure group $G \subset \mathrm{Aut}(A)$. The chain complex $C_n(X; E)$ is defined using finite sums $\sum_i n_i \sigma_i$ where each $\sigma_i : \triangle^n \to X$ is a singular n-simplex in X and $n_i : \triangle^n \to E$ a lifting of σ_i. More precisely, if $e_0 \in \triangle^n$ is the base point $(1, 0, \cdots, 0)$ of $\triangle^n$, then n_i is an element of the group

$E_{\sigma_i(e_0)}$. The differential can be defined in a similar way, using the local coefficient system to identify fibers over different points of the simplex [15, Theorem 5.8].

In any case, the homology groups obtained this way are isomorphic to the groups $H_\bullet(X; A_\rho)$, where $\rho : \pi_1 X \to \mathrm{Aut}(A)$ is the homomorphism determined by the local coefficient system $p : E \to X$. To define $H^n(X; E)$ one works with the cochain group $C^n(X; E)$ consisting of all functions φ assigning to each singular simplex $\sigma : \triangle^n \to X$ a lift $\varphi(\sigma) : \triangle^n \to E$, and the differential is defined similar to the untwisted case. Again $H^\bullet(X; E)$ can be identified with $H^\bullet(X; A_\rho)$.

As a special case, when A is a $\mathbb{Z}[\pi]$ module for $\pi = \pi_1(X)$, then [15, Proposition 5.14]

$$H^0(X; A) \simeq A^\pi$$

where A^π denotes the **group of invariants,** namely

$$A^\pi = \{a \in A \mid \gamma \cdot a = a \text{ for all } \gamma \in \pi\}$$

Usually we will think of the cohomology with local coefficients as being specified by a flat connection of some vector bundle ξ over X. In the case where ξ is a $\mathbb{C}$ bundle there are some useful formulas in simple cases which we recall for the reader's sake [79, Appendix].

A local coefficient system with fiber $\mathbb{C}$ is completely specified by a map $\mu : H_1(X) \to \mathbb{C}^*$, where $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$ is the multiplicative group of $\mathbb{C}$. This map μ can be regarded as an element of $H^1(X; \mathbb{C}^*)$. We will usually write $H^\bullet(X; \mathbb{C}^\mu)$ instead of $H^\bullet(X; \xi)$ in this special case.

When $X = S^1$, we will write $\zeta = \mu(1) \in \mathbb{C}^*$ for $1 \in H_1(S^1) = \mathbb{Z}$. It is then straightforward to show the following useful properties, which can be found in [79, Appendix B], although there is a small typo in that paper for the last two properties we state.

Lemma 63. ***(Twisted Acyclicity of the circle)*** *Let $\mu : H_1(S^1) \to \mathbb{C}^*$ determine a local coefficient system over S^1 and $\zeta = \mu(1) \in \mathbb{C}^*$. Then $H_\bullet(S^1; \mathbb{C}^\mu)$ vanishes if and only if $\zeta \neq 1$.*

(Vanishing of twisted cohomology with circle factors) *Let X be a path connected space and $\mu : H_1(S^1 \times X) \to \mathbb{C}^*$. Let ζ be the image under μ of the homology class realized by $S^1 \times \{pt\}$. Then $H_\bullet(S^1 \times X; \mathbb{C}^\mu)$ vanishes if $\zeta \neq 1$.*

(Vanishing of twisted cohomology for S^1 bundles) *Let B be a path connected space, $p : X \to B$ a locally trivial fibration with fiber S^1. Let $\mu : H_1(X) \to \mathbb{C}^*$ be a homomorphism and ζ the image under μ of the homology class realized by a fiber of p. Then $H_\bullet(X; \mathbb{C}^\mu)$ if $\zeta \neq 1$.*

In this case there are also some useful duality theorems which stem from the fact that when $\mu, \nu : H_1(X) \to \mathbb{C}^*$ are homomorphisms then $\mathbb{C}^\mu \otimes \mathbb{C}^\nu$ can be regarded as the local coefficient system $\mathbb{C}^{\mu\nu}$ corresponding to $\mu\nu : H_1(X) \to \mathbb{C}^*$, so that taking $\nu = \mu^{-1}$, we get back to the untwisted case [79, Appendix D].

Lemma 64. ***Pairings and Poincare Duality:*** *Let X be an oriented connected compact manifold of dimension n. For local coefficient systems ξ, η and ζ with fiber $\mathbb{C}$ on X, and a pairing $\xi \oplus \eta \to \zeta$, we obtain pairings*

$$\cup : H^p(X; \xi) \times H^q(X; \eta) \to H^{p+q}(X; \zeta)$$
$$\cap : H^{p+q}(X; \xi) \times H^q(X; \eta) \to H^p(X; \zeta)$$

In particular, the vector spaces $H_p(X; \mathbb{C}^{\mu^{-1}})$ and $H^p(X; \mathbb{C}^\mu)$ are dual. More generally, we have the Poincare-Lefschetz duality isomorphisms

$$[X]\cap : H^p(X; \mathbb{C}^\mu) \to H_{n-p}(X; \partial X; \mathbb{C}^\mu)$$
$$[X]\cap : H^p(X, \partial X; \mathbb{C}^\mu) \to H_{n-p}(X; \mathbb{C}^\mu)$$

Now we recall the definition of the Alexander polynomial of a CW complex [76, Chapter II, 11]. Let G be a free abelian group with $b \geq 1$ generators $t_1, \cdots, t_b$. The group ring $\mathbb{Z}[G]$ is isomorphic to the ring of Laurent polynomials on b indeterminates $\mathbb{Z}[t_1^{\pm}, \cdots, t_b^{\pm 1}]$. It is a domain with invertible elements

$$(\mathbb{Z}[G])^* = \pm G = \{\pm t_1^{k_1} \cdots t_b^{k_b} \mid k_1, \cdots, k_b \in \mathbb{Z}\}$$

While the ring $\mathbb{Z}[G]$ is not a principal ideal domain, it is Noetherian and a unique factorization domain. In particular, we can consider the quotient field $Q(G)$ of $\mathbb{Z}[G]$. Clearly $\mathbb{Z}[G] \subset Q(G)$.

If X is a finite connected CW complex and $H = H_1(X; \mathbb{Z})$, then we can take $G = H/\mathrm{Tor}H$. There is a tower of coverings

$$\hat{X} \to \bar{X} \to X$$

where $\hat{X}$ is the maximal abelian covering of X with group of covering transformations H and $\bar{X}$ the maximal free abelian covering of X with group of covering transformations G.

The **i-th Alexander module of** X is the $\mathbb{Z}[G]$ module

$$A_i(X) = H_i(\bar{X}; \mathbb{Z})$$

Since $\mathbb{Z}[G]$ is Noetherian, the module $A_i(X)$ is finitely generated over $\mathbb{Z}[G]$ for all $i \geq 0$.

In general, for a finitely generated R-module M over a commutative ring with unit R, we can choose a presentation of M, i.e, an exact sequence

$$R^m \to R^n \to M \to 0$$

where $n \in \mathbb{N}$, and m may be infinite. A **presentation matrix** of M is the $m \times n$ matrix of the homomorphism $R^m \to R^n$ with respect to the standard bases in R^m and R^n. For $k \geq 0$, the **k-th elementary ideal of** M is the ideal

$$E_k(M) = E_k(A) \subset R$$

generated by the $n-k$ minors of A (i.e, the $(n-k) \times (n-k)$ sub-determinants of A). If $n - k \leq 0$, then we set $E_k(M) = R$ and if $n - k > m$, then $E_k(M) = 0$. We have that

$$E_0(M) \subset E_1(M) \subset E_2(M) \subset \cdots$$

and the ideals $E_k(M)$ do not depend on the choice of A [76, Lemma 4.4].

Now, given a subset E of a unique factorization domain R, its greatest common divisor $\gcd(E) \in R$ is the generator of the smallest principal ideal containing E. It is well defined up to multiplication by invertible elements of R. When M is a finitely generated R-module we set

$$\triangle_k(M) = \gcd(E_k(M)) \in R$$

We have the inclusions for all $k = 0, 1, 2, \cdots$

$$\triangle_{k+1}(M) \mid \triangle_k(M)$$

and $\triangle_0(M)$ is denoted the **order of** M**, ord**M**.**

If we set $\pi = \pi_1(X)$ and $\bar{\pi} = \pi_1(\bar{X})$, the $\mathbb{Z}[G]$ module $A_1(X) = H_1(\bar{X}) = \bar{\pi}/[\bar{\pi}, \bar{\pi}]$ depends only on π and the order

$$\mathrm{ord}A_1(X) = \triangle_0(H_1(\bar{X})) \in \mathbb{Z}[G]/\pm G$$

is called the **Alexander polynomial of** $\pi = \pi_1(X)$**,** which we denote as $\triangle_\pi$ or $\triangle_X$[76, Definition 11.7].

There is a close relation between the Alexander polynomial of a space and the local coefficient systems $H^\bullet(X; \mathbb{C}^\mu)$ determined by (twisted) complex line bundles. To state this somewhat in more generality than what we will need, suppose that (X, x) is a path-connected, pointed CW complex with finitely many 1-cells (this is equivalent to $\pi_1(X,x)$ being a finitely generated group). Then the parameter space for rank N locally constant sheaves on X can be identified with the $GL_N(\mathbb{C})$ representation variety of $\pi = \pi_1(X)$, which can be given a natural structure of an affine variety. The different twisted cohomology groups on X are encoded by filtrations of these varieties [63, Sections 1 and 4.5].

More precisely, if V is a vector space and $\imath : \mathbb{B} \to GL(V)$ a rational representation of the linear algebraic group $\mathbb{B}$, then for each $i, k \geq 0$, the **embedded jump locus of X with respect to $\imath$** is the set

$$\mathscr{V}^i_k(X,\imath) = \{\rho \in \hom(\pi,\mathbb{B}) \mid \dim H^i(X; V_{\imath\circ\rho}) \geq k\}$$

As long as the r-skeleton of X is finite, the sets $\mathscr{V}^i_k(X,\imath)$ are Zariski closed subsets of $\hom(\pi,\mathbb{B})$ for all $i \leq r$ and $k \geq 1$ [16, Section 1.1]. When $\mathbb{B} = GL_1(\mathbb{C}) = \mathbb{C}^*$ and $\imath = \mathrm{id}_{\mathbb{B}}$, these varieties are typically written $\mathscr{V}^i_k(X)$. In fact, we will care mostly about the case $i = 1$, where the jump loci depend only on $\pi = \pi_1(X)$, so we will just denote this as $\mathscr{V}_k(\pi)$. That is, we will take

$$\mathscr{V}_k(\pi) = \{\mu \in \hom(\pi,\mathbb{C}^*) \mid \dim H^1(X; \mathbb{C}^\mu) \geq k\}$$

Notice that $\hat{\pi} = \hom(\pi,\mathbb{C}^*)$ is the group of characters of π. If π_{ab} denotes the abelianization of π, then $\hat{\pi}$ has the structure of an algebraic group with coordinate ring $\mathbb{C}[\pi_{ab}]$, since the closed points in $\mathrm{Spec}(\mathbb{C}[\pi_{ab}])$ correspond to homomorphisms $\pi_{ab} \to \mathbb{C}^*$ [34, Section 2]. If π is given a presentation $\langle F_n \mid \mathcal{R}\rangle$, where F_n denotes the free group in n generators $x_1, \cdots, x_n$ and $\mathcal{R}$ the set of relations, then there is an embedding of $\hat{\pi}$ in $\hat{F_n}$. The latter can be identified with the affine torus $(\mathbb{C}^*)^n$, by assigning to $\mu \in \hat{F_n}$ the point $(\mu(x_1), \cdots, \mu(x_n))$. The image of $\hat{\pi}$ in $(\mathbb{C}^*)^r$ is the zero set of the subset of $\mathbb{C}[F_n^{ab}]$ defined by

$$\{\mathrm{ab}(R) - 1 \mid R \in \mathcal{R}\} \subset \mathbb{C}[F_n^{ab}]$$

When $\alpha : \pi' \to \pi$ is a homomorphism between two finitely presented groups and $\hat{\alpha} : \hat{\pi} \to \hat{\pi}'$ is defined by composition, then there is an induced map

$$\hat{\alpha}^* : \mathbb{C}[\pi'_{ab}] \to \mathbb{C}[\pi_{ab}]$$

obtained by extending α_{ab} linearly. If we present π as a sequence of homomorphisms

$$F_s \to^{\psi} F_n \to^{q} \pi$$

where q us onto and the normalization of the image of ψ is the kernel of q, then the **Alexander matrix** of π is the $n \times s$ matrix of partials

$$M(F_n,\mathcal{R}) = [(\hat{q})^* D_i(R_j)]$$

where $D_i : F_n \to \mathbb{Z}[F_n^{ab}]$ is the Fox derivative given by

$$D_i(x_j) = \delta_{ij}$$
$$D_i(fg) = D_i(f) + \mathrm{ab}(f) D_i(g)$$

For any $\mu \in \hat{\pi}$, $M(F_n,\mathcal{R})(\mu)$ is the $n \times s$ complex matrix given by evaluation on μ so we can define the **Alexander d-set** as

$$\mathcal{A}_d(\pi) = \{\mu \in \hat{\pi} \mid \mathrm{rank} M(F_r,\mathcal{R})(\mu) < n - d\}$$

These are subvarieties of $\hat{\pi}$ defined by the ideals of $(n-d)\times(n-d)$ minors of $M(F_n,\mathcal{R})$. This gives the **Alexander stratification,** i.e, the descending series

$$\hat{\pi} \supset \mathcal{A}_1(\pi) \supset \cdots \supset \mathcal{A}_n(\pi)$$

As the notation suggest, the $\mathcal{A}_d(\pi)$ are independent of the group presentation. On the other hand, we have the **jump loci stratification**

$$\hat{\pi} \supset \mathscr{V}_1(\pi) \supset \cdots \supset \mathscr{V}_n(\pi)$$

In fact, [34, Corollary 2.4.3, Lemma 2.2.3] shows that

$$\mathcal{A}_d(\pi) = \mathscr{V}_d(\pi) \quad (d > 1), \qquad \mathcal{A}_1(\pi) \cup \{\mathbf{1}\} = \mathscr{V}_1(d)$$

where $\mathbf{1}$ denotes the trivial representation. This theorem has the following consequences ([34, Lemma 2.5.4], [58, Section 3]).

Corollary 65. ***(Vanishing of the twisted cohomology in terms of the Alexander polynomial)*** *Suppose that $\mu \in \hat{\pi}$, where $\pi = \pi_1(X)$. Then*

$$H^1(X; \mathbb{C}^\mu) = 0 \iff \triangle_X(\mu) \neq 0$$

By the latter condition we are using the interpretation of the Alexander polynomial in terms of the order of the ideal given before, so that $\triangle_X(\mu) \neq 0$ means there is an $(n-1) \times (n-1)$ minor of a presentation matrix $M(F_n, \mathcal{R})$ for π so that μ does not evaluate to zero on this minor.

Now we specialize to the case of knots K inside integer homology spheres Y [60, Section 13.2]. Let μ_K be a meridian of K. Let $\mathcal{R}_{K,N}$ the set of N -dimensional complex representations of the knot group $G_K = \pi_1(Y \backslash K)$:

$$\mathcal{R}_{K,N} = \hom(G_K, GL_N(\mathbb{C})) = \{\rho : G_K \to GL_N(\mathbb{C}) \mid \rho \text{ is a homomorphism}\}$$

If $G_K =< x_1, \cdots, x_n \mid r_1 = \cdots = r_{n-1} = 1 >$ is a presentation for the knot group we can identify $\mathcal{R}_{K,N}$ with

$$\mathcal{R}_{K,N} = \{(X_1, \cdots, X_n) \in GL_N(\mathbb{C})^n \mid r_1(X_1, \cdots, X_n) = \cdots = r_{n-1}(X_1, \cdots, X_n) = I\}$$

If $R_{K,N}$ is the coordinate ring of $\mathcal{R}_{K,N}$ and $R_{K,N}^{GL_N(\mathbb{C})}$ the ring invariant under the conjugate action. The **character variety**

$$\mathcal{X}_{K,N} = \mathcal{R}_{K,N} // GL_N(\mathbb{C})$$

has coordinate ring $R_{K,N}^{GL_N(\mathbb{C})}$. In the case that $N = 1$, then the correspondence $\rho \to \rho(\mu_K)$ gives an isomorphism [60, Theorem 13.1]

$$\mathcal{X}_{K,1} \simeq \mathbb{C}^*$$

The **d-th Alexander set in** $\mathcal{X}_{K,1}$ is defined as

$$\mathcal{A}_K(d) = \{\rho \in \mathcal{X}_{K,1} \mid f(\rho(\mu_K)) = 0 \text{ for any } f \in E_{d-1}(H_1(\bar{X}))\}$$

This gives a descending series

$$\mathcal{X}_{K,1} \supset \mathcal{A}_K(1) \supset \cdots \supset \mathcal{A}_K(d) \supset \cdots$$

For $\rho \in \mathcal{X}_{K,1}$, define the G_K module $\mathbb{C}(\rho)$ by the additive group $\mathbb{C}$ equipped with the G_K action given by $g \cdot z = \rho(g)z$ for $g \in G_K, z \in \mathbb{C}$. The **d-th cohomology jumping set in** $\mathcal{X}_{K,1}$ is

$$\mathcal{C}_K(d) = \{\rho \in \mathcal{X}_{K,1} \mid \dim_{\mathbb{C}} H^1(G_K; \mathbb{C}(\rho)) \geq d\}$$

We also have the descending series

$$\mathcal{X}_{K,1} \supset \mathcal{C}_K(1) \supset \cdots \supset \mathcal{C}_K(d) \supset \cdots$$

and in fact [60, Theorem 13.2]

$$\mathcal{A}_K(d) = \mathcal{C}_K(d) \quad (d > 1), \qquad \mathcal{A}_K(1) \cup \{\mathbf{1}\} = \mathcal{C}_K(1)$$

where $\mathbf{1}$ stands for the trivial representation of G_K. Moreover, if we set

$$A_K(\rho) = A_K \otimes_{\mathbb{Z}[G_K^{ab}]} \mathbb{C}(\rho)$$

then

$$\dim H^1(G_K, \mathbb{C}(\rho)) = \begin{cases} \dim A_K(\rho) - 1 & \rho \neq \mathbf{1} \\ 1 & \rho = \mathbf{1} \end{cases}$$

and this can be used to show that [60, Corollary 13.4]:

Corollary 66. *Suppose that $\rho \neq \mathbf{1}$. Then*

$$\triangle_K(\rho(\mu_K)) \neq 0 \iff H^1(G_K, \mathbb{C}(\rho)) = 0$$

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