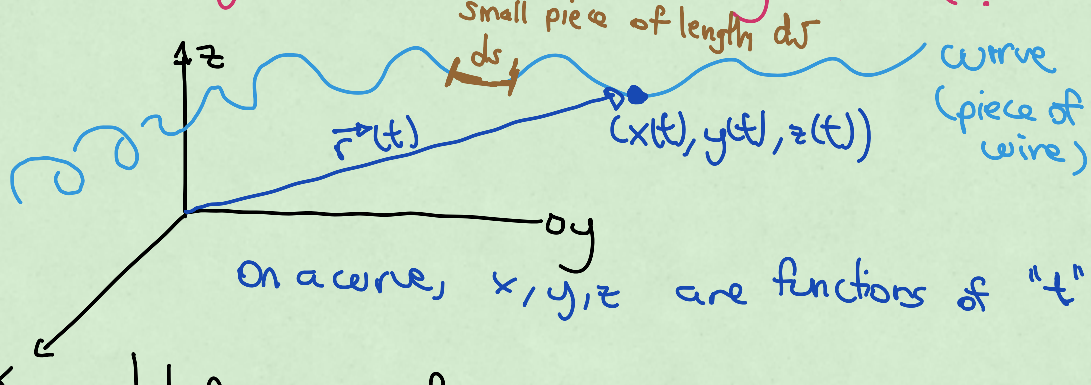
Lecture 22 (16.1 - 16.2)

Integrals along curves (Line integrals)

How to integrate a function along a curve?



On a curve, x, y, z are functions of "t"

what is total mass if you know the density function of the wire?

units $\frac{\text{mass}}{\text{length}}$ ← f = mass density of the wire = mass per unit length

piece of wire size ds, mass of this piece is = density · length $= f\,ds$

$$\text{Total mass} = \int f\,ds = \int_{t_i}^{t_f} f \boxed{\frac{ds}{dt}}\,dt$$

$\frac{ds}{dt}$ = rate of change of $\frac{\text{distance}}{\text{time}}$ = speed of curve

$$= \int_{t_i}^{t_f} f \underbrace{|\vec{v}|}_{\text{speed}}\,dt$$

$$\text{speed} = |\vec{v}| = \left|\frac{d\vec{r}}{dt}\right|$$

Key Formula #1

If you want to integrate a function f along a curve

① Find equation/parametrization of the curve

$$\vec{r}(t) = (x(t), y(t), z(t))$$

② Find velocity $\vec{v} = \frac{d\vec{r}}{dt}$

③ Find speed $|\vec{v}|$.

④ Do the integral

$$\boxed{\int_{t_i}^{t_f} f\, |\vec{v}|\, dt}$$

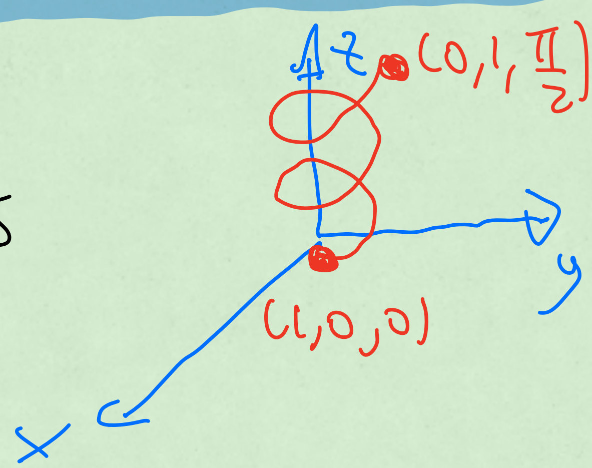
t_i = initial time , t_f = final time.

Remark! that integral is sometimes written as $\int_C f\, ds$

C stands for curve.

Example:

Find $\int_C f\, ds$



$f = xy + 1$,

C is the part of the helix

$\vec{r}(t) = (\cos t, \sin t, t)$

from the point $(1, 0, 0)$

to the point $(0, 1, \pi/2)$

steps

#1 $\vec{r}(t) = (\underbrace{\cos t}_{x(t)}, \underbrace{\sin t}_{y(t)}, \underbrace{t}_{z(t)})$

#2 $\vec{v}(t) = \frac{d\vec{r}}{dt} = (-\sin t, \cos t, 1)$

#3 $|\vec{v}| = \sqrt{(-\sin t)^2 + \cos^2 t + 1^2} = \sqrt{2}$

#4

$$\int f \, |\vec{v}| \, dt$$

$$= \int (xy + 1) \sqrt{2} \, dt$$

$$= \int_{t=0 \text{ since } \vec{r}(0)=(1,0,0)}^{t=\pi/2 \text{ since } \vec{r}(\pi/2)=(0,1,\pi/2)} (\cos t \sin t + 1)\sqrt{2}\, dt$$

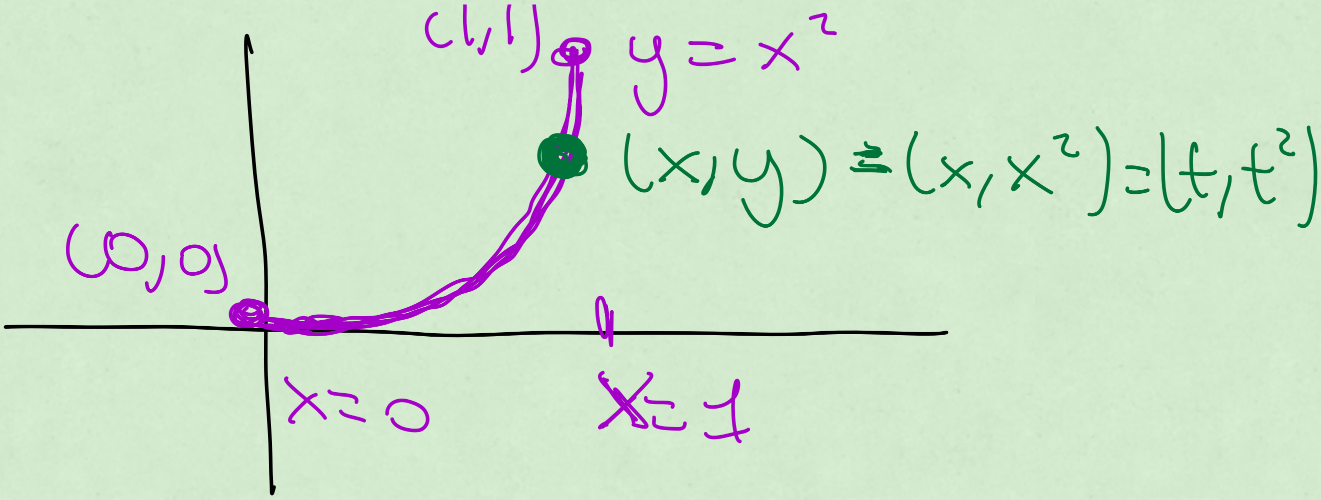
$$= \int_0^{\pi/2} (1+\cos t \sin t)\sqrt{2}\, dt$$

$$= \frac{\sqrt{2}}{2}(\pi+1)$$

Example 2

Find $\int_C f\,ds$

$f = x$, C which is the part of the parabola $y = x^2$ between $x=0$ and $x=1$



#1 find $\vec{r}(t)$

$$\vec{r}(t) = (t, t^2)$$

#2 $\vec{v}(t) = (1, 2t)$

#3 $|\vec{v}| = \sqrt{1+4t^2}$

#4 $$\int f \, |\vec{v}| \, dt$$

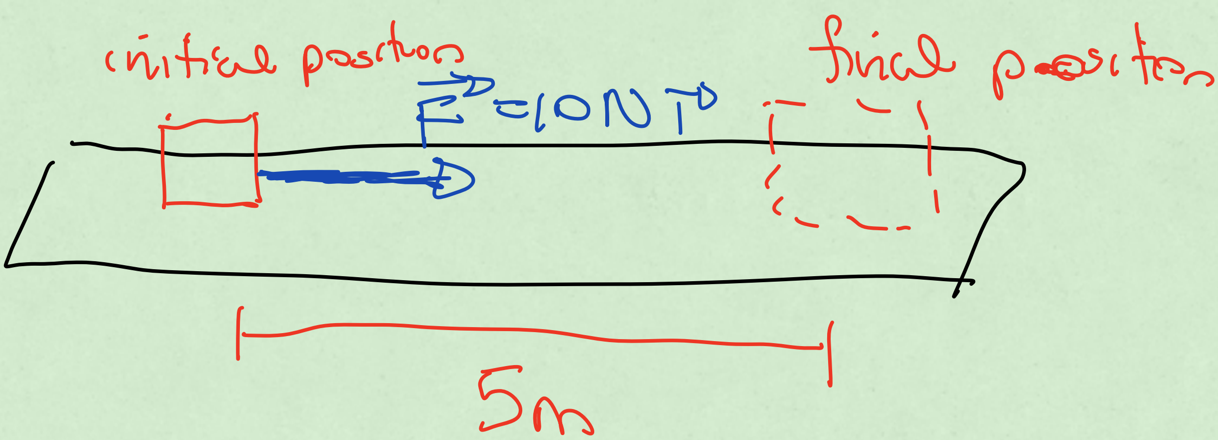
$$= \int x \sqrt{1+4t^2} \, dt$$

$$= \int_{t=0}^{t=1} t\sqrt{1+4t^2}\,dt$$

$$= \frac{1}{12}\left(5\sqrt{5}-1\right)$$

The end of 16.1

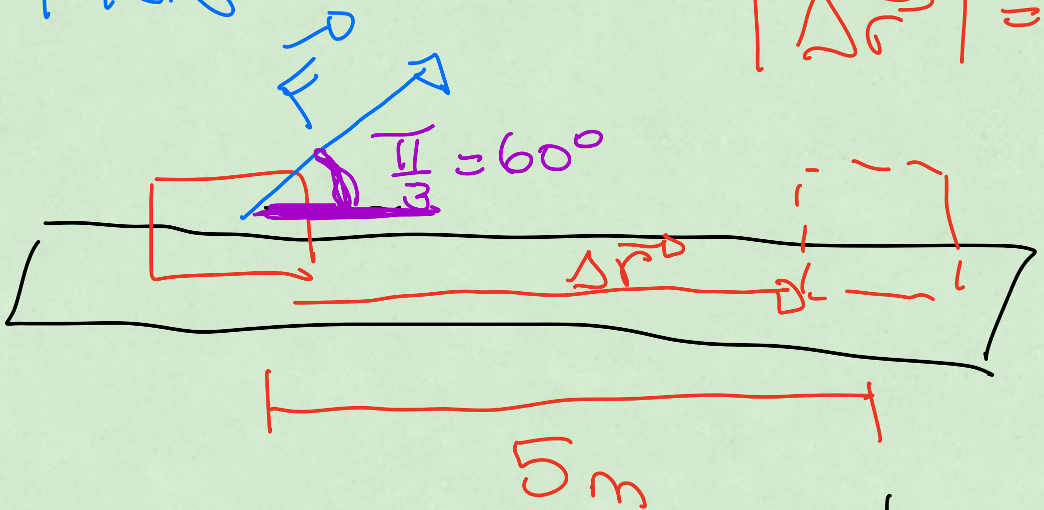
Section 16.2



work $W = 10N \cdot 5m = 50J$

$|\vec{F}| = 10N$

$|\Delta\vec{r}| = 5m$



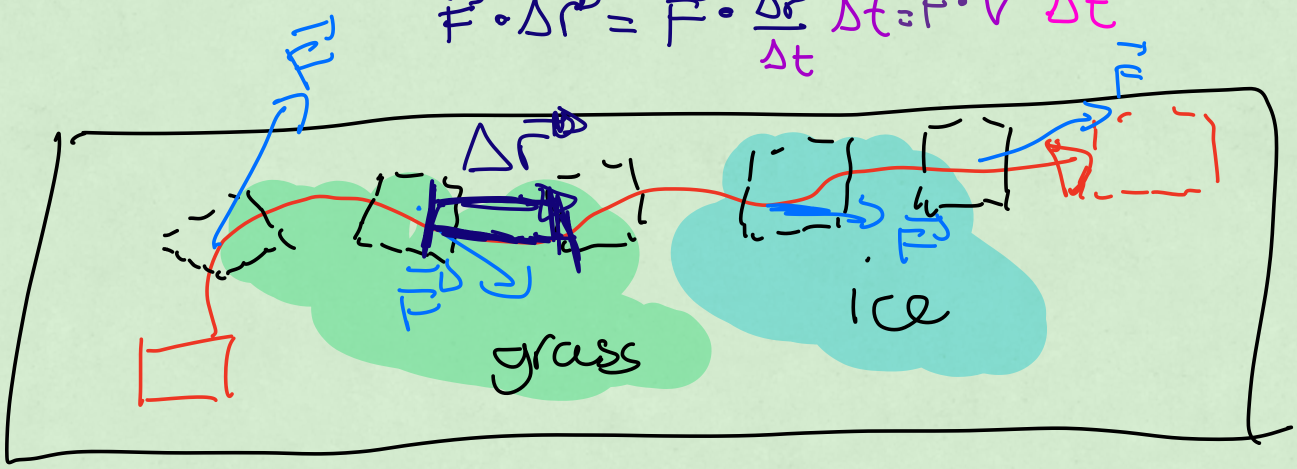
$$\text{Work} = 10N \cdot 5m \cdot \overbrace{\cos(60°)}^{\frac{1}{2}} = 25J$$

$$= |\vec{F}|\,|\Delta\vec{r}| \cos\theta$$

$$\text{work} = \vec{F} \cdot \Delta\vec{r}$$ (dot product)

work for this small piece

$$\vec{F} \cdot \Delta\vec{r} = \vec{F} \cdot \frac{\Delta\vec{r}}{\Delta t} \Delta t = \vec{F} \cdot \vec{v}\, \Delta t$$



total work $= \boxed{\int \vec{F} \cdot \vec{v}\, dt}$

key Formula #2

To find the integral of a vector $\vec{F}$ along a curve C:

#1 parametrize curve

$$\vec{r}(t) = (x(t), y(t), z(t))$$

#2 find velocity

$$\vec{v} = \frac{d\vec{r}}{dt}$$

#3 (different) do the integral

$$\int_{t_i}^{t_f} \vec{F} \cdot \vec{v}\, dt$$

Remark: this integral is called work or circulation, and it is written as

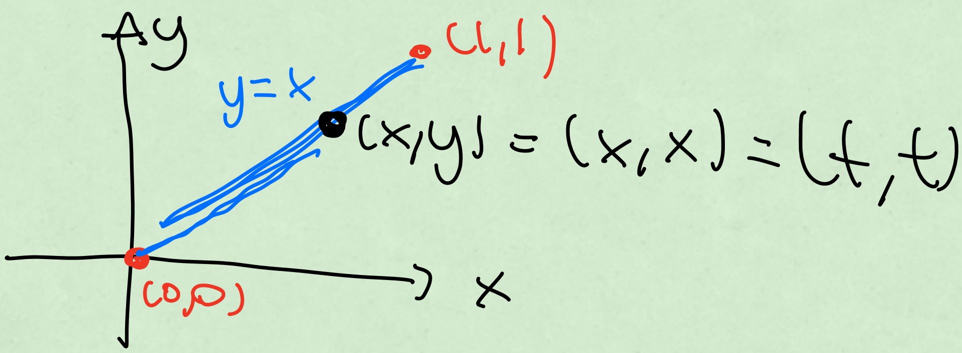
$$\int_C \vec{F} \cdot d\vec{r}$$

Example 1

Find the work done by

$$\vec{F} = xy\,\vec{i} + 3\,\vec{j} = (xy, 3)$$

along the curve $y = x$ from $(0,0)$ to $(1,1)$



#1 $\vec{r}(t) = (t, t)$

#2 $\vec{v}(t) = (1, 1)$

#3 $\int \vec{F} \cdot \vec{v}\, dt$

$$= \int (xy, 3) \cdot (1,1)\, dt$$

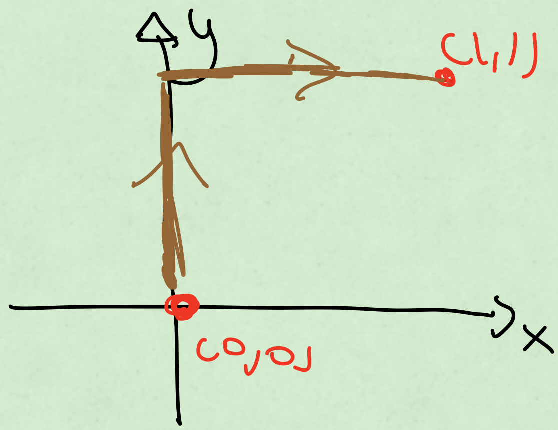
$$= \int (xy + 3)\, dt$$

$$= \int_{t=0}^{t=1} (t^2 + 3)\, dt = \boxed{\frac{10}{3}}$$

example 2

$\vec{F} = (xy, 3)$

find work along this new path



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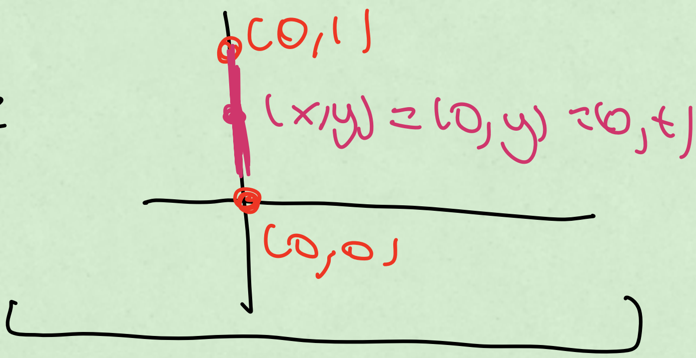

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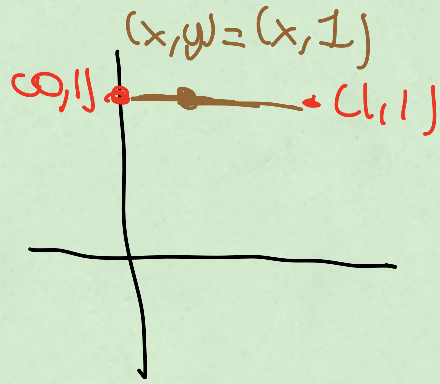


$\vec{r}(t) = (0, t)$

$\vec{v}(t) = (0, 1)$

$\int \vec{F} \cdot \vec{v}\, dt$

$= \int (xy, 3) \cdot (0, 1)\, dt$

$= \int_{t=0}^{t=1} 3\, dt$

$= 3$

$\vec{r}(t) = (t, 1)$

$\vec{v}(t) = (1, 0)$

$\int \vec{F} \cdot \vec{v}\, dt$

$= \int (xy, 3) \cdot (1, 0)\, dt$

$= \int xy\, dt$

$= \int_{t=0}^{t=1} t\, dt = \frac{1}{2}$

total work $= 3 + 1/2 = 5/2$

which is not the same as the one from before !!

moral:

work depends on the trajectory not just the endpoints.

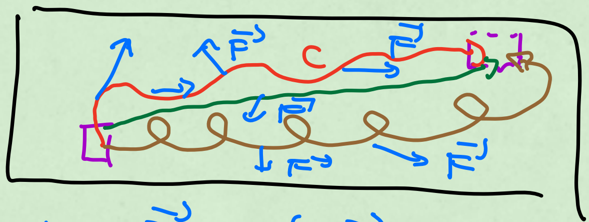
Thursday:

conservative forces

$\vec{F}$: work only depends on endpoints not trajectory.

Lecture 23 (16.2-16.3)

Conservative vector fields and potential functions



work done by $\vec{F} = \int_C \vec{F} \cdot d\vec{r} = \int \vec{F} \cdot \vec{v}\, dt$

$\vec{F}$ is conservative if the work done by $\vec{F}$ depends only on the endpoints and not the curve (trajectory) which connects two points.

(1) How can you verify whether $\vec{F}$ is conservative or not?

$\vec{F}$ will be conservative if $\nabla \times \vec{F}$ (called the curl of $\vec{F}$) vanishes.

what is the curl of $\vec{F}$? (black box)

$$\nabla \times \vec{F} = \text{curl of } \vec{F} = \text{a new vector}$$

$$\vec{F} = F_1 \vec{i} + F_2 \vec{j} + F_3 \vec{k} = (F_1, F_2, F_3)$$

$$\nabla \times \vec{F} = \det \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{pmatrix}$$

Example

$$\vec{F} = (xy^2z,\ x^2yz,\ z^2)$$

$$\nabla \times \vec{F} = \det \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy^2z & x^2yz & z^2 \end{pmatrix}$$

$$= \vec{i}\left(\underbrace{\frac{\partial}{\partial y} z^2}_{0} - \underbrace{\frac{\partial}{\partial z} x^2yz}_{x^2y}\right) - \vec{j}\left(\underbrace{\frac{\partial}{\partial x} z^2}_{0} - \underbrace{\frac{\partial}{\partial z} xy^2z}_{xy^2}\right)$$

$$+ \vec{k}\left(\underbrace{\frac{\partial}{\partial x} x^2yz}_{2xyz} - \underbrace{\frac{\partial}{\partial y} xy^2z}_{2xyz}\right)$$

$$= -x^2y\,\vec{i} + xy^2\,\vec{j} + 0\,\vec{k}$$

$$\nabla \times \vec{F} = (-x^2y,\ xy^2,\ 0)$$

since not every entry of $\nabla \times \vec{F}$ is zero, $\vec{F}$ is not conservative.

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example: $\vec{F} = (-y, x, 0)$

$$\nabla \times \vec{F} = \det \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & 0 \end{pmatrix}$$

$$= \vec{i}\left(\frac{\partial}{\partial y}0 - \frac{\partial}{\partial z}x\right) - \vec{j}\left(\frac{\partial}{\partial x}0 + \frac{\partial}{\partial z}y\right) + \vec{k}\left(\frac{\partial}{\partial x}x + \frac{\partial}{\partial y}y\right)$$

$$= 0\vec{i} + 0\vec{j} + 2\vec{k}$$

$$= (0, 0, 2)$$

$$\vec{F} = (x, y, 0)$$

$$\nabla \times \vec{F} = \det \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & 0 \end{pmatrix}$$

$$= \vec{i}\left(\frac{\partial}{\partial y}0 - \frac{\partial}{\partial z}y\right) - \vec{j}\left(\frac{\partial}{\partial x}0 - \frac{\partial}{\partial z}x\right) + \vec{k}\left(\frac{\partial}{\partial x}y - \frac{\partial}{\partial y}x\right)$$

$$= 0\vec{i} + 0\vec{j} + 0\vec{k}$$

$$= (0, 0, 0)$$

$\vec{F}$ is conservative.

$$\nabla \times (-\vec{F}) = -\nabla \times \vec{F} = (0, 0, 0)$$

$$\nabla \times (2\vec{F}) = 2\nabla \times \vec{F} = (0, 0, 0)$$

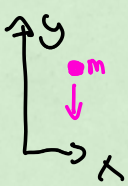
Step 2!

If $\vec{F}$ is conservative, then one can find a potential function for $\vec{F}$.

This is a function f which satisfies

$$\boxed{\nabla f = \vec{F}}$$

example



$\vec{F} = -mg\vec{j} = (0, -mg, 0)$, m, g constants

$f = -mgy$, $\nabla f = (0, -mg, 0)$

How to find the potential function f of $\vec{F} = (F_1, F_2, F_3)$

You must solve

$$\begin{cases} \dfrac{\partial f}{\partial x} = F_1 \\ \dfrac{\partial f}{\partial y} = F_2 \\ \dfrac{\partial f}{\partial z} = F_3 \end{cases}$$

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example: Find a potential function for

$$\vec{F} = (\overbrace{2x\ln y}^{F_1}, \overbrace{\frac{x^2}{y} + z^2}^{F_2}, \overbrace{2yz}^{F_3})$$

(Remark: check that $\nabla \times \vec{F} = (0,0,0)$ so that $\vec{F}$ is conservative)

Let's find a potential function f:

Must solve the system

$$\begin{cases} (1) & \dfrac{\partial f}{\partial x} = 2x\ln y \\ (2) & \dfrac{\partial f}{\partial y} = \dfrac{x^2}{y} + z^2 \\ (3) & \dfrac{\partial f}{\partial z} = 2yz \end{cases}$$

analogy

$\dfrac{df}{dx} = x^2 + 1$

$f = \dfrac{x^3}{3} + x + C$

∂z

"Partial" integral with respect to x:

$$f = \int 2x \ln y \, dx = x^2 \ln y + \underbrace{c(y,z)}_{\text{our "constant" is a function of all the variables we didn't use for the integration}}$$

new "f"

$$\boxed{f = x^2 \ln y + c(y,z)}$$

equation (2)

$$\frac{\partial f}{\partial y} = \frac{x^2}{y} + z^2$$

$$\frac{\partial}{\partial y}\left(x^2 \ln y + c(y,z)\right) = \frac{x^2}{y} + z^2$$

$$\frac{x^2}{y} + \frac{\partial}{\partial y} c = \frac{x^2}{y} + z^2$$

$$\frac{\partial c}{\partial y} = z^2$$

$$\boxed{C(y,z)} = \int z^2\, dy = \boxed{yz^2 + b(z)}$$

f version 2.0

$$\boxed{f = x^2 \ln y + yz^2 + b(z)}$$

equation (3)

$$\frac{\partial f}{\partial z} = 2yz$$

$$\frac{\partial}{\partial z}\left(x^2 \ln y + yz^2 + b(z) \right) = 2yz$$

$$0 + 2yz + \frac{db}{dz} = 2yz$$

$$\frac{db}{dz} = 0$$

actual number

$$b(z) = \int 0\, dz = 0 + \boxed{a}$$

not a function anymore!

$$\boxed{f = x^2 \ln y + yz^2 + a}$$

arbitrary constant

$$\nabla f = \left(2x\ln y,\ \frac{x^2}{y} + z^2,\ 2yz \right)$$

= entries of $\vec{F}$ ☺

so good ✓

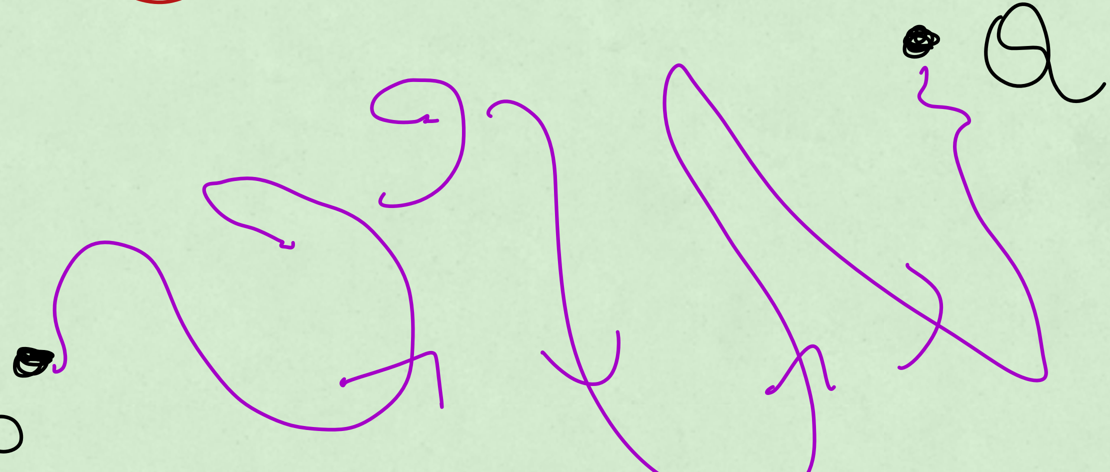
Remark

$$f = \underbrace{-V}$$

V is the notation physicist's use

for potentia

③ Knowing potential f, makes finding the work entirely trivial!



work of $\vec{F}$ along this path (when $\vec{F}$ is conservative)

$$\underbrace{\int_C \vec{F}\cdot d\vec{r}}\ = f(Q) - f(P)$$

work equals difference in potential enery

example:

$$\vec{F} = \left(2x\ln y, \frac{x^2}{y} + z^2, 2yz\right)$$

work done by $\vec{F}$

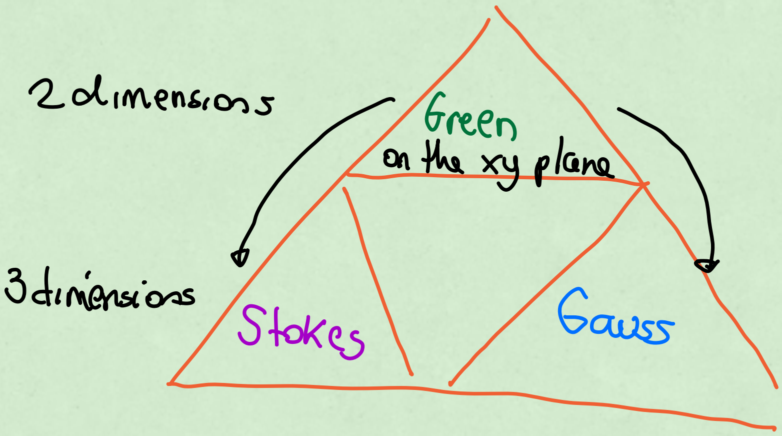
$$f = x^2 \ln y + \cancel{y} z^2 + a$$

$$\text{work} = f(1,1,1) - f(0,1,0)$$
$$= (1\ln 1 + 1 + a) - (0 + 0 + a)$$
$$= 1 + a - a$$
$$= \boxed{1}$$

Lecture 24 (16.4: Green's Theorem)

and a bit from 16.2

Calculus Triforce

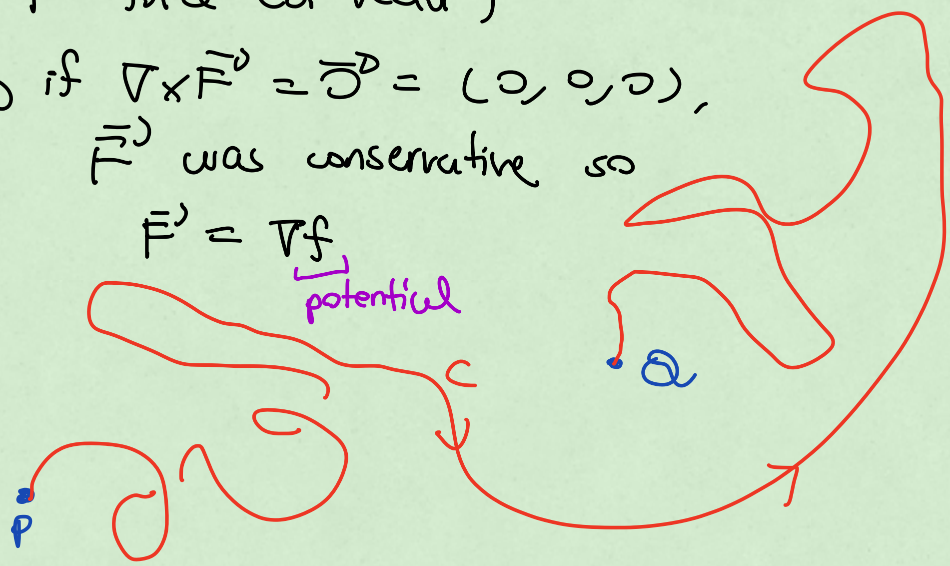


Last time

- $\vec{F}$ force (or vector)
- if $\nabla \times \vec{F} = \vec{0} = (0,0,0)$,

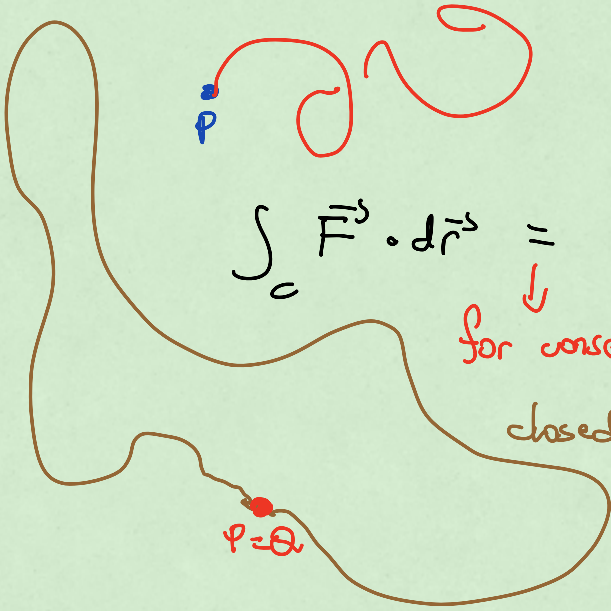
  $\vec{F}$ was conservative so

  $\vec{F} = \underbrace{\nabla f}_{\text{potential}}$



$$\int_C \vec{F} \cdot d\vec{r} = f(Q) - f(P)$$

for conservative forces

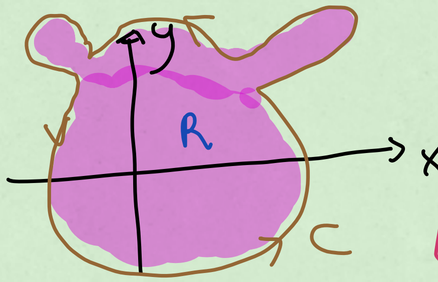


closed loop
(closed path)
(round trip)

work for a conservative force is zero!

Green's Theorem (for regions on the xy plane)

If C is a closed curve on the xy plane and R is the region it encloses then the work done by a force (or vector field $\vec{F}$) along the curve C equals



$$\underbrace{\int_C \vec{F}\cdot d\vec{r}}_{\text{work}} \overset{\text{Green}}{=} \iint_R (\nabla\times\vec{F})\cdot\vec{k}\, dA$$

Remarks:

1. Here $\vec{k} = (0,0,1)$
2. The curve is traveled (oriented) in a counterclockwise way for this theorem to hold
3. How it appears in the book!

if $\vec{F} = M\vec{i} + N\vec{j} = (M, N, 0)$, so

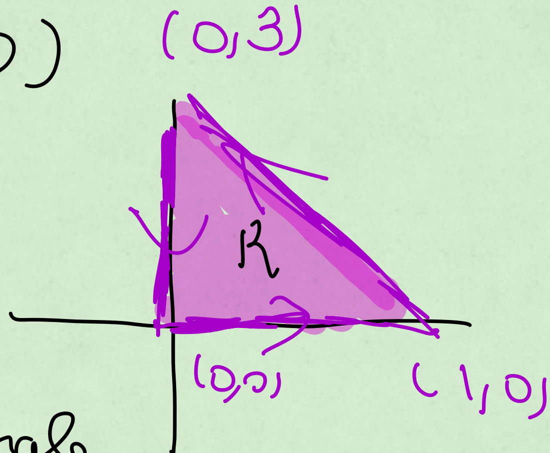
$\nabla\times\vec{F} = \left(0, 0, \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$ and so the theorem can be written as

$$\int_C \vec{F}\cdot d\vec{r} = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$$

how we will use it

Example:

$$\vec{F} = (\underbrace{x^2 + e^y}_{M}, \underbrace{x+y}_{N}, 0)$$



curve
C = sides of the triangle
R = inside of the triangle

$\underbrace{\text{work done by } \vec{F}}$

directly: one integral for each side of the triangle and then add them up.

Green

$$\text{work} = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA$$

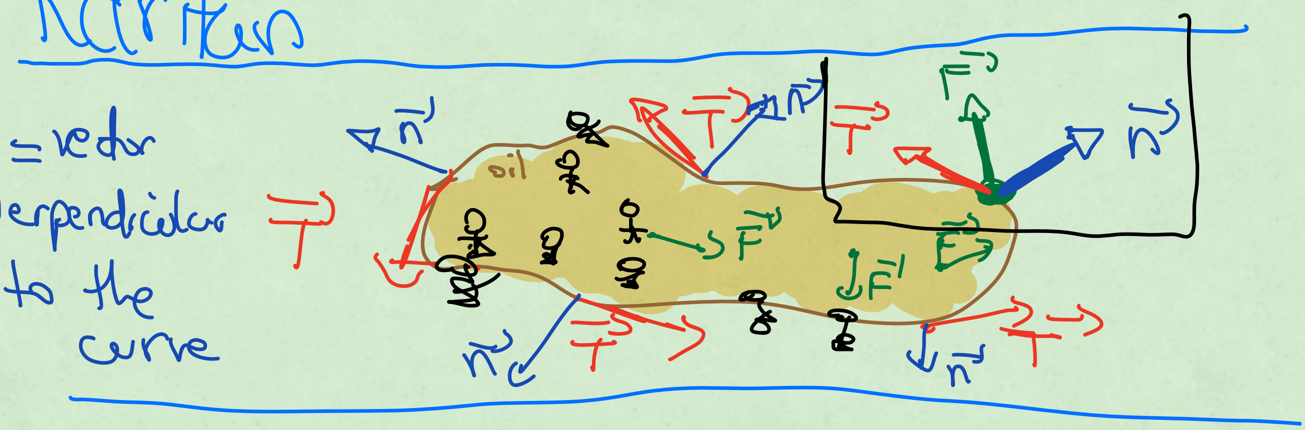
$$= \int_0^1 \int_0^{3-3x} (1 - e^y)\, dy\, dx$$

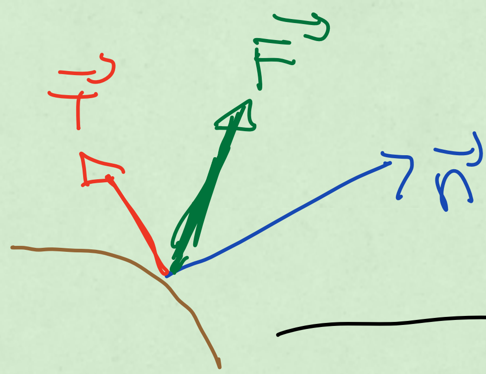
$$= \frac{1}{6}(17 - 2e^3)$$

Flux and second version of Green's Theorem

Raritan

$\vec{n}$ = vector perpendicular to the curve

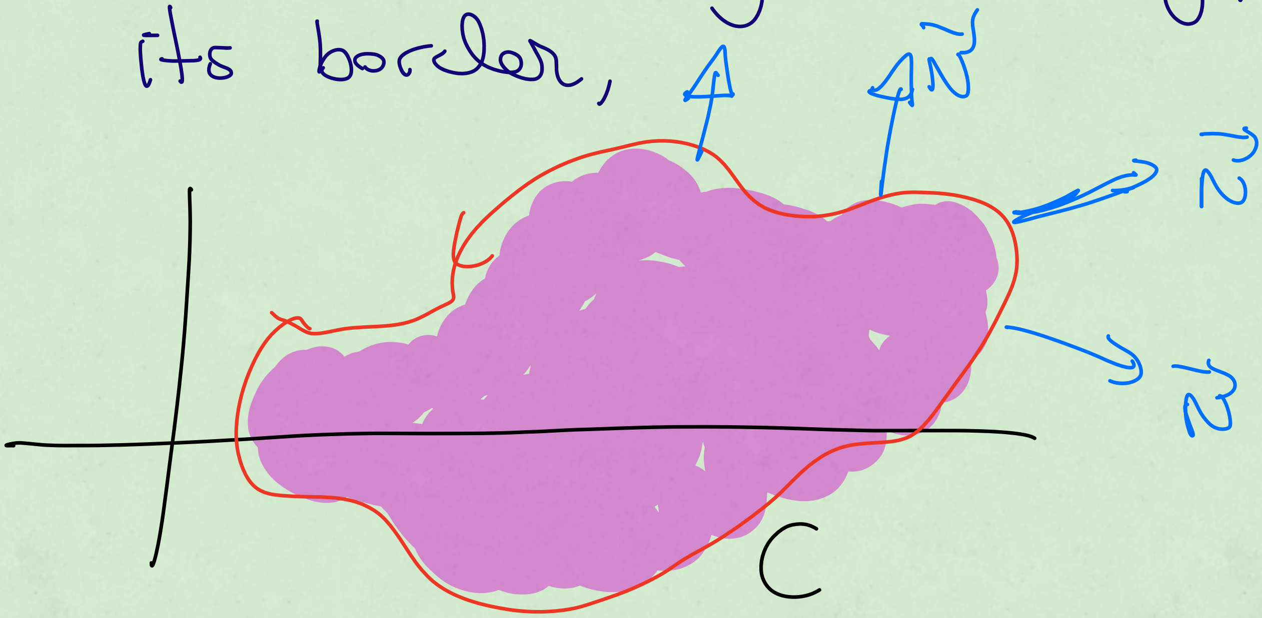




$\text{proj}_{\vec{T}} \vec{F}$ or $\boxed{\text{proj}_{\vec{n}} \vec{F}}$

this is the one that helps you exit the region

Flux: how much "stuff" exits a region on the xy plane through its border,



curve C with equation
$\vec{r}(t) = (x(t), y(t))$

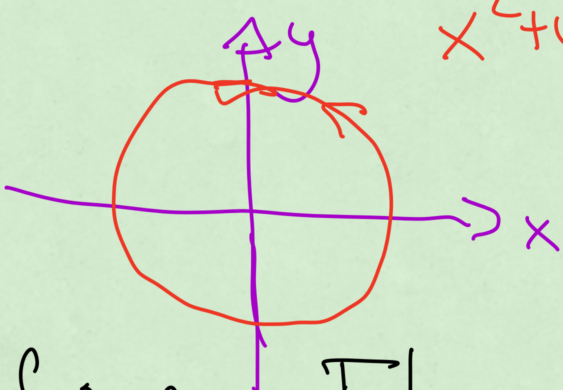
$\vec{N}$ = normal vector to the curve
$$\vec{N} = \left(\frac{dy}{dt}, -\frac{dx}{dt} \right)$$

$$\text{Flux of } \vec{F} = \int_C \vec{F} \cdot \vec{N} \, dt$$

so here we use the dot product

with the normal vector instead of the velocity vector

exemple



$x^2+y^2=4$

$\vec{F}=x\vec{i}+y\vec{j}$

find Flux of $\vec{F}$:

① Find equation (parametrization of the curve)

$$\vec{r}(t)=(r\cos\theta, r\sin\theta)=(2\cos\theta, 2\sin\theta)$$

$$\vec{r}(t)=(\underbrace{2\cos t}_{x(t)}, \underbrace{2\sin t}_{y(t)})$$

$$\vec{N}=\left(\frac{dy}{dt}, -\frac{dx}{dt}\right)=(2\cos t, 2\sin t)$$

Flux

$$= \int \vec{F} \cdot \vec{N} \, dt$$

$$= \int (x, y) \cdot (2\cos t, 2\sin t) \, dt$$

$$= \int 2x \cos t + 2y \sin t \, dt$$

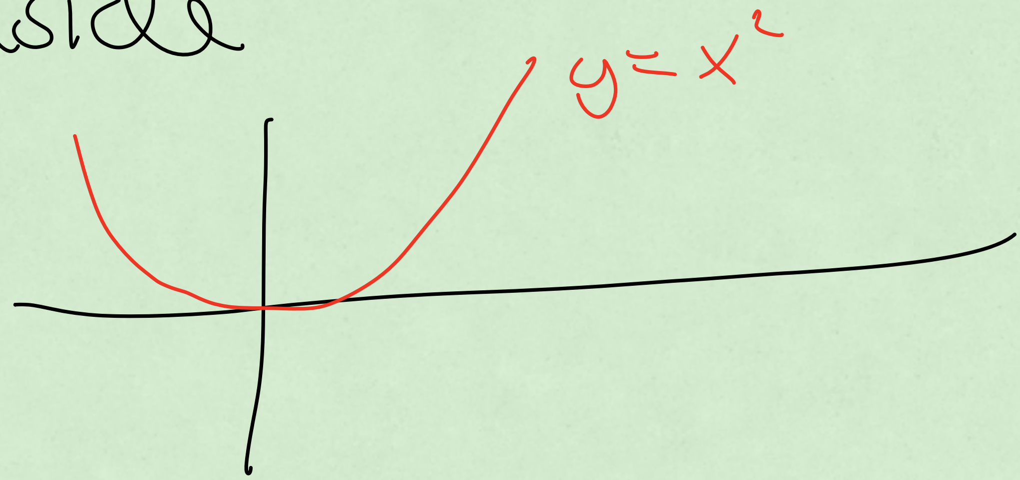
$$= \int_0^{2\pi} 2(2\cos t)\cos t + 2(2\sin t)\sin t \, dt$$

$$= \int_0^{2\pi} 4\cos^2 t + 4\sin^2 t \, dt$$

$$= \int_0^{2\pi} 4 \, dt$$

$$= 8\pi$$

aside



$$\vec{r}(t) = (x, y) = (x, x^2)$$

$$\vec{r}(t) = (\underbrace{t}_{x(t)}, \underbrace{t^2}_{y(t)})$$

$$\vec{N} = \left(\frac{dy}{dt}, -\frac{dx}{dt}\right) = (2t, -1)$$

$$y = x^2$$

$$r\sin\theta = r^2\cos^2\theta$$

$$\boxed{\frac{\sin\theta}{\cos^2\theta} = r}$$

$$\vec{r} = (x, y) = (r\cos\theta, r\sin\theta)$$

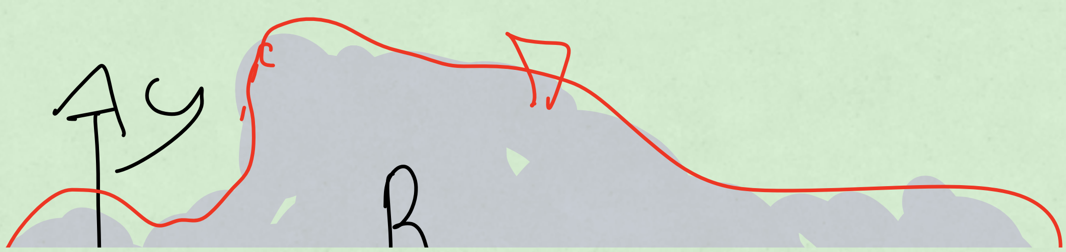
$$\vec{r} = \left(\frac{\sin\theta}{\cos^2\theta}\cos\theta, \frac{\sin\theta}{\cos^2\theta}\sin\theta \right)$$

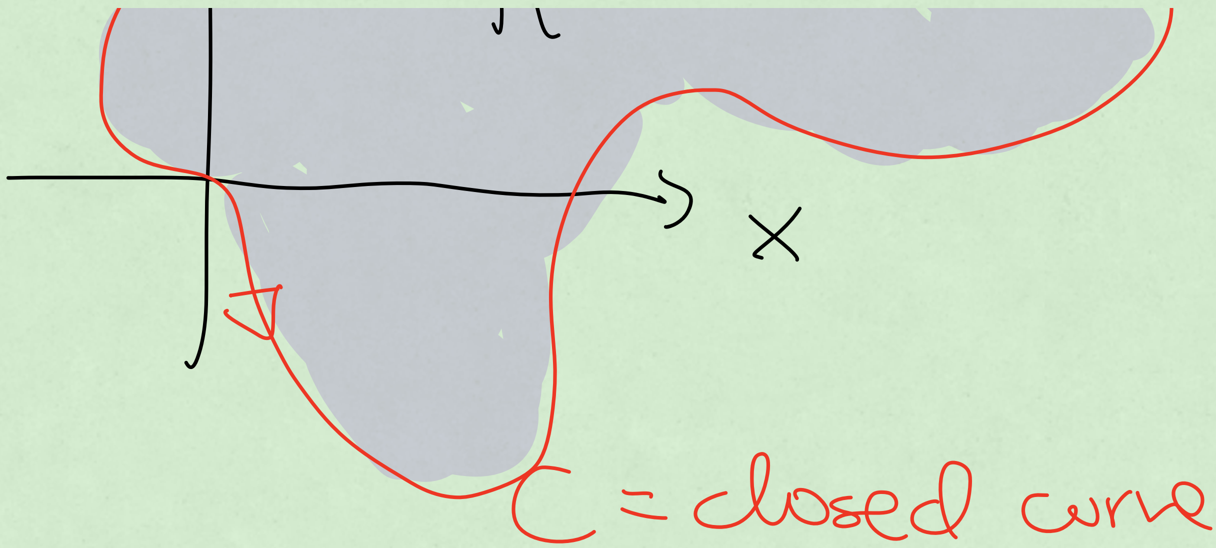
$$\vec{r} = (\tan\theta, \tan^2\theta)$$

$$\underbrace{\vec{r}(t)}_{\text{alternative valid}} = (\tan t, \tan^2 t)$$

Green's Theorem for Flux

(regions on the xy plane)



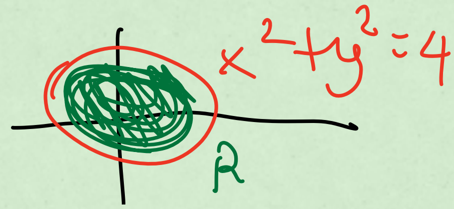


$\vec{F} = (M, N, 0)$ R = enclosed region

$$\boxed{\text{Flux} = \int_C \vec{F}\cdot\vec{N}\,dt \underset{\text{Green}}{=} \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dA}$$

Back to the circle example:

$\vec{F} = (\underbrace{x}_{M}, \underbrace{y}_{N})$



Flux of $\vec{F}$

$$\underset{\text{Green}}{=} \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dA$$

$$= \iint_R (1 + 1)\, dA$$

$$= \int_0^{2\pi}\int_0^2 2r\, dr\, d\theta$$

$$= \int_0^{2\pi} r^2 \Big|_0^2 \, d\theta$$

$$= \int_0^{2\pi} 4\, d\theta$$

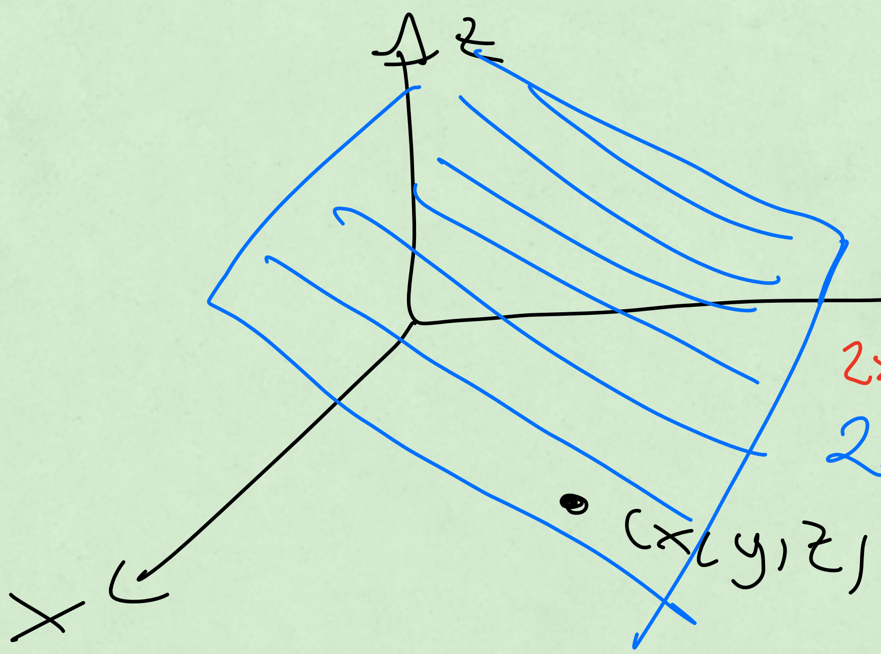
$$= 8\pi$$

16.5 (Preview)

Parametrize SURFACES

$\vec{r}(t)$: curve uses one parameter t

$\vec{r}(u,v)$: surface: uses two parameters



$2x + z - 5 = y$

$2x - y + z = 5$

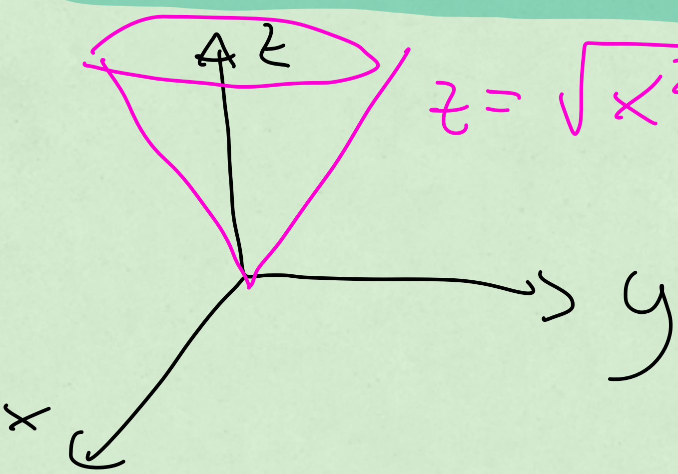
$z = 5 + y - 2x$

$$(x,y,z) = \underbrace{(x, y, 5+y-2x)}_{\vec{r}(x,y)}$$

parametrization plane in terms of x, y

$$(x,y,z) = \underbrace{\left(x, 2x+z-5, z\right)}_{\vec{r}(x,z)}$$

alternative parametrization of the SAME plane



$$z = \sqrt{x^2+y^2} = r$$

$$(x,y,z) = \underbrace{\left(x, y, \sqrt{x^2+y^2}\right)}_{\vec{r}(x,y)}$$

$$(x,y,z) = (r\cos\theta, r\sin\theta, z)$$

$$= (r\cos\theta, r\sin\theta, r)$$

$\vec{r}\,(r, \theta)$

Lecture 25 (16.5, $\underbrace{16.6}_{\text{not on midterm}}$)

Parametrizing surfaces

- you write the position of a point on the surface in terms of two variables (parameters)
- to find a parametrization, you use the equation of the surface to find a relation between x, y, z and so you can write one of them in terms of the others

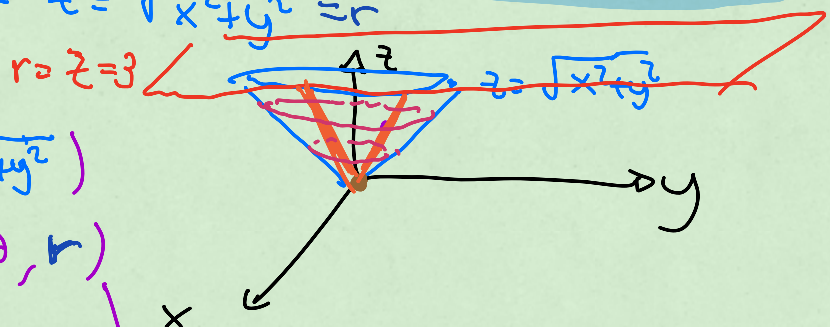
example: cone $z = \sqrt{x^2+y^2} = r$

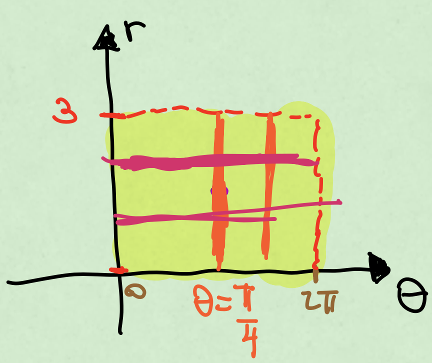
$\vec{r} = (x, y, z)$

$\vec{r}(x,y) = (x, y, \sqrt{x^2+y^2})$

$\vec{r}(r,\theta) = (r\cos\theta, r\sin\theta, r)$

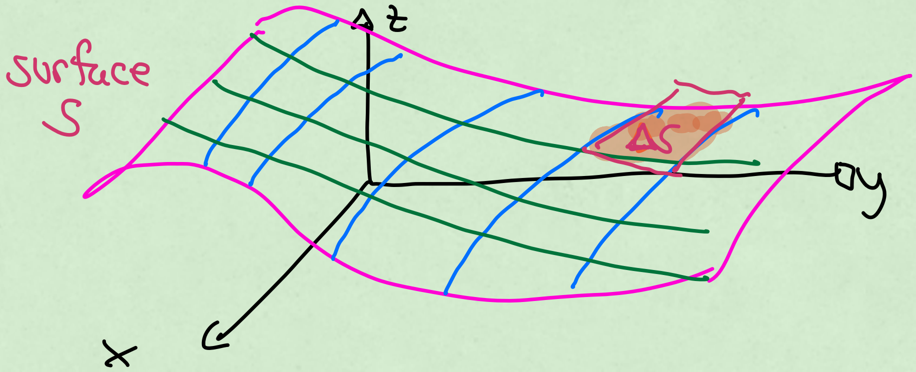
$0 \le \theta \le 2\pi \qquad 0 \le r \le 3$





$\vec{r}(r, \frac{\pi}{4}) = (\frac{r}{\sqrt{2}}, \frac{r}{\sqrt{2}}, r)$

Last formula for the exam [surface Jacobian]



ΔS = tiny area of the surface

$$= \underbrace{\left| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right|}\, dv\, du$$

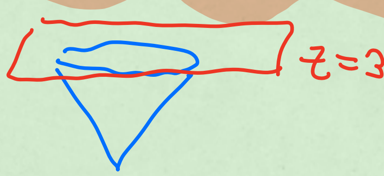
surface Jacobian

$$= \left| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right|$$

Last formula for exam

$$\text{area surface } S = \int_{u \text{ bounds}} \int_{v \text{ bounds}} \left| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right| dv\, du$$

here $\vec{r}(u,v)$ is the parametrization of the surface



$\vec{r}(\theta, r) = (r\cos\theta, r\sin\theta, r)$

$$\frac{\partial \vec{r}}{\partial \theta} = (-r\sin\theta, r\cos\theta, 0)$$

$$\frac{\partial \vec{r}}{\partial r} = (\cos\theta, \sin\theta, 1)$$

$$\underbrace{\frac{\partial \vec{r}}{\partial \theta} \times \frac{\partial \vec{r}}{\partial r}}_{\text{vector perpendicular to the cone}} = \det \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ -r\sin\theta & r\cos\theta & 0 \\ \cos\theta & \sin\theta & 1 \end{pmatrix}$$

$$= (r\cos\theta,\ r\sin\theta,\ -r)$$

"surface" Jacobian $= \left| \frac{\partial \vec{r}}{\partial \theta} \times \frac{\partial \vec{r}}{\partial r} \right|$

$$= \sqrt{r^2\cos^2\theta + r^2\sin^2\theta + r^2}$$

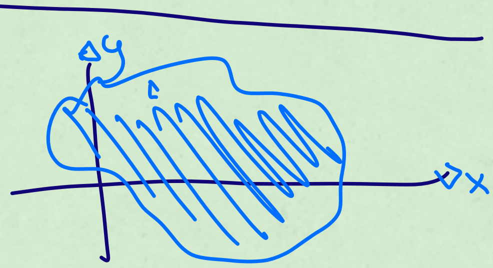
$$= \sqrt{2r^2}$$

$$= \sqrt{2}\, r$$

area cone $= \displaystyle\int_0^{2\pi} \int_0^{3} \sqrt{2}\, r \, dr \, d\theta$

$$= \boxed{9\sqrt{2}\,\pi}$$

$\vec{r}(r,\theta) = (r\cos\theta,\ r\sin\theta,\ 0)$



$\frac{\partial \vec{r}}{\partial \theta} = (-r\sin\theta,\ r\cos\theta,\ 0)$

$\frac{\partial \vec{r}}{\partial r} = (\cos\theta,\ \sin\theta,\ 0)$

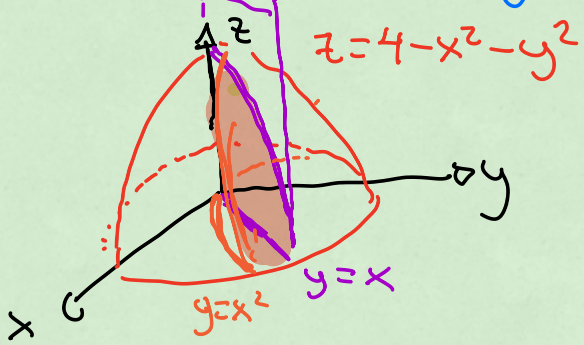
$$\frac{\partial \vec{r}}{\partial \theta} \times \frac{\partial \vec{r}}{\partial \theta} = (0,\ 0,\ -r)$$

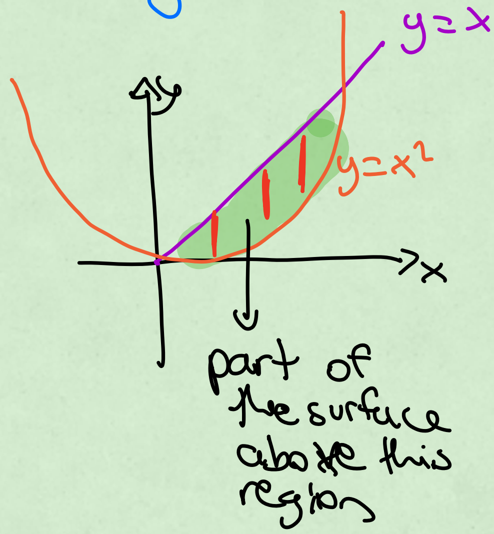
$$\left| \frac{\partial \vec{r}}{\partial \theta} \times \frac{\partial \vec{r}}{\partial \theta} \right| = \sqrt{0^2 + 0^2 + r^2} = r$$

Find the area of the piece of the surface

$$\boxed{z = 4 - x^2 - y^2}$$

- above the xy plane
- between the curves $y = x$ and $y = x^2$





$$\vec{r}(x,y) = (x, y, 4 - x^2 - y^2)$$

$$\frac{\partial \vec{r}}{\partial x} = (1, 0, -2x)$$

$$\frac{\partial \vec{r}}{\partial y} = (0, 1, -2y)$$

$$\frac{\partial \vec{r}}{\partial x} \times \frac{\partial \vec{r}}{\partial y} = \det\begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -2x \\ 0 & 1 & -2y \end{pmatrix} = (2x, 2y, 1)$$

$$\left| \frac{\partial \vec{r}}{\partial x} \times \frac{\partial \vec{r}}{\partial y} \right| = \sqrt{1 + 4x^2 + 4y^2}$$

$$\text{area} = \int_0^1 \int_{x^2}^{x} \sqrt{1 + 4x^2 + 4y^2}\, dy\, dx$$

Steps

① Find parametrization $\vec{r}(u,v)$ of surface

② Find $\frac{\partial \vec{r}}{\partial u}$, $\frac{\partial \vec{r}}{\partial v}$

③ Find $\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}$

④ Find $\left| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right|$

⑤ $$\text{area} = \int\limits_{\substack{u \\ \text{bounds}}} \int\limits_{\substack{v \\ \text{bounds}}} \left| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right| dv\,du$$

Section 16.6 (for final exam)

want to integrate a function f over the surface.

rewrite function in terms of u, v

$$\underbrace{\iint f\,dS}_{\text{integral of } f \text{ over}} = \int\limits_{\substack{\text{bounds} \\ u}} \int\limits_{\substack{\text{bounds} \\ v}} f \underbrace{\left| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right| dv\,du}_{\text{same as}}$$

surface

before

example

$f = \frac{\text{mass}}{\text{unit area}} \rightarrow \iint f\,dS$ = total mass of the surface

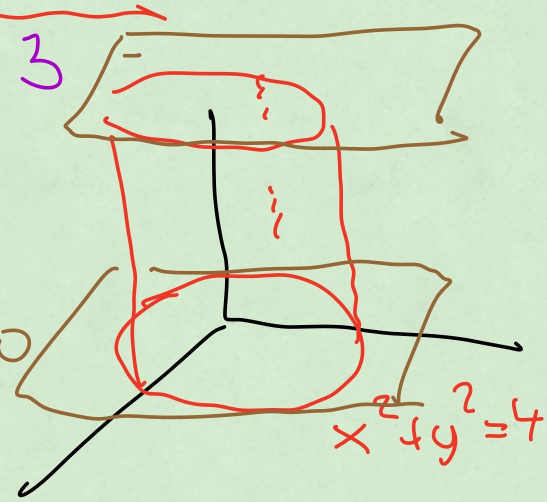
---

example:

integrate $f(x,y,z) = x+y^2$

over the cylinder $\boxed{x^2+y^2=4}$

between $z=0$ and $z=3$



① Step 1 : parametrize cylinder

$\vec{r} = (x, y, z)$

$\vec{r} = (r\cos\theta, r\sin\theta, z)$

$x^2+y^2=4$

$r^2=4$

$\boxed{r=2}$

$\vec{r}(\theta,z) = (2\cos\theta, 2\sin\theta, z)$

②

$\frac{\partial\vec{r}}{\partial\theta} = (-2\sin\theta, 2\cos\theta, 0)$

$\frac{\partial\vec{r}}{\partial z} = (0, 0, 1)$

③ $\frac{\partial \vec{r}}{\partial \theta} \times \frac{\partial \vec{r}}{\partial z} = (2\cos\theta, 2\sin\theta, 0)$

④ $\left|\frac{\partial \vec{r}}{\partial \theta} \times \frac{\partial \vec{r}}{\partial z}\right| = \sqrt{4\cos^2\theta + 4\sin^2\theta} = \boxed{2}$

⑤ (new step)

$f = x + y^2$

$f = 2\cos\theta + (2\sin\theta)^2$

$f = 2\cos\theta + 4\sin^2\theta$

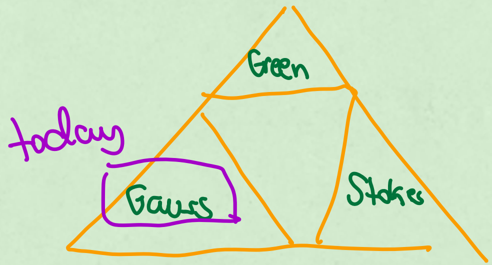
$\iint f \, dS$

$= \iint (2\cos\theta + 4\sin^2\theta) \left|\frac{\partial \vec{r}}{\partial \theta} \times \frac{\partial \vec{r}}{\partial z}\right| dz\, d\theta$

$= \int_0^{2\pi} \int_0^3 (2\cos\theta + 4\sin^2\theta) \, 2 \, dz\, d\theta$

$= 24\pi$

Lecture 26 (16.6, 16.8)



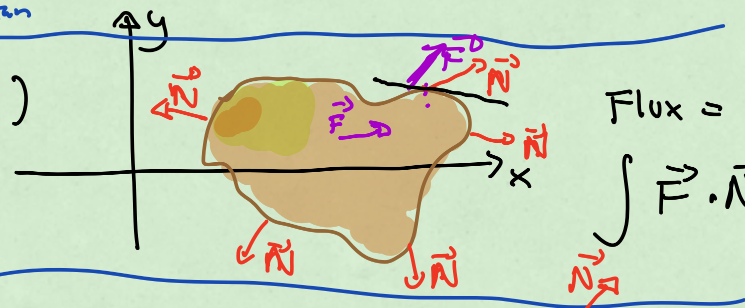
Flux

curve C

$\vec{r}(t) = (x(t), y(t))$

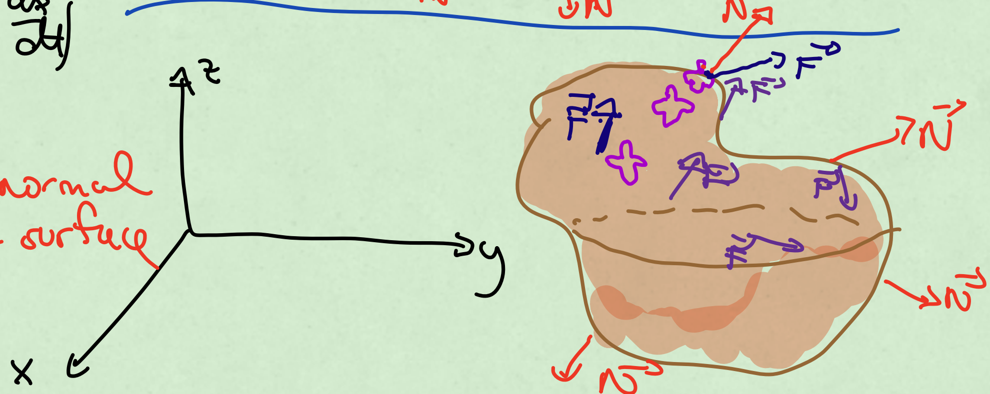
$\vec{v}(t) = \left(\frac{dx}{dt}, \frac{dy}{dt}\right)$

$\vec{N} = \left(\frac{dy}{dt}, -\frac{dx}{dt}\right)$



Flux = $\int \vec{F} \cdot \vec{N} \, dt$

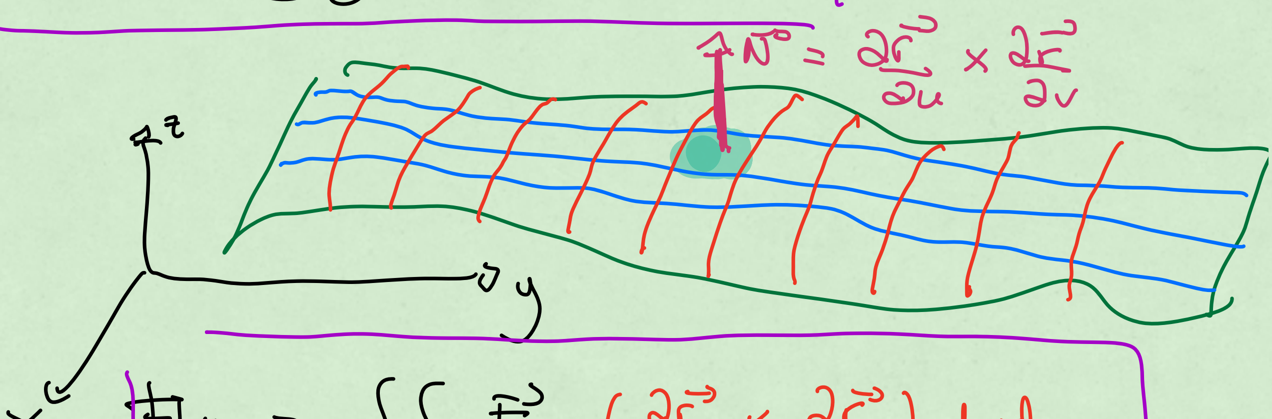
$\vec{N}$ = vector normal to the surface



$$\boxed{\text{Flux} = \iint \vec{F} \cdot \vec{N} \, dv\, du}$$

u, v parameters of parametrization of the surface

$\vec{N} = \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}$

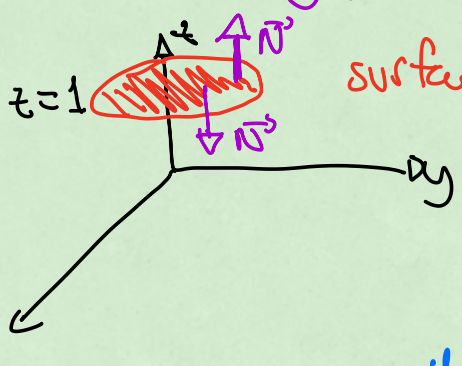


$$\boxed{\text{Flux} = \iint \vec{F} \cdot \left(\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}\right) dv\, du}$$

related to how much goes "stuff" passes through a surface

example $\vec{F} = (x-y, x+z, z-y)$

two choices for $\vec{N}$ and you are told which one to use



surface unit disk $x^2+y^2 \leq 1$, but at height 1,

Flux of $\vec{F}$ through S [the one with normal vector having third positive entry]

First step: parametrize the surface

$$\vec{r}(\theta, r) = (r\cos\theta, r\sin\theta, 1)$$

$$\frac{\partial \vec{r}}{\partial \theta} = (-r\sin\theta, r\cos\theta, 0)$$

$$\frac{\partial \vec{r}}{\partial r} = (\cos\theta, \sin\theta, 0)$$

$$\frac{\partial \vec{r}}{\partial \theta} \times \frac{\partial \vec{r}}{\partial r} = \det \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ -r\sin\theta & r\cos\theta & 0 \\ \cos\theta & \sin\theta & 0 \end{pmatrix}$$

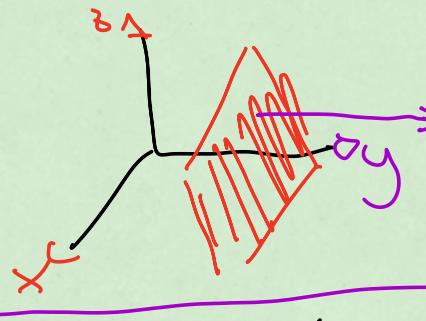
$$= (0, 0, -r\sin^2\theta - r\cos^2\theta)$$

$$= (0, 0, \underbrace{-r}_{\text{negative third entry}})$$

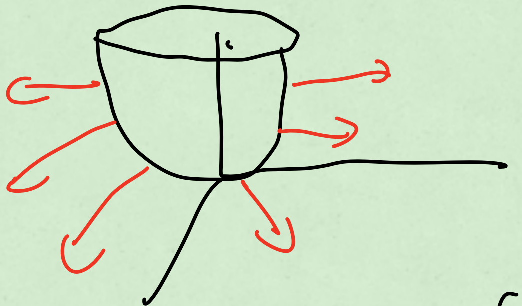
we should have done

$$\frac{\partial \vec r}{\partial r} \times \frac{\partial \vec r}{\partial \theta} = -(0, 0, -r) = (0, 0, r)$$

Aside

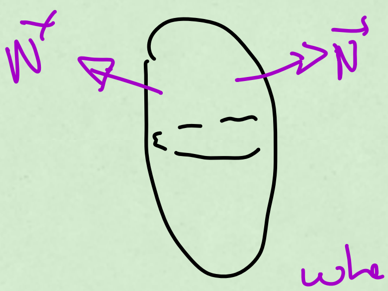


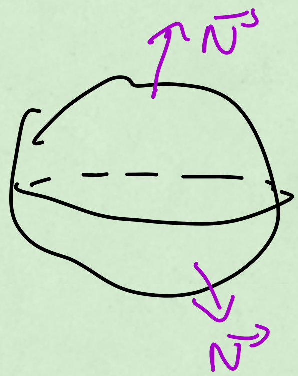
$\vec N = (0, 1, 0)$

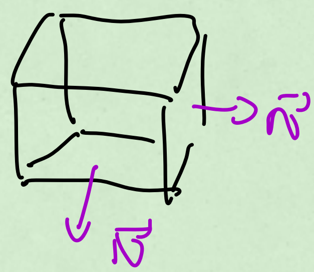


$x^2 + y^2 - z = 0$

$\nabla f = (2x, 2y, -1)$







when surface is closed we will choose the normal vector that points away from the surface

$$\vec r = (r\cos\theta, r\sin\theta, 1)$$

$$\frac{\partial \vec r}{\partial r} \times \frac{\partial \vec r}{\partial \theta} = (0, 0, r)$$

$$\vec F = (x-y, x+z, z-y)$$

$$\vec{F} = (r\cos\theta - r\sin\theta,\ r\cos\theta + 1,\ \boxed{1 - r\sin\theta})$$

$$\text{Flux} = \int_0^{2\pi}\int_0^{1} \vec{F} \cdot \underbrace{\left(\frac{\partial \vec{r}}{\partial r} \times \frac{\partial \vec{r}}{\partial \theta}\right)}_{(0,0,r)} dr\,d\theta$$

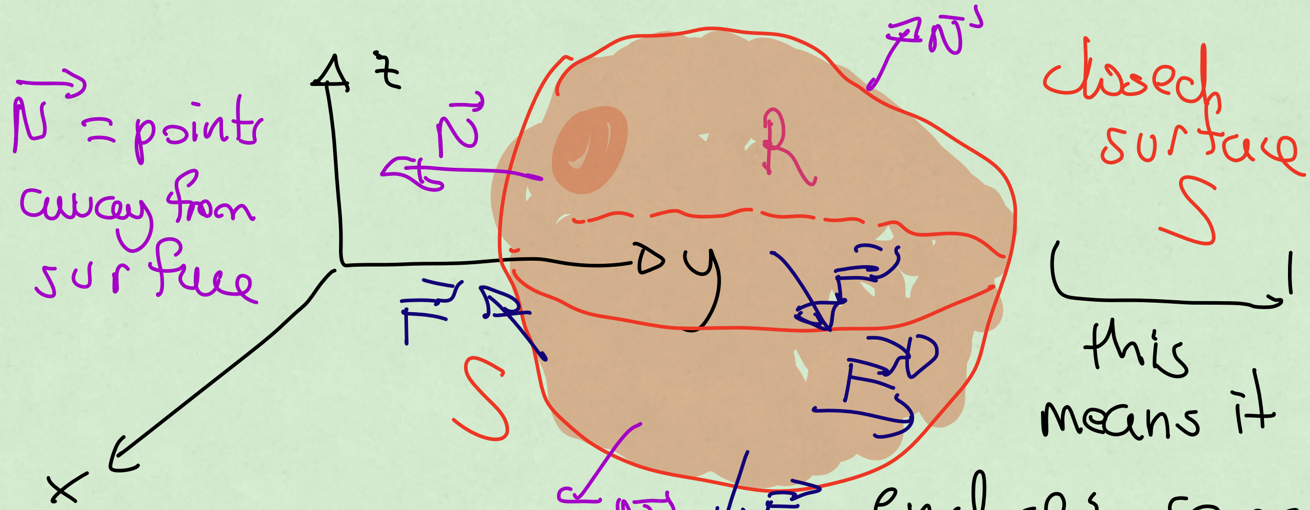
$$= \int_0^{2\pi}\int_0^{1} \boxed{(1 - r\sin\theta)}\, r \, dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{1} (r - r^2\sin\theta)\,dr\,d\theta$$

$$= \pi$$

Section 16.8

Divergence (or Gauss) Theorem



this means it encloses some

$\vec{F} = (F_1, F_2, F_3)$ (like a closed box or container) 3d region R

poke ball

Divergence Theorem

Flux of $\vec{F}$

$$= \iint \vec{F} \cdot \left(\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right) dv\, du$$

Theorem

$$= \iiint_R \underbrace{\left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right)}_{\text{called divergence of } \vec{F}} dV$$

divergence of $\vec{F} = \nabla \cdot \vec{F}$

$$= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \cdot (F_1, F_2, F_3)$$

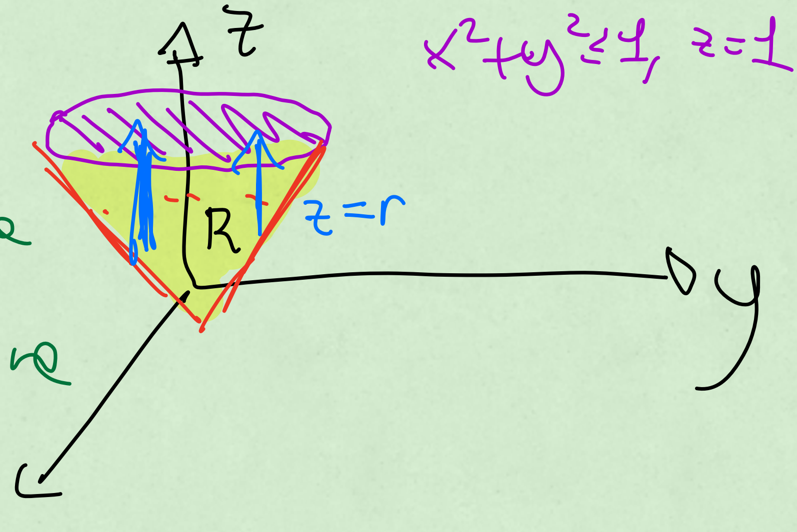
$$= \frac{\partial F_1}{} + \frac{\partial F_2}{} + \partial F_3$$

$\partial x \qquad \partial y \qquad \partial z$

Example:

$\vec{F} = (x-y,\ x+z,\ z-y)$

surface =
disk at height 1
+ part of cone
cone $z=\sqrt{x^2+y^2}$
below $z=1$ plane



divergence theorem
= Flux (disk + cone)

$\overset{\text{Theorem}}{=} \iiint_R \nabla \cdot \vec{F}\, dV$

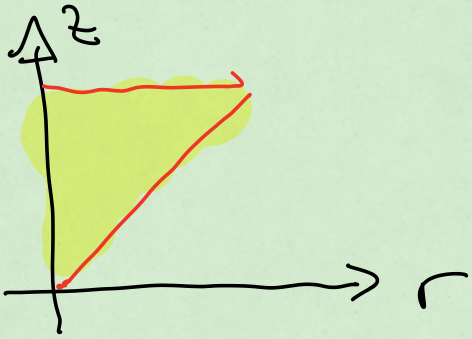
$$= \iiint \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot (x-y, x+z, z-y) \, dV$$

$$= \iiint \frac{\partial}{\partial x}(x-y) + \frac{\partial}{\partial y}(x+z) + \frac{\partial}{\partial z}(z-y) \, dV$$

$$= \iiint (1 + 0 + 1) \boxed{dV}$$

$$= \int_0^{2\pi} \int_0^1 \int_r^1 2 \; r \, dz \, dr \, d\theta$$

cylindrical coordinates



answer

$$= \boxed{\frac{2\pi}{3}}$$

flux (disk + cone)

Remark

$$\frac{2\pi}{3} = \underbrace{\text{flux of disk}}_{\substack{\text{found it on last} \\ \text{example and it was} \\ \pi}} + \text{flux of the cone}$$

$$\frac{2\pi}{3} = \pi + \text{flux of the cone}$$

$$\text{flux through cone} = \frac{2\pi}{3} - \pi = \boxed{-\frac{\pi}{3}}$$

---

if we want to check this you can find flux through cone $\boxed{z = \sqrt{x^2+y^2}}$ directly.

parametrization of cone

$$\vec{r} = (x, y, \sqrt{x^2+y^2})$$

$$\vec{r}(\theta, r) = (r\cos\theta, r\sin\theta, r)$$

$$\frac{\partial \vec{r}}{\partial \theta} = (-r\sin\theta,\ r\cos\theta,\ 0)$$

$$\frac{\partial \vec{r}}{\partial r} = (\cos\theta,\ \sin\theta,\ 1)$$

$$\frac{\partial \vec{r}}{\partial \theta} \times \frac{\partial \vec{r}}{\partial r} = (r\cos\theta,\ r\sin\theta,\ -r)$$

Flux through cone

$$= \int_0^{2\pi}\int_0^1 (x-y,\ x+z,\ z-y)\cdot(r\cos\theta,\ r\sin\theta,\ -r)\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^1 (r\cos\theta - r\sin\theta,\ r\cos\theta + r,\ r - r\sin\theta) \cdot (r\cos\theta,\ r\sin\theta,\ -r)\, dr\, d\theta$$

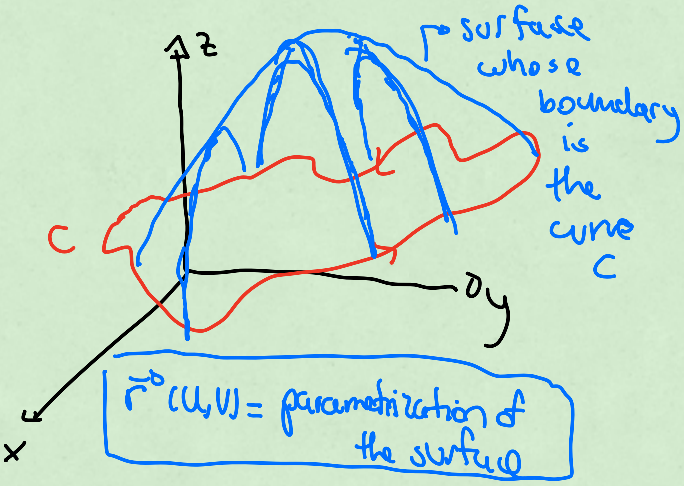
check

$$= -\frac{\pi}{3}$$

Lecture 27 (16.7)
Stokes Theorem

C = closed curve in space

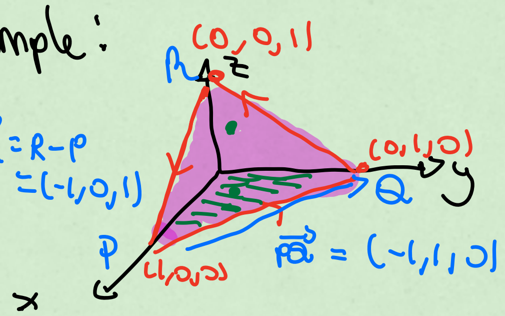
$\vec{F}$ = some vector field



$\vec{r}(u,v)$ = parametrization of the surface

$$\underbrace{\int_C \vec{F}\cdot d\vec{r}}_{\text{work of } \vec{F} \text{ along curve } C} \overset{\text{Stokes}}{=} \underbrace{\iint_S (\nabla\times\vec{F})\cdot\left(\frac{\partial\vec{r}}{\partial u}\times\frac{\partial\vec{r}}{\partial v}\right)dv\,du}_{\text{flux of the curl of } \vec{F}}$$

example:



$\vec{PR} = R-P = (-1,0,1)$

$\vec{PQ} = (-1,1,0)$

curve C = 3 segments connecting the points $(0,0,1)$, $(1,0,0)$, $(0,1,0)$

$\vec{F} = (z^2, y^2, x)$

Find work of $\vec{F}$ along C using Stokes' theorem

Before stokes: 3 separate line integrals and add the answers

After Stokes

surface: triangle shown in the picture

to parametrize the triangle, you parametrize the plane to which it belongs.

$$\vec{n} = \vec{PQ} \times \vec{PR} = \det\begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

$$= (1, 1, 1)$$

$$\boxed{x + y + z = 1 = (1,1,1)\cdot(1,0,0)}$$

$$\boxed{z = 1 - x - y}$$

$\vec{r}(x,y) = (x, y, 1-x-y)$ parametrization plane

$$\frac{\partial \vec{r}}{\partial x} = (1, 0, -1) \qquad \frac{\partial \vec{r}}{\partial x} \times \frac{\partial \vec{r}}{\partial y} = \boxed{(1,1,1)}$$

$$\frac{\partial \vec{r}}{\partial y} = (0, 1, -1)$$

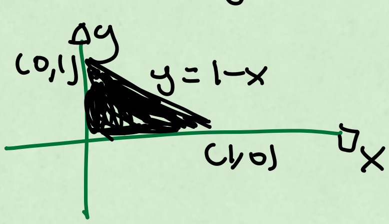
Stokes : $\boxed{\int_C \vec{F}\cdot d\vec{r} = \iint_S \boxed{(\nabla \times \vec{F})} \cdot \boxed{\left(\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}\right)} \, dv\, du}$

$$\vec{F} = (z^2, y^2, x)$$

$$\nabla \times \vec{F} = \det\begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z^2 & y^2 & x \end{pmatrix} = \boxed{(0, 2z-1, 0)}$$

Stokes

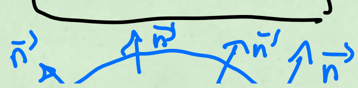
$$= \int_0^1 \int_0^{1-x} (0, 2z-1, 0) \cdot (1,1,1) \, dy\, dx$$



$$= \int_0^1 \int_0^{1-x} (0, 2(1-x-y)-1, 0)\cdot(1,1,1)\, dy\, dx$$

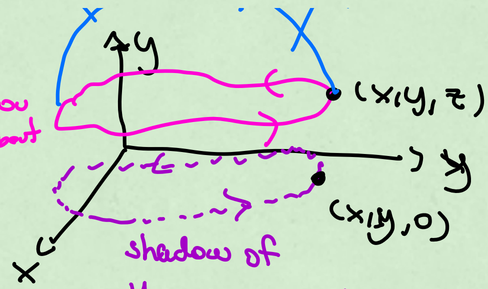
$$= \int_0^1 \int_0^{1-x} (2 - 2x - 2y - 1)\, dy\, dx$$

$$\boxed{= -\frac{1}{6}}$$



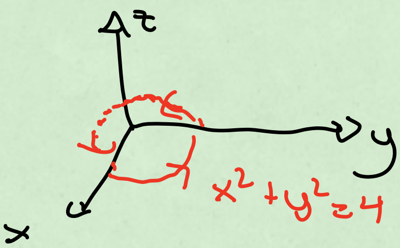
Orientation convention:

if you are moving counterclockwise on the (shadow) curve, then the normal vector to the surface has third positive entry.



Exhaustive example:

$\vec{F} = (y, 2z, x^2)$



Find work of $\vec{F}$ along circle in 3 different ways:

(1) Directly as a line integral $\int_C \vec{F}\cdot d\vec{r} = \int \vec{F}\cdot\vec{v}\,dt$

parametrize circle

$\vec{r}(t) = (2\cos t, 2\sin t, 0)$

$\vec{v}(t) = (-2\sin t, 2\cos t, 0)$

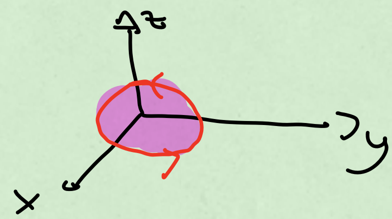
$\vec{F} = (y, 2z, x^2) = (2\sin t, 0, 4\cos^2 t)$

$$\int_0^{2\pi} (2\sin t, 0, 4\cos^2 t)\cdot(-2\sin t, 2\cos t, 0)\,dt$$

$$= \int_0^{2\pi} -4\sin^2 t\,dt$$

$$\boxed{= -4\pi}$$

(2) option 2: use Stokes with disk on the xy plane as surface S



parametrize disk:

$$\vec{r}(\theta, r) = (r\cos\theta, r\sin\theta, 0) \quad \begin{array}{l} 0 \le \theta \le 2\pi \\ 0 \le r \le 2 \end{array}$$

$$\frac{\partial \vec{r}}{\partial \theta} = (-r\sin\theta, r\cos\theta, 0)$$

$$\frac{\partial \vec{r}}{\partial r} = (\cos\theta, \sin\theta, 0)$$

$$\frac{\partial \vec{r}}{\partial \theta} \times \frac{\partial \vec{r}}{\partial r} = \det \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ -r\sin\theta & r\cos\theta & 0 \\ \cos\theta & \sin\theta & 0 \end{pmatrix}$$

$$= (0, 0, -r\sin^2\theta - r\cos^2\theta)$$

$$= (0, 0, \boxed{-r})$$

negative third entry

so we have to use

$$\frac{\partial \vec{r}}{\partial r} \times \frac{\partial \vec{r}}{\partial \theta} = (0, 0, r)$$ instead due to the conventions,

$$\text{Stokes work} = \iint (\nabla \times \vec{F}) \cdot \left(\frac{\partial \vec{r}}{\partial r} \times \frac{\partial \vec{r}}{\partial \theta}\right) dr\,d\theta$$

$$\vec{F} = (y, 2z, x^2)$$

$$\boxed{\nabla \times \vec{F}} = \det \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & 2z & x^2 \end{pmatrix} = (-2, -2x, -1)$$

$$= (-2, -2r\cos\theta, -1)$$

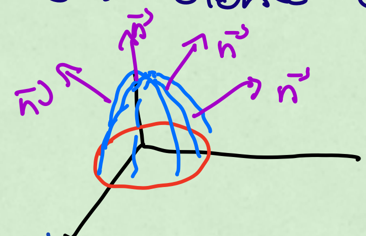
find it first in terms of x, y, z and then rewrite in terms of parameters

$$= \int_0^{2\pi}\int_0^2 (-2, -2r\cos\theta, -1)\cdot(0,0,r)\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^2 -r\,dr\,d\theta$$

$$= \boxed{-4\pi}$$

option 3 : Stokes with a different surface



surface

$z = 4 - x^2 - y^2$ (paraboloid)

$z = 4 - r^2$ $\quad z \geq 0$

parametrization paraboloid

$$\vec{r}(r,\theta) = (r\cos\theta, r\sin\theta, 4 - r^2) \quad 0 \leq \theta \leq 2\pi$$

$$0 \leq r \leq 2$$

$$\frac{\partial \vec{r}}{\partial r} = (\cos\theta, \sin\theta, -2r)$$

$$\frac{\partial \vec{r}}{\partial \theta} = (-r\sin\theta, r\cos\theta, 0)$$

$$\frac{\partial \vec{r}}{\partial r} \times \frac{\partial \vec{r}}{\partial \theta} \overset{\det}{=} \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos\theta & \sin\theta & -2r \\ -r\sin\theta & r\cos\theta & 0 \end{pmatrix}$$

$$= (2r^2\cos\theta, 2r^2\sin\theta, \boxed{r})$$

important that it was with positive sign

$$\nabla \times \vec{F} = (-2, -2x, -1)$$

$$= (-2, -2r\cos\theta, -1)$$

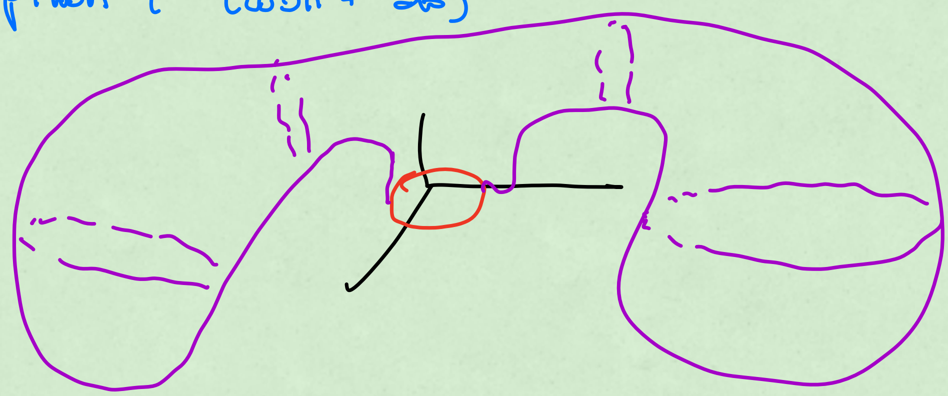
$$\int_0^{2\pi}\int_0^2 (-2, -2r\cos\theta, -1)\cdot(2r^2\cos\theta, 2r^2\sin\theta, r)\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^2 (\underbrace{-4r^2\cos\theta}_{\text{integrates to } 0} - \underbrace{4r^3\cos\theta\sin\theta}_{\text{integrates to}} - r)\,dr\,d\theta$$

$$= \int_0^{2\pi} \int_0^{2} -r \, dr \, d\theta$$

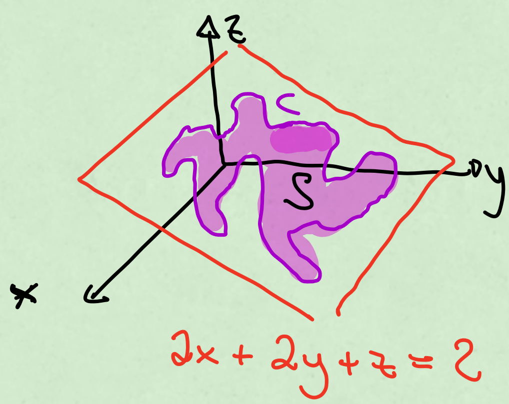
$$= \boxed{-4\pi}$$

option 4 (won't do)



Last Day !!!
Review

Stokes' Theorem:



Use Stokes' theorem to show that

$$\int_C 2y\,dx - 3z\,dy - x\,dz$$

only depends on the area of the region enclosed by the curve.

$$\int_C 2y\,dx - 3z\,dy - x\,dz = \int_C \underbrace{(2y, -3z, -x)}_{\vec{F}} \cdot \underbrace{(dx, dy, dz)}_{d\vec{r}}$$

so we are trying to find the work of $\vec{F} = (2y, -3z, -x)$ along this curve C,

S = surface for Stokes theorem: region of the plane whose boundary is the curve C.

$$= \iint_S (\nabla \times \vec{F}) \cdot \left(\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right) du\,dv$$

curl of $\vec{F}$:

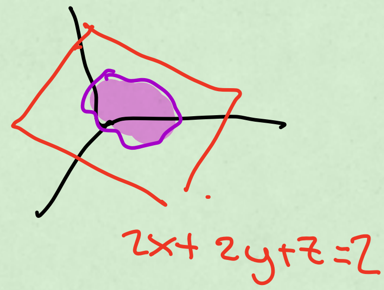
$$\vec{F} = (2y, 3z, -x)$$

$$\nabla \times \vec{F} = \det \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial & \partial & \partial \end{pmatrix}$$

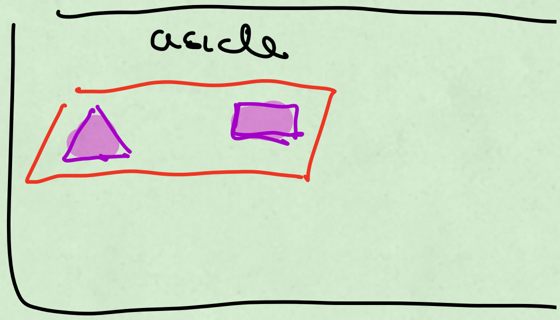
$$\begin{pmatrix} \partial x & \partial y & \partial z \\ 2y & 3z & -x \end{pmatrix}$$

$$\nabla \times \vec{F} = (-3, 1, -2)$$

Reminder: curl of $\vec{F}$ is always computed in terms of x, y, z.



Parametrize the surface so need to parametrize the plane



$$\vec{r}(x,y) = (x, y, 2-2x-2y)$$

$$\frac{\partial \vec{r}}{\partial x} = (1, 0, -2)$$

$$\frac{\partial \vec{r}}{\partial y} = (0, 1, -2)$$

$$\frac{\partial \vec{r}}{\partial x} \times \frac{\partial \vec{r}}{\partial y} = (2, 2, 1)$$

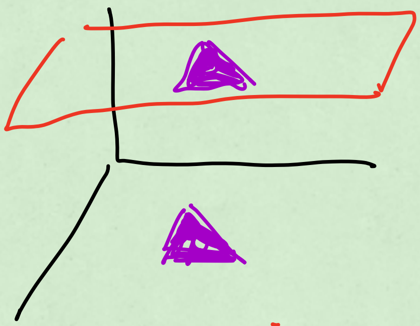
Stokes Theorem

$$\int_C \vec{F} \cdot d\vec{r} = \iint (\nabla \times \vec{F}) \cdot \left(\frac{\partial \vec{r}}{\partial x} \times \frac{\partial \vec{r}}{\partial y} \right) dy\,dx$$

$$= \iint (-3, 1, -2) \cdot (2, 2, 1)\, dy\,dx$$

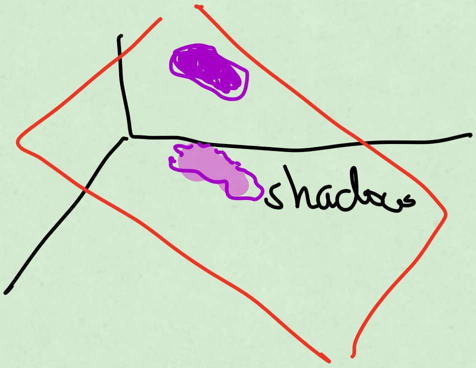
$$= \iint -6 \, dy\,dx$$

$$= -6 \underset{\substack{\text{Bounds}\\ \text{for } x,y}}{\iint} 1 \, dy\,dx$$



$$= -6 \text{ area} \begin{pmatrix} \text{Shadow} \\ \text{on } xy \\ \text{plane} \end{pmatrix}$$

work = -6 area (shadow)

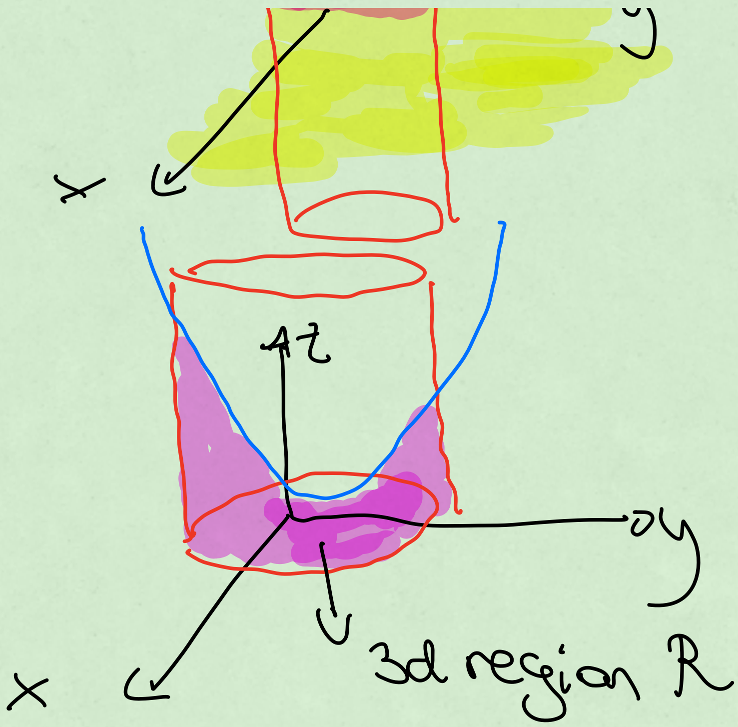


Remark

$$\text{area shadow} = \boxed{-\frac{1}{6} \text{ work}}$$

Another example:

3d region = • stuff inside cylinder $x^2+y^2=4$
• below paraboloid $z=x^2+y^2$
• above the xy plane



xy plane = floor

$z = x^2 + y^2$ roof

cylinder = lateral walls

Boundary of R = surface S consisting of

- disk on the xy plane plus
- part of the cylinder plus
- part of the paraboloid.

$\vec{F} = (2y, 9xy, -4z)$

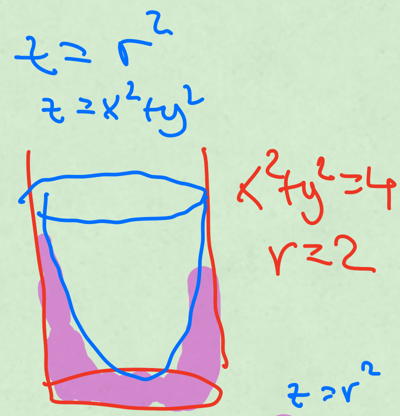
Use divergence theorem to find the outward flux of $\vec{F}$ through surface S.

Flux of $\vec{F}$ $\underset{\text{divergence theorem}}{=}$ $\iiint_R \nabla \cdot \vec{F}\, dV$

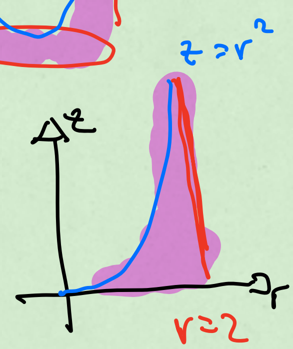
$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(2y) + \frac{\partial}{\partial y}(9xy) + \frac{\partial}{\partial z}(-4z)$$

$$= 0 \quad + 9x \quad - 4$$

$$\nabla \cdot \vec{F} = 9x - 4$$



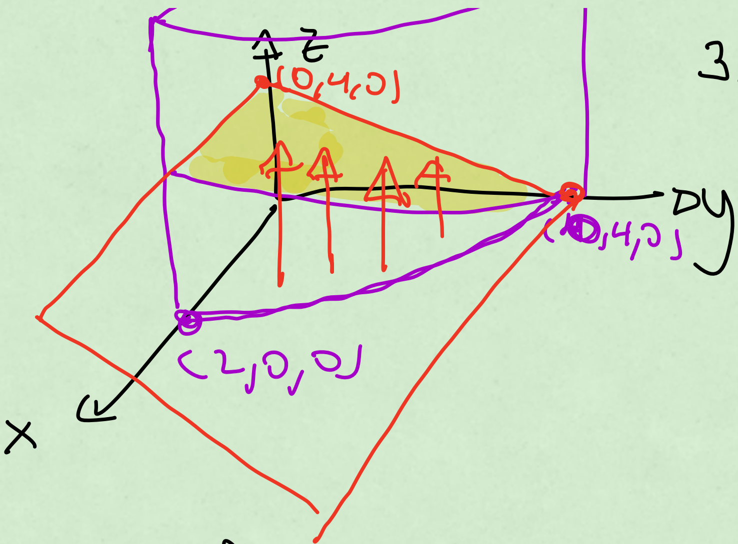
$$\iiint (9x-4)\,dV$$



$$\int_0^{2\pi}\int_0^{2}\int_0^{r^2} (9r\cos\theta - 4)\, r\,dz\,dr\,d\theta$$

$$= \boxed{-32\pi}$$

Another example



3d Region:

- inside first octant $(x, y, z \geq 0)$
- inside the elliptical cylinder $\boxed{4x^2 + y^2 = 16}$
- below plane $\boxed{y + z = 4}$

$z = 4 - y$

surface S = boundary of this region

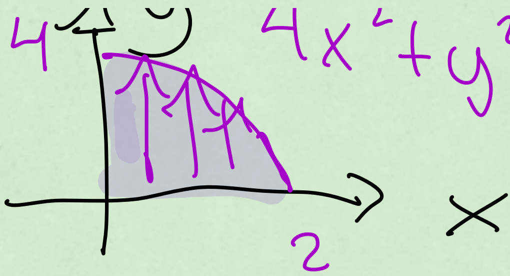
$$\vec{F} = (4xz, -3xy, -2z^2)$$

use divergence theorem to find the flux of $\vec{F}$ through S

$$\nabla \cdot \vec{F} = 4z - 3x - 4z = -3x$$

$$\text{Flux} = \iiint \nabla \cdot \vec{F} \, dV = \int_0^2 \int_0^{\sqrt{16-4x^2}} \int_0^{4-y} -3x \, dz\,dy\,dx$$

$$\boxed{= -40}$$



$4x^2+y^2=16$

$y=\sqrt{16-4x^2}$

$\vec{F}$ units U

$\nabla\cdot\vec{F}$ units $\frac{U}{\text{length}}$

$\iiint \nabla\cdot\vec{F}\,dV$ has units

$$\frac{U}{\text{length}}\cdot\text{volume}$$

$$= U\cdot\text{area}$$

Units of flux or this integral $=$ units of $\vec{F}$ $\cdot$ units of area