INTERIOR CURVATURE ESTIMATES AND THE ASYMPTOTIC PLATEAU PROBLEM IN HYPERBOLIC SPACE

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Abstract. We show that for a very general class of curvature functions defined in the positive cone, the problem of finding a complete strictly locally convex hypersurface in $\mathbb{H}^{n+1}$ satisfying $f(\kappa) = \sigma \in (0,1)$ with a prescribed asymptotic boundary Γ at infinity has at least one smooth solution with uniformly bounded hyperbolic principal curvatures. Moreover if Γ is (Euclidean) starshaped, the solution is unique and also (Euclidean) starshaped while if Γ is mean convex the solution is unique. We also show via a strong duality theorem that analogous results hold in De Sitter space. A novel feature of our approach is a "global interior curvature estimate".

1. Introduction

Let $\mathbb{H}^{n+1}$ be the the hyperbolic space of dimension $n+1$, $n \geq 2$, and $\partial_\infty \mathbb{H}^{n+1}$ denote the ideal boundary of $\mathbb{H}^{n+1}$ at infinity. In this paper we are concerned with the problem of finding complete hypersurfaces of constant curvature in $\mathbb{H}^{n+1}$ with prescribed asymptotic boundary at infinity. More precisely, given a disjoint collection of closed embedded smooth $n-1$ dimensional submanifolds $\Gamma = \{\Gamma_1, \ldots, \Gamma_m\} \subset \partial_\infty \mathbb{H}^{n+1}$, we seek a complete hypersurface Σ in $\mathbb{H}^{n+1}$ satisfying

$$f(\kappa[\Sigma]) = \sigma \tag{1.1}$$

with the asymptotic boundary

$$\partial\Sigma = \Gamma \tag{1.2}$$

where f is a smooth symmetric function of n variables, $\kappa[\Sigma] = (\kappa_1, \ldots, \kappa_n)$ denotes the induced (positive) hyperbolic principal curvatures of Σ and σ is a constant.

The problem was first studied by Anderson [1], [2], Hardt-Lin [11] for area-minimizing varieties using geometric measure theory; their results were extended by Tonegawa [18] to hypersurfaces of constant mean curvature. In [13], Lin first used PDE methods to prove the existence of smooth complete minimal hypersurfaces which are graphs

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in the upper half space model over mean convex domains, followed by work of Nelli-Spruck [14] and Guan-Spruck [7] for hypersurfaces of constant mean curvature. For Gauss curvature, the asymptotic Plateau problem was initiated by Labourie [12] in $\mathbb{H}^3$ and by Rosenberg-Spruck [16] in $\mathbb{H}^{n+1}$. In recent work [10], [8], [9], [17] the authors considered the problem for more general curvature functions. In this paper we shall focus on locally strictly convex hypersurfaces and give a complete solution to problem (1.1)-(1.2) under very general assumptions on f.

Accordingly, we shall assume the curvature function f to be defined on the positive cone $K_n^+ := \big\{\lambda \in \mathbb{R}^n : \text{each component } \lambda_i > 0\big\}$ with

$$f = 0 \ \text{ on } \partial K_n^+, \tag{1.3}$$

and satisfy the fundamental structure conditions [4]:

$$f_i(\lambda) \equiv \frac{\partial f(\lambda)}{\partial \lambda_i} > 0 \ \text{ in } K_n^+, \ \ 1 \le i \le n, \tag{1.4}$$

$$f \text{ is a concave function in } K_n^+. \tag{1.5}$$

Consequently,

$$f > 0 \ \text{ in } K_n^+. \tag{1.6}$$

For convenience we shall assume in addition that f is normalized

$$f(1, \dots, 1) = 1 \tag{1.7}$$

and is homogeneous of degree one:

$$f(t\kappa) = tf(\kappa), \ \ \forall\, t \ge 0, \ \kappa \in K_n^+. \tag{1.8}$$

A hypersurface Σ in $\mathbb{H}^{n+1}$ is said to be *locally strictly convex* if $\kappa[\Sigma] \in K_n^+$, i.e. the principal curvatures of Σ are positive everywhere.

In order to state our main results, it is convenient (without loss of any generality) to use the upper half-space model

$$\mathbb{H}^{n+1} = \{(x, x_{n+1}) \in \mathbb{R}^{n+1} : x_{n+1} > 0\}$$

equipped with the hyperbolic metric

$$ds^2 = \frac{1}{x_{n+1}^2} \sum_{i=1}^{n+1} dx_i^2. \tag{1.9}$$

Thus $\partial_\infty \mathbb{H}^{n+1}$ is naturally identified with $\mathbb{R}^n = \mathbb{R}^n \times \{0\} \subset \mathbb{R}^{n+1}$ and (1.2) may be understood in the Euclidean sense.

The first main result of this paper may be stated as follows.

Theorem 1.1. *Suppose $\Gamma = \partial\Omega \in C^2$ for a bounded domain $\Omega \subset \mathbb{R}^n = \mathbb{R}^n \times \{0\}$ and $0 < \sigma < 1$. Under conditions (1.3)-(1.5) and (1.7)-(1.8), there exists a complete locally strictly convex hypersurface Σ in $\mathbb{H}^{n+1}$ satisfying (1.1)-(1.2) with uniformly bounded principal curvatures*

$$C^{-1} \leq \kappa_i \leq C \quad \text{on } \Sigma. \tag{1.10}$$

Moreover, Σ is the vertical graph of $u \in C^\infty(\Omega) \cap C^{0,1}(\overline{\Omega})$, $u > 0$ in Ω, $u = 0$ on $\partial\Omega$, $u^2 \in C^{1,1}(\overline{\Omega})$ and

$$u|D^2u| + |Du| \leq C \text{ in } \overline{\Omega}, \quad \sqrt{1 + |Du|^2} = \frac{1}{\sigma} \text{ on } \partial\Omega. \tag{1.11}$$

Theorem 1.1 substantially improves our earlier results in [10], [9] where we needed the additional more technical assumption

$$\lim_{R \to +\infty} f(\lambda_1, \cdots, \lambda_{n-1}, \lambda_n + R) \geq 1 + \varepsilon_0 \quad \text{uniformly in } B_{\delta_0}(\mathbf{1}) \tag{1.12}$$

for some fixed $\varepsilon_0 > 0$ and $\delta_0 > 0$, where $B_{\delta_0}(\mathbf{1})$ is the ball of radius δ_0 centered at $\mathbf{1} = (1, \ldots, 1) \in \mathbb{R}^n$, which was used in the proof of boundary estimates for curvature. We achieve this by deriving a novel "global interior curvature bound" (Theorem 1.3) which also allows us to prove uniqueness of the solution for mean convex or starshaped asymptotic boundary (Theorem 1.4 and Theorem 1.5).

An advantage in using the upper half space model of $\mathbb{H}^{n+1}$ is due to the fact that there is a remarkably simple relation between the hyperbolic (κ_i) and Euclidean (κ_i^e) principal curvatures of a hypersurface Σ:

$$\kappa_i = x_{n+1}\kappa_i^e + \nu^{n+1}, \quad 1 \leq i \leq n \tag{1.13}$$

at $(x, x_{n+1}) \in \Sigma$, where ν is Euclidean unit normal vector to Σ and $\nu^{n+1} = \nu \cdot e_{n+1}$.

One important consequence of (1.13) is the following result of [10].

Theorem 1.2. *Let Σ be a complete locally strictly convex C^2 hypersurface in $\mathbb{H}^{n+1}$ with compact asymptotic boundary at infinity. Then Σ is the (vertical) graph of a*

function $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$, $u > 0$ *in* Ω *and* $u = 0$ *on* $\partial\Omega$, *for some domain* $\Omega \subset \mathbb{R}^n$. *Moreover, the function* $u^2 + |x|^2$ *is strictly (Euclidean) convex.*

For convenience we say Σ has compact asymptotic boundary if $\partial\Sigma \subset \partial_\infty \mathbb{H}^{n+1}$ is compact with respect to the Euclidean metric in $\mathbb{R}^n$.

According to Theorem 1.2, the asymptotic Plateau problem (1.1)-(1.2) for locally strictly convex hypersurfaces reduces to the Dirichlet problem for a fully nonlinear equation of the form

$$G(D^2u, Du, u) = \sigma, \quad u > 0 \quad \text{in } \Omega \subset \mathbb{R}^n \tag{1.14}$$

with the boundary condition $u = 0$ on $\partial\Omega$. In particular, the asymptotic boundary Γ must be the boundary of some bounded domain in $\mathbb{R}^n$. Moreover, it is also necessary to assume $0 < \sigma < 1$ in Theorem 1.1.

The graph of a solution u of equation (1.14) is locally strictly convex in $\mathbb{H}^{n+1}$ if and only if $|x|^2 + u^2$ is a strictly convex function on Ω. We shall call such solutions *admissible*. Condition (1.4) ensures that equation (1.14) is elliptic for admissible solutions while assumption (1.5) implies that the function G is concave with respect to D^2u; see [4]. By (1.3), equation (1.14) becomes uniformly elliptic for admissible solutions with *a priori* bounds in C^2 norm and therefore allows us to apply Evans-Krylov Theorem to derive $C^{2,\alpha}$ and higher order estimates.

From the above discussion we see that Theorem 1.1 is essentially optimal as far as locally strictly convex hypersurfaces are concerned. It is worthwhile to remark that we could remove condition (1.8) from Theorem 1.1. (The only conclusion that might need adjustment would be $\sqrt{1 + |Du|^2} = \frac{1}{\sigma}$ on $\partial\Omega$.) We keep this assumption in Theorem 1.1 in order to apply results from [10], [9] which allow us to significantly shorten the proof. By an approximation argument we could also remove the smoothness assumption on $\Gamma = \partial\Omega$ and instead assume a uniform exterior ball condition.

The main new technical tool used in this paper is a global curvature estimate *which is obtained from an interior curvature estimate.* More precisely we have

Theorem 1.3. *Suppose* f *satisfies conditions* (1.3)-(1.8) *and* $0 < \sigma < 1$. *Let* Σ *be a smooth locally strictly convex hypersurface in* $\mathbb{H}^{n+1}$ *satisfying* (1.1), (1.2) *with*

uniformly bounded principal curvatures $0 < \kappa_i \leq C$. *In the upper half space model, let* $a > 0$ *satisfy*

$$\nu^{n+1} \geq 2a > 0 \text{ on } \Sigma \tag{1.15}$$

and, for $\mathbf{x} \in \Sigma$, *let* $\kappa_{\max}(\mathbf{x})$ *denote the largest principal curvature of* Σ *at* $\mathbf{x}$. *Then for* $0 < b \leq \frac{a}{4}$,

$$\sup_{\Sigma} \frac{x_{n+1}^b \kappa_{\max}}{\nu^{n+1} - a} \leq \frac{8}{a^{\frac{5}{2}}} \big(\sup_{\Sigma} x_{n+1} \big)^b. \tag{1.16}$$

In particular,

$$\kappa_{\max} \leq 8a^{-\frac{5}{2}} \quad \text{on } \Sigma. \tag{1.17}$$

The existence of such constant $a > 0$ follows from the global gradient estimates in [9], see Corollary 2.6.

Theorem 1.1 follows from Theorem 1.3 and the existence result of [9]. To see this we apply Theorem 1.2 of [9] to the curvature function $f^\theta := \theta H_n^{\frac{1}{n}} + (1-\theta) f$ which satisfies conditions (1.3)-(1.8) as well as (1.12), where $H_n(\kappa_1, \ldots, \kappa_n) = \kappa_1 \cdots \kappa_n$ corresponds to the Gauss curvature. We obtain a complete strictly locally convex hypersurface $\Sigma^\theta = \text{graph}(u^\theta)$ in $\mathbb{H}^{n+1}$ satisfying (1.1)-(1.2) with f replaced by f^θ. The principal curvatures of Σ^θ admit an upper bound depending on θ. Moreover, $u^\theta \in C^{0,1}(\overline{\Omega})$, $(u^\theta)^2 \in C^\infty(\Omega) \cap C^{1,1}(\overline{\Omega})$ and $u^\theta + |Du^\theta| \leq C$ independent of θ. Using Theorem 1.3, we find that $u^\theta |D^2 u^\theta| \leq C$ where C is independent of θ. We can now let θ tend to 0 to complete the proof of Theorem 1.1.

An important issue is the uniqueness of solutions to problem (1.1)-(1.2). This is a complicated question even in the case of locally strictly convex hypersurfaces. From the PDE point of view, the main difficulty come from the fact that the linearized operator of equation (1.14) may have non-trivial kernel. In this paper we are able to prove the following general uniqueness when Γ is mean convex in $\mathbb{R}^n$. Throughout the rest of this paper, we assume $\Gamma = \partial\Omega \times \{0\} \subset \mathbb{R}^{n+1}$ where Ω is a bounded domain in $\mathbb{R}^n$. Unless otherwise stated, we also assume $\partial\Omega$ is smooth.

Theorem 1.4. *Assume* Ω *is a* $C^{2,\alpha}$ *mean convex domain, that is, the Euclidean mean curvature* $\mathcal{H}_{\partial\Omega} \geq 0$. *Then the solution* Σ *of Theorem 1.1 is unique.*

There is also uniqueness if $\partial\Omega$ is strictly (Euclidean) starshaped about the origin. This is a well-known fact. In the following theorem we give a quantitative description in terms of the starshapedness of the boundary; See Theorem 4.3 for more details.

Theorem 1.5. *Let $\partial\Omega \in C^1$ be strictly (Euclidean) starshaped about the origin. Then the unique solution given in Theorem 1.1 is strictly (Euclidean) starshaped about the origin. Indeed, $\mathbf{x} \cdot \nu \geq c_0$ for some $c_0 > 0$.*

Remark 1.6. The reader should note that in Theorem 1.3 we are not claiming that all possible locally strictly convex solutions of (1.1), (1.2) satisfy the global curvature bound (1.17), rather only those which are a priori known to have uniformly bounded principal curvatures $0 < \kappa_i \leq C$. When we have uniqueness, for instance in a $C^{2,\alpha}$ mean convex domain (by Theorem 1.4) , then this is the case. The following example may be helpful in understanding some of the subtlety of existence and regularity issues.

Example 1.7. Take a convex domain Ω with a boundary point P with a conical singularity and $\partial\Omega \setminus \{P\}$ smooth. Then we can approximate Ω by smooth convex domains Ω^ε. Applying Theorem 1.1 we find locally strictly convex complete hypersurfaces $\Sigma^\varepsilon = \text{graph}(u^\varepsilon)$ satisfying (1.1)-(1.2) with uniformly bounded principal curvatures $\frac{1}{C^\varepsilon} \leq \kappa_i^\varepsilon| \leq C^\varepsilon$. Moreover by Corollary 2.6, $|u^\varepsilon| + |Du^\varepsilon| \leq C$, $(\nu^\varepsilon)^{n+1} \geq 2a > 0$ with C, a independent of ε. We can now apply Theorem 1.3 to conclude that $u^\varepsilon|D^2u^\varepsilon| \leq C$ where C is independent of ε. We can now let ε tend to 0 and obtain a smooth limiting locally strictly convex $\Sigma = \text{graph}(u)$ a solution satisfying all the conditions of Theorem 1.1. This means that u is globally Lipschitz and $u|D^2u| \leq C$ so u satisfies interior estimates similar to those satisfies by the solution of a uniformly elliptic equation. Even in this case we do not know if there is uniqueness.

We end with an application of Theorem 1.1 to the existence of constant curvature spacelike hypersurfaces in de Sitter space. There is a natural asymptotic Plateau problem dual to (1.1)-(1.2) for strictly spacelike hypersurfaces [17] which takes place in the steady state subspace $\mathcal{H}^{n+1} \subset dS_{n+1}$ of de Sitter space. Following Montiel [15], there is a halfspace model which identifies $\mathcal{H}^{n+1}$ with $\mathbb{R}^{n+1}_+$ endowed with the Lorentz metric

$$ds^2 = \frac{1}{y_{n+1}^2}(dy^2 - dy_{n+1}^2). \tag{1.18}$$

It is important to note that *the isometry from $\mathcal{H}^{n+1}$ to the halfspace model reverses the time orientation.* The dual asymptotic Plateau problem seeks to find a strictly spacelike hypersurface S satisfying

$$f(\kappa[S]) = \sigma > 1, \quad \partial S = \Gamma \tag{1.19}$$

where $\kappa[S]$ denotes the principal curvatures of S in the induced de Sitter metric.

If S is a complete spacelike hypersurface in $\mathcal{H}^{n+1}$ with compact asymptotic boundary at infinity, then the normal vector field N of S is chosen to be the one pointing to the unique unbounded region in $\mathbb{R}^{n+1}_+ \setminus S$, and the de Sitter principal curvatures of S are calculated with respect to this normal vector field.

Because S is strictly spacelike, we are essentially forced to take $\Gamma = \partial V$ where $V \subset \mathbb{R}^n$ is a bounded domain and seek S as the graph of a "spacelike" function v

$$S = \{(y, y_{n+1}) : \ y_{n+1} = v(y), \ y \in V\}, \quad |\nabla v| < 1 \text{ in } \overline{V}. \tag{1.20}$$

In [17] we have computed the first and second fundamental forms of S with respect to the induced de Sitter metric. We use

$$X_i = e_i + v_i e_{n+1}, \quad N = v\nu = v\frac{v_i e_i + e_{n+1}}{w},$$

where $w = \sqrt{1 - |\nabla v|^2}$ and ν is the normal vector field of S viewed as a Minkowski space $R^{n,1}$ graph. The first and second fundamental forms g_{ij} and h_{ij} are given by

$$g_{ij} = \langle X_i, X_j \rangle_D = \frac{1}{v^2}(\delta_{ij} - v_i v_j), \tag{1.21}$$

$$h_{ij} = \langle \nabla_{X_i} X_j, v\nu \rangle_D = \frac{1}{v^2 w}(\delta_{ij} - v_i v_j - v v_{ij}) \tag{1.22}$$

respectively. Note that from (1.22), S is locally strictly convex if and only if

$$|y|^2 - v^2 \text{ is a (Euclidean) locally strictly convex function.} \tag{1.23}$$

There is a well known Gauss map duality for locally strictly convex hypersurfaces in dS_{n+1}. For our purposes we will need a very concrete formulation of this duality [17]. Montiel [15] showed that if we use the upper halfspace representation for both $\mathcal{H}^{n+1}$ and $\mathbb{H}^{n+1}$, the Gauss map N corresponds to the map $L : S \to \mathbb{H}^{n+1}$ defined by

$$L((y, v(y))) = (y - v(y)\nabla v(y), v(y)\sqrt{1 - |\nabla v|^2}), \quad y \in V. \tag{1.24}$$

We now identify the map L in terms of a hodograph map and its associated Legendre transform. Let $p(y) = \frac{1}{2}(|y|^2 - v(y)^2)$; since p is strictly convex in the Euclidean sense by (1.23), its gradient map $\nabla p : V \subset \mathbb{R}^n \to \mathbb{R}^n$ is globally one to one. Define

$$x = \nabla p(y), \quad u(x) := v(y)\sqrt{1 - |\nabla v(y)|^2}, \quad y \in V. \tag{1.25}$$

Then u is well defined in $\Omega := \nabla p(V)$. The associated Legendre transform is the function $q(x)$ defined in Ω by $p(y) + q(x) = x \cdot y$ or $q(x) = -p(y) + y \cdot \nabla p(y)$.

Theorem 1.8. [17]. *Let L be defined by* (1.24) *and x by* (1.25). *Then the image of S under L is the hyperbolic locally strictly convex graph in $\mathbb{H}^{n+1}$*

$$\Sigma = \{(x, u(x)) \in \mathbb{R}^{n+1}_+ : u \in C^\infty(\overline{\Omega}),\ u(x) > 0\}$$

with principal curvatures $\kappa_i^* = \kappa_i^{-1}$. Here $\kappa_1, \dots \kappa_n$ are the principal curvatures of S with respect to the induced de Sitter metric. Moreover the inverse map $L^{-1} : \Sigma \to S$

$$L^{-1}((x, u(x))) = (x + u(x)Du(x), u(x)\sqrt{1 + |Du(x)|^2}), \quad x \in \Omega$$

is the dual Legendre transform and hodograph map $y = Dq(x)$, $q(x) = \frac{1}{2}(|x|^2 + u(x)^2)$.

Note that when $\Sigma = \text{graph}(u)$ over Ω is a strictly locally convex solution of the asymptotic Plateau problem (1.1)-(1.2) in $\mathbb{H}^{n+1}$, then its Gauss image $S = \text{graph}(v)$ is a locally strictly convex spacelike graph also defined over Ω which solves the asymptotic Plateau problem $f^*(\kappa) = \frac{1}{\sigma} > 1$. We now define f^*.

Definition 1.9. Given a curvature function $f(\kappa)$ in the positive cone K_n^+, define the dual curvature function $f^*(\kappa)$ by

$$f^*(\kappa) := \frac{1}{f(\kappa_1^{-1}, \dots, \kappa_n^{-1})}, \quad \kappa \in K_n^+. \tag{1.26}$$

Note that f^* may in fact be naturally defined in a cone $K \supseteq K_n^+$. For example if $f(\kappa) = \left(\frac{H_n}{H_l}\right)^{\frac{1}{n-l}}$, $n > l \geq 0$ defined in K_n^+, then

$$f^*(\kappa) = \left(H_{n-l}\right)^{\frac{1}{n-l}}$$

is in fact defined in the standard Garding cone $K = \Gamma_{n-l}$.

Using the duality Theorem 1.8 we can transplant Theorem 1.1 to $\mathcal{H}^{n+1}$.

Theorem 1.10. *Suppose $\Gamma = \partial\Omega \in C^2$ for a bounded domain $\Omega \subset \mathbb{R}^n = \mathbb{R}^n \times \{0\}$ and $f(\kappa)$ satisfies conditions (1.3)-(1.5) and (1.7)-(1.8). Then for $\sigma > 1$, there exists a*

complete locally strictly convex spacelike hypersurface S in $\mathcal{H}^{n+1}$ satisfying $f^*(\kappa) = \sigma$ and $\partial S = \Gamma$ with uniformly bounded principal curvatures

$$C^{-1} \leq \kappa_i \leq C \quad \text{on } S. \tag{1.27}$$

Moreover $S = graph(v)$ with $v \in C^\infty(\Omega) \cap C^{0,1}(\overline{\Omega})$, $v^2 \in C^{1,1}(\overline{\Omega})$, $v|D^2v| + |Dv| \leq C$ and

$$\sqrt{1 - |Dv|^2} = \frac{1}{\sigma} \quad \text{on } \partial\Omega. \tag{1.28}$$

Corollary 1.11. *Under the assumptions of Theorem 1.10, there exists a complete locally strictly convex spacelike hypersurface S in $\mathcal{H}^{n+1}$ satisfying*

$$(H_l)^{\frac{1}{l}} = \sigma > 1, \quad 1 \leq l \leq n$$

with $\partial S = \Gamma$ and having uniformly bounded principal curvatures $C^{-1} \leq \kappa_i \leq C$ on S. Moreover, $S = graph(v)$ with $v \in C^\infty(\Omega) \cap C^{0,1}(\bar{\Omega})$, $v^2 \in C^{1,1}(\overline{\Omega})$, $v|D^2v| + |Dv| \leq C$. Further, if $l = 1$ or $l = 2$ (corresponding to mean curvature and normalized scalar curvature) or if $\partial\Omega$ is mean convex, we have uniqueness among convex solutions and even among all solutions (convex or not) if Ω is simply-connected.

The uniqueness part of Corollary 1.11 follows from Theorem 1.6 of [9] or Theorem 1.4 and a continuous deformation argument as used in [16]. Montiel [15] proved existence for $H = \sigma > 1$ (mean curvature) assuming $\partial\Omega$ is mean convex. Our result shows that for arbitrary Ω there is always a unique locally strictly convex solution. If Ω is mean convex the solutions constructed by Montiel must agree with the ones we construct.

An outline of the paper is as follows. In Section 2 we recall some important identities and estimates, most of them from [9], needed in the proof of our main technical result (Theorem 3.1), the "global interior curvature estimate". These identities and formulas are interesting and important in themselves and will orient the reader to our point of view. The proof of Theorem 3.1 is carried our in Section 3; Theorem 1.3 follows immediately. Theorem 1.5 and Theorem 1.4 are proved in Sections 4 and 5, respectively; the use of Theorem 1.3 is essential in these proofs.

In the following sections, f is always assumed to satisfy (1.3)-(1.8) in K_n^+.

2. Formulas on hypersurfaces and some basic identities

In this section we recall some basic properties of solutions of (1.1) derived in [9] that will be needed in the following sections to prove our main results.

In this paper all hypersurfaces in $\mathbb{H}^{n+1}$ we consider are assumed to be connected and orientable. If Σ is a complete hypersurface in $\mathbb{H}^{n+1}$ with compact asymptotic boundary at infinity, then the normal vector field of Σ is chosen to be the one pointing to the unique unbounded region in $\mathbb{R}^{n+1}_+ \setminus \Sigma$, and the (both hyperbolic and Euclidean) principal curvatures of Σ are calculated with respect to this normal vector field.

Let Σ be a hypersurface in $\mathbb{H}^{n+1}$. We shall use g and ∇ to denote the induced hyperbolic metric and Levi-Civita connection on Σ, respectively.

Let $\mathbf{x}$ and ν be the position vector and Euclidean unit normal vector of Σ in $\mathbb{R}^{n+1}$, respectively and set

$$u = \mathbf{x} \cdot \mathbf{e}, \quad \nu^{n+1} = \mathbf{e} \cdot \nu$$

where $\mathbf{e}$ is the unit vector in the positive x_{n+1} direction in $\mathbb{R}^{n+1}$, and '$\cdot$' denotes the Euclidean inner product in $\mathbb{R}^{n+1}$. We refer u as the *height function* of Σ. The hyperbolic unit normal vector is $\mathbf{n} = u\nu$.

Let $\tau_1, \ldots, \tau_n$ be local frames. The metric and second fundamental form of Σ are respectively given by

$$g_{ij} = \langle \tau_i, \tau_j \rangle, \quad h_{ij} = \langle D_{\tau_i} \tau_j, \mathbf{n} \rangle = -\langle D_{\tau_i} \mathbf{n}, \tau_j \rangle \tag{2.1}$$

where D denotes the Levi-Civita connection of $\mathbb{H}^{n+1}$. Throughout the paper we assume $\tau_1, \ldots, \tau_n$ are orthonormal so $g_{ij} = \delta_{ij}$. The principal curvatures of Σ are the eigenvalues of the second fundamental form $\{h_{ij}\}$ with respect to the metric $\{g_{ij}\}$. The following formula is derived in [9]

$$\nabla_{ij} \frac{1}{u} = \frac{1}{u}(g_{ij} - \nu^{n+1} h_{ij}). \tag{2.2}$$

Let $\mathcal{S}$ be the space of $n \times n$ symmetric matrices and $\mathcal{S}^+ = \{A \in \mathcal{S} : \lambda(A) \in K_n^+\}$, where $\lambda(A) = (\lambda_1, \ldots, \lambda_n)$ are the eigenvalues of A. Let F be the function defined by

$$F(A) = f(\lambda(A)), \quad A \in \mathcal{S}^+ \tag{2.3}$$

and denote

$$F^{ij}(A) = \frac{\partial F}{\partial a_{ij}}(A), \quad F^{ij,kl}(A) = \frac{\partial^2 F}{\partial a_{ij} \partial a_{kl}}(A). \tag{2.4}$$

We have $F^{ij}(A) = f_i(\lambda(A))\delta_{ij}$ when A is diagonal. Moreover,

$$F^{ij}(A)a_{ij} = \sum f_i(\lambda(A))\lambda_i = F(A), \tag{2.5}$$

$$F^{ij}(A)a_{ik}a_{jk} = \sum f_i(\lambda(A))\lambda_i^2. \tag{2.6}$$

Equation (1.1) can therefore be rewritten locally in the form

$$F(h_{ij}) = \sigma. \tag{2.7}$$

Denote $F^{ij} = F^{ij}(h_{ij})$, $F^{ij,kl} = F^{ij,kl}(h_{ij})$.

Lemma 2.1 ([9])**.** *Let Σ be a smooth hypersurface in $\mathbb{H}^{n+1}$ satisfying* (1.1)*. Then*

$$F^{ij}\nabla_{ij}\frac{1}{u} = -\frac{\sigma\nu^{n+1}}{u} + \frac{1}{u}\sum f_i, \tag{2.8}$$

$$F^{ij}\nabla_{ij}\frac{\nu^{n+1}}{u} = \frac{\sigma}{u} - \frac{\nu^{n+1}}{u}\sum f_i\kappa_i^2. \tag{2.9}$$

Using Lemma 2.1 one derives the following important maximum principle.

Theorem 2.2 ([9])**.** *Let Σ be a smooth strictly locally convex hypersurface in $\mathbb{H}^{n+1}$ satisfying equation* (1.1)*. Suppose Σ is globally a graph: $\Sigma = \{(x, u(x)) : x \in \Omega\}$ where Ω is a domain in $\mathbb{R}^n \equiv \partial\mathbb{H}^{n+1}$. Then*

$$F^{ij}\nabla_{ij}\frac{\sigma - \nu^{n+1}}{u} \geq \sigma(1-\sigma)\frac{(\sum f_i - 1)}{u} \geq 0. \tag{2.10}$$

Upper and lower bounds on $\partial\Omega$ for $\eta := \frac{\sigma-\nu^{n+1}}{u}$ follow from the following lemma which is based on comparisons with equidistant sphere solutions.

Lemma 2.3. *Assume that $\partial\Sigma$ satisfies a uniform interior and/or exterior ball condition and let u denote the height function of Σ with $u = \varepsilon$ on $\partial\Omega$. Then for $\varepsilon \geq 0$ sufficiently small,*

$$-\frac{\varepsilon\sqrt{1-\sigma^2}}{r_2} - \frac{\varepsilon^2(1+\sigma)}{r_2^2} < \nu^{n+1} - \sigma < \frac{\varepsilon\sqrt{1-\sigma^2}}{r_1} + \frac{\varepsilon^2(1-\sigma)}{r_1^2} \quad \text{on } \partial\Sigma \tag{2.11}$$

where r_2 and r_1 are the maximal radii of exterior and interior spheres to $\partial\Omega$, respectively. In particular, $\nu^{n+1} \to \sigma$ on $\partial\Sigma$ as $\varepsilon \to 0$.

Corollary 2.4.

$$\eta := \frac{\sigma - \nu^{n+1}}{u} \leq \sup_{\partial\Sigma}\frac{\sigma - \nu^{n+1}}{u} \quad \text{on } \Sigma. \tag{2.12}$$

Moreover, if $u = \epsilon > 0$ on $\partial\Omega$ (satisfying a uniform exterior ball condition), then there exists $\epsilon_0 > 0$ depending only on $\partial\Omega$, such that for all $\epsilon \leq \epsilon_0$,

$$\frac{\sigma - \nu^{n+1}}{u} \leq \frac{\sqrt{1-\sigma^2}}{r_2} + \frac{\varepsilon(1+\sigma)}{r_2^2} \quad \text{on } \Sigma \tag{2.13}$$

where r_2 is the maximal radius of exterior tangent spheres to $\partial\Omega$.

Proposition 2.5. *Let Σ be a smooth strictly locally convex graph*

$$\Sigma = \{(x, u(x)) : x \in \Omega\}$$

in $\mathbb{H}^{n+1}$ satisfying $u \geq \varepsilon$ in Ω, $u = \varepsilon$ on $\partial\Omega$. Then at an interior maximum of $\frac{u}{\nu^{n+1}}$ we have $\frac{u}{\nu^{n+1}} \leq \max_\Omega u$. Hence for ε small compared to σ,

$$\nu^{n+1} \geq \frac{u}{\max_\Omega u} \quad \text{in } \Omega \tag{2.14}$$

Proof. Let $h = \frac{u}{\nu^{n+1}} = uw$ and suppose that h assumes its maximum at an interior point x_0. Then at x_0,

$$\partial_i h = u_i w + u\frac{u_k u_{ki}}{w} = (\delta_{ki} + u_k u_i + u u_{ki})\frac{u_k}{w} = 0 \ \ \forall\, 1 \leq i \leq n.$$

Since Σ is strictly locally convex, this implies that $\nabla u = 0$ at x_0 so the proposition follows immediately from Corollary 2.4. □

Combining Theorem 2.2 and Proposition 2.5 gives

Corollary 2.6. *Let Σ be a smooth strictly locally convex graph*

$$\Sigma = \{(x, u(x)) : x \in \Omega\}$$

in $\mathbb{H}^{n+1}$ satisfying $u \geq \varepsilon$ in Ω, $u = \varepsilon$ on $\partial\Omega$. Assume that $\partial\Omega$ satisfies a uniform exterior ball condition. Then for ε sufficiently small compared to σ

$$\nu^{n+1} \geq 2a := \frac{\sigma}{1 + M \max_\Omega u} \tag{2.15}$$

where $M = \frac{\sqrt{1-\sigma^2}}{r_2} + \frac{\varepsilon(1+\sigma)}{r_2^2}$.

Proof. By Theorem 2.2 we have $\nu^{n+1} \geq \sigma - Mu$ while by Proposition 2.5 we have $\nu^{n+1} \geq \frac{u}{\max_\Omega u}$. Hence if $u \leq \lambda\sigma$ we find $\nu^{n+1} \geq \sigma(1 - \lambda M)$ while if $u \geq \lambda\sigma$ we find $\nu^{n+1} \geq \frac{\lambda\sigma}{\max_\Omega u}$. Choosing $\lambda = \frac{\max_\Omega u}{1 + M \max_\Omega u}$ completes the proof. □

3. The global interior curvature estimate

In this section we prove an interior curvature estimate (see Theorem 3.1 below) for the largest principal curvature of locally strictly convex graphs with uniformly bounded principal curvatures $0 < \kappa_i \leq C$ satisfying $f(\kappa) = \sigma$. What is remarkable is that the bound we obtain is independent of C and the "cutoff" function u^b which vanishes at $\partial\Omega$. Hence we can let b tend to zero to prove the global estimate Theorem 1.3.

Let Σ be a smooth strictly locally convex hypersurface in $\mathbb{H}^{n+1}$ satisfying $f(\kappa) = \sigma$ with $\partial\Sigma \subset \partial_\infty \mathbb{H}^{n+1}$. For a fixed point $\mathbf{x}_0 \in \Sigma$ we choose a local orthonormal frame $\tau_1, \ldots, \tau_n$ around $\mathbf{x}_0$ such that $h_{ij}(\mathbf{x}_0) = \kappa_i \delta_{ij}$. The calculations below are done at $\mathbf{x}_0$. For convenience we shall write $v_{ij} = \nabla_{ij} v$, $h_{ijk} = \nabla_k h_{ij}$, $h_{ijkl} = \nabla_{lk} h_{ij} = \nabla_l \nabla_k h_{ij}$, etc.

Since $\mathbb{H}^{n+1}$ has constant sectional curvature -1, by the Codazzi and Gauss equations we have $h_{ijk} = h_{ikj}$ and

$$h_{iijj} = h_{jjii} + (h_{ii}h_{jj} - 1)(h_{ii} - h_{jj}) = h_{jjii} + (\kappa_i\kappa_j - 1)(\kappa_i - \kappa_j). \tag{3.1}$$

Consequently for each fixed j,

$$F^{ii} h_{jjii} = F^{ii} h_{iijj} + (1 + \kappa_j^2) \sum f_i \kappa_i - \kappa_j \sum f_i - \kappa_j \sum \kappa_i^2 f_i. \tag{3.2}$$

Theorem 3.1. *Let Σ be a smooth strictly locally convex graph in $\mathbb{H}^{n+1}$ with uniformly bounded principal curvatures $0 < \kappa_i \leq C$ satisfying $f(\kappa) = \sigma$, $\partial_\infty \Sigma \subset \partial_\infty \mathbb{H}^{n+1}$ and*

$$\nu^{n+1} \geq 2a > 0 \text{ on } \Sigma. \tag{3.3}$$

For $\mathbf{x} \in \Sigma$ let $\kappa_{\max}(\mathbf{x})$ be the largest principal curvature of Σ at $\mathbf{x}$. Then for $0 < b \leq \frac{a}{4}$,

$$\max_\Sigma u^b \frac{\kappa_{\max}}{\nu^{n+1} - a} \leq \frac{8}{a^{\frac{5}{2}}} (\sup_\Sigma u)^b. \tag{3.4}$$

Proof. Let

$$M_0 = \sup_{\mathbf{x} \in \Sigma} u^b \frac{\kappa_{\max}(x)}{\nu^{n+1} - a}. \tag{3.5}$$

Since $\kappa_{\max}(\mathbf{x}) \leq C$, $M_0 > 0$ is attained at an interior point $\mathbf{x}_0 \in \Sigma$. Let $\tau_1, \ldots, \tau_n$ be a local orthonormal frame around $\mathbf{x}_0$ such that $h_{ij}(\mathbf{x}_0) = \kappa_i \delta_{ij}$, where $\kappa_1, \ldots, \kappa_n$ are the principal curvatures of Σ at $\mathbf{x}_0$. We may assume $\kappa_1 = \kappa_{\max}(\mathbf{x}_0)$. Thus, at $\mathbf{x}_0$, $u^b \frac{h_{11}}{\nu^{n+1} - a}$ has a local maximum and so

$$\frac{h_{11i}}{h_{11}} + b\frac{u_i}{u} - \frac{\nabla_i \nu^{n+1}}{\nu^{n+1} - a} = 0, \tag{3.6}$$

$$\frac{h_{11ii}}{h_{11}} + b\frac{u_{ii}}{u} - \frac{\nabla_{ii}\nu^{n+1}}{\nu^{n+1}-a} - (b+b^2)\frac{u_i^2}{u^2} + 2b\frac{u_i}{u}\frac{\nabla_i\nu^{n+1}}{\nu^{n+1}-a} \le 0. \tag{3.7}$$

Using (3.2), we find after differentiating the equation $F(h_{ij}) = \sigma$ twice that at $\mathbf{x}_0$,

$$F^{ii}h_{11ii} = -F^{ij,rs}h_{ij1}h_{rs1} + \sigma(1+\kappa_1^2) - \kappa_1\Big(\sum f_i + \sum \kappa_i^2 f_i\Big). \tag{3.8}$$

By Lemma 2.1 we immediately derive

$$\begin{aligned} F^{ij}\nabla_{ij}\nu^{n+1} =& \frac{2}{u}F^{ij}\nabla_i u\nabla_j\nu^{n+1} + \sigma(1+(\nu^{n+1})^2) \\ & - \nu^{n+1}\Big(\sum f_i + \sum f_i\kappa_i^2\Big), \end{aligned} \tag{3.9}$$

$$F^{ij}\frac{\nabla_{ij}u}{u} = 2\sum f_i\frac{u_i^2}{u^2} + \sigma\nu^{n+1} - \sum f_i. \tag{3.10}$$

By (3.7)-(3.10) we find

$$\begin{aligned} 0 \ge & -F^{ij,rs}h_{ij1}h_{rs1} + \sigma\Big(1+\kappa_1^2 - \frac{1+(\nu^{n+1})^2}{\nu^{n+1}-a}\kappa_1\Big) \\ & + \frac{a\kappa_1}{\nu^{n+1}-a}\Big(\sum f_i + \sum \kappa_i^2 f_i\Big) - b\kappa_1\sum f_i \\ & + (b-b^2)\kappa_1\sum f_i\frac{u_i^2}{u^2} - \frac{(2-2b)\kappa_1}{\nu^{n+1}-a}F^{ij}\frac{u_i}{u}\nabla_j\nu^{n+1}. \end{aligned} \tag{3.11}$$

Next we use an inequality due to Andrews [3] and Gerhardt [6] which states

$$-F^{ij,kl}h_{ij1}h_{kl,1} \ge \sum_{i\ne j}\frac{f_i-f_j}{\kappa_j-\kappa_i}h_{ij1}^2 \ge 2\sum_{i\ge 2}\frac{f_i-f_1}{\kappa_1-\kappa_i}h_{i11}^2. \tag{3.12}$$

Recall that (see [9])

$$\nabla_i\nu^{n+1} = \frac{u_i}{u}(\nu^{n+1}-\kappa_i).$$

Thus at $\mathbf{x}_0$ we obtain from (3.6)

$$h_{11i} = \kappa_1\frac{u_i}{u}\Big(\frac{\nu^{n+1}-\kappa_i}{\nu^{n+1}-a} - b\Big). \tag{3.13}$$

Inserting this into (3.12) we derive

$$-F^{ij,kl}h_{ij1}h_{kl,1} \ge 2{\kappa_1}^2\sum_{i\ge 2}\frac{f_i-f_1}{\kappa_1-\kappa_i}\frac{u_i^2}{u^2}\Big(\frac{\kappa_i-\nu^{n+1}}{\nu^{n+1}-a} + b\Big)^2. \tag{3.14}$$

Note that we may write

$$\sum f_i + \sum \kappa_i^2 f_i = (1-(\nu^{n+1})^2)\sum f_i + \sum(\kappa_i-\nu^{n+1})^2 f_i + 2\sigma\nu^{n+1}. \tag{3.15}$$

Combining (3.12), (3.14) and (3.15) gives at $\mathbf{x}_0$

$$
\begin{aligned}
0 \geq\, & \sigma\Big(1+\kappa_1^2-\frac{1+(\nu^{n+1})^2}{\nu^{n+1}-a}\kappa_1\Big)-b\kappa_1\sum f_i \\
& +(b-b^2)\sum f_i\frac{u_i^2}{u^2}+\frac{a\kappa_1}{2(\nu^{n+1}-a)}\Big(\sum f_i+\sum\kappa_i^2 f_i\Big) \\
& +\frac{a\kappa_1}{2(\nu^{n+1}-a)}\Big((1-(\nu^{n+1})^2)\sum f_i+\sum(\kappa_i-\nu^{n+1})^2 f_i+2\sigma\nu^{n+1}\Big) \\
& +2{\kappa_1}^2\sum_{i\geq 2}\frac{f_i-f_1}{\kappa_1-\kappa_i}\frac{u_i^2}{u^2}\Big(\frac{\kappa_i-\nu^{n+1}}{\nu^{n+1}-a}+b\Big)^2 \\
& +(2-2b)\kappa_1\sum f_i\frac{u_i^2}{u^2}\frac{\kappa_i-\nu^{n+1}}{\nu^{n+1}-a}.
\end{aligned}
\tag{3.16}
$$

Note that (assuming $\kappa_1 \geq \frac{2}{a}$ and $b \leq \frac{a}{4}$) all the terms of (3.16) are positive except possibly the ones in the last sum involving $(\kappa_i - \nu^{n+1})$ and only if $\kappa_i < \nu^{n+1}$.

For $\theta \in (0,1)$ to be chosen later, define

$$
\begin{aligned}
J &= \{i : \kappa_i - \nu^{n+1} < 0,\ f_i < \theta^{-1} f_1\}, \\
L &= \{i : \kappa_i - \nu^{n+1} < 0,\ f_i \geq \theta^{-1} f_1\}.
\end{aligned}
$$

Since $\sum u_i^2/u^2 = |\tilde{\nabla} u|^2 = 1-(\nu^{n+1})^2 \leq 1$, $\nu^{n+1} \geq 2a$ and $\kappa_i f_i \leq \sigma$ for each i, we derive

$$
\sum_{i\in J}(\kappa_i-\nu^{n+1})f_i\frac{u_i^2}{u^2} \geq -\frac{f_1}{\theta} \geq -\frac{\sigma}{\theta\kappa_1},
\tag{3.17}
$$

and

$$
\begin{aligned}
& 2\kappa_1^2\sum_{i\in L}\frac{f_i-f_1}{\kappa_1-\kappa_i}\frac{u_i^2}{u^2}\Big(\frac{\kappa_i-\nu^{n+1}}{\nu^{n+1}-a}+b\Big)^2+(2-2b)\kappa_1\sum_{i\in L}f_i\frac{u_i^2}{u^2}\frac{\kappa_i-\nu^{n+1}}{\nu^{n+1}-a} \\
& \geq 2(1-\theta)\kappa_1\sum_{i\in L}f_i\frac{u_i^2}{u^2}\Big(\frac{\kappa_i-\nu^{n+1}}{\nu^{n+1}-a}\Big)^2+(2+2b-4b\theta)\kappa_1\sum_{i\in L}f_i\frac{u_i^2}{u^2}\frac{(\kappa_i-\nu^{n+1})}{\nu^{n+1}-a} \\
& \geq \frac{2\kappa_1}{(\nu^{n+1}-a)^2}\sum_{i\in L}f_i\frac{u_i^2}{u^2}(\kappa_i^2-(a+\nu^{n+1})\kappa_i+a\nu^{n+1}) \\
& \quad -\frac{2\theta}{a}\frac{\kappa_1}{\nu^{n+1}-a}\sum_{i\in L}f_i(\kappa_i-\nu^{n+1})^2+2b(1-2\theta)\kappa_1\sum_{i\in L}f_i\frac{u_i^2}{u^2}\frac{(\kappa_i-\nu^{n+1})}{\nu^{n+1}-a} \\
& \geq -\frac{6\sigma}{a}\kappa_1-\frac{2b\kappa_1(1-(\nu^{n+1})^2)}{\nu^{n+1}-a}\sum f_i-\frac{2\theta\kappa_1}{a(\nu^{n+1}-a)}\sum_{i\in L}f_i(\kappa_i-\nu^{n+1})^2.
\end{aligned}
\tag{3.18}
$$

We now fix $\theta = \frac{a^2}{4}$ and $0 < b \leq \frac{a}{4}$. From (3.17) and (3.18) we see that the right hand side of (3.16) at $\mathbf{x}_0$ is strictly greater than

$$\sigma\Big(1 + \kappa_1^2 - \frac{8}{a}\kappa_1 - \frac{8}{a^3}\Big). \tag{3.19}$$

Then (3.19) is strictly positive if for example $\kappa_1 \geq 8a^{-\frac{3}{2}}$. Therefore $\kappa_1 \leq 8a^{-\frac{3}{2}}$ at $\mathbf{x}_0$, completing the proof of Theorem 3.1. □

4. Strict Euclidean starshapedness for convex solutions

In this section we prove Theorem 1.5 by direct construction in Theorem 4.3 below of a strictly starshaped locally strictly convex solution with boundary in the horosphere $\{x_{n+1} = \varepsilon\}$. By compactness and uniqueness we can then pass to the limit as ε tends to zero. We use the continuity method by deforming from the horosphere solution $u \equiv \varepsilon$ for $\sigma = 1$. Under this deformation we will show that the property of being strictly sharshaped, i.e. $\mathbf{x}\cdot\nu > 0$, persists as long as a solution exists. This property is intertwined with the demonstration that the full linearized operator has trivial kernel.

Suppose Σ is locally represented as the graph of a function $u \in C^2(\Omega)$, $u > 0$, in a domain $\Omega \subset \mathbb{R}^n$: $\Sigma = \{(x, u(x)) \in \mathbb{R}^{n+1} : \ x \in \Omega\}$, oriented by the upward (Euclidean) unit normal vector field ν to Σ:

$$\nu = \Big(\frac{-Du}{w}, \frac{1}{w}\Big), \ \ w = \sqrt{1 + |Du|^2}.$$

The Euclidean metric and second fundamental form of Σ are given respectively by

$$g^e_{ij} = \delta_{ij} + u_i u_j, \ h^e_{ij} = \frac{u_{ij}}{w}.$$

According to [5], the Euclidean principal curvatures $\kappa^e[\Sigma]$ are the eigenvalues of the symmetric matrix $A^e[u] = \{a^e_{ij}\}$:

$$a^e_{ij} := \frac{1}{w}\gamma^{ik}u_{kl}\gamma^{lj}, \ \ \gamma^{ij} = \delta_{ij} - \frac{u_i u_j}{w(1+w)}. \tag{4.1}$$

Note that the matrix $\{\gamma^{ij}\}$ is invertible and equal to the inverse square root of $\{g^e_{ij}\}$, i.e., $\gamma^{ik}\gamma^{kj} = (g^e)^{ij}$. By (1.13) the hyperbolic principal curvatures $\kappa[u]$ of Σ are the

eigenvalues of the matrix $A[u] = \{a_{ij}[u]\}$:

$$a_{ij}[u] := ua^e_{ij} + \frac{\delta_{ij}}{w} = \frac{1}{w}\Big(\delta_{ij} + u\gamma^{ik}u_{kl}\gamma^{lj}\Big). \tag{4.2}$$

Problem (1.1)-(1.2) reduces to the Dirichlet problem for a fully nonlinear second order equation which we shall write in the form

$$G(D^2u, Du, u) = \sigma, \quad u > 0 \quad \text{in } \Omega \subset \mathbb{R}^n \tag{4.3}$$

with the boundary condition

$$u = 0 \quad \text{on } \partial\Omega. \tag{4.4}$$

The function G in equation (4.3) is determined by $G(D^2u, Du, u) = F(A[u])$ where $A[u] = \{a_{ij}[u]\}$ is given by (4.2). Let

$$\mathcal{L} = G^{st}\partial_s\partial_t + G^s\partial_s + G_u \tag{4.5}$$

be the linearized operator of G at u, where

$$G^{st} = \frac{\partial G}{\partial u_{st}},\ G^s = \frac{\partial G}{\partial u_s},\ G_u = \frac{\partial G}{\partial u}. \tag{4.6}$$

We shall not need the exact formula for G^s but note that

$$G^{st} = \frac{u}{w}F^{ij}\gamma^{is}\gamma^{jt}, \quad G^{st}u_{st} = uG_u = G - \frac{1}{w}\sum F^{ii} \tag{4.7}$$

where $F^{ij} = F^{ij}(A[u])$, etc. Under condition (1.4) equation (4.3) is elliptic for u if $A[u] \in \mathcal{S}^+$, while (1.5) implies that $G(D^2u, Du, u)$ is concave with respect to D^2u.

Since $\mathbf{x}\cdot\nu = \frac{u-\sum x_k u_k}{w}$, the following lemma is important.

Lemma 4.1. *We have* $\mathcal{L}(u - \sum x_k u_k) = 0$.

Proof. Write $\mathcal{L} = L + G_u$. Note that $\mathcal{L}(u_k) = 0$ since horizontal translation is an isometry. We have

$$\mathcal{L}(x_k u_k) = x_k\mathcal{L}(u_k) + u_k L(x_k) + 2G^{ij}\delta_{ki}u_{kj} = u_k G^k + 2G^{ij}u_{ij} = \mathcal{L}u$$

since $G^{ij}u_{ij} = uG_u$. □

Lemma 4.2. *Suppose* $\mathcal{L}\phi = 0$ *in* Ω, $\phi = 0$ *on* $\partial\Omega$ *and there exists* $v > 0$ *in* $\overline{\Omega}$ *satisfying* $\mathcal{L}v = 0$. *Then* $\phi \equiv 0$.

Proof. Set $h = \frac{\phi}{v}$. A simple computation shows that

$$Lh + 2G^{ij}\frac{v_i}{v}h_j = 0 \text{ in } \Omega, \quad h = 0 \text{ on } \partial\Omega.$$

The lemma now follows by the maximum principle. □

Theorem 4.3. *Let Ω be a strictly starshaped $C^{2,\alpha}$ domain with respect to the origin. Suppose f satisfies* (1.12) *in addition to* (1.3)-(1.8)*. There exists a unique solution $u \in C^\infty(\overline{\Omega})$ of the Dirichlet problem*

$$G(D^2u, Du, u) = \sigma \ \text{ in } \Omega, \ \ u = \varepsilon \ \text{ on } \partial\Omega. \tag{4.8}$$

Moreover, the hypersurface $\Sigma = graph(u)$ is strictly starshaped with respect to the origin. More precisely, there exist constants $c_0, \varepsilon_0 > 0$ such that for all $0 < \varepsilon \le \varepsilon_0$,

$$\mathbf{x} \cdot \nu \ge \frac{c_0 \nu^{n+1}\sqrt{1-\sigma^2}}{\sigma} \min_{x\in\partial\Omega} x \cdot \mathbf{N} \ \text{ on } \Sigma \tag{4.9}$$

where $\mathbf{N}$ is the exterior unit normal to $\partial\Omega$.

Proof. Consider for $0 \le t \le 1$, the family of Dirichlet problems

$$\begin{aligned} G(D^2u^t, Du^t, u^t) = \sigma^t := t\sigma + (1-t) \ \text{ in } \Omega, \\ u^t = \varepsilon \ \text{ on } \partial\Omega. \end{aligned} \tag{4.10}$$

Starting from $u^0 \equiv \varepsilon$ we shall use the continuity method to prove for any $t \in [0,1]$ that the Dirichlet problem (4.10) has a unique solution $u^t \in C^\infty(\overline{\Omega})$. Let S be the set of all such t; we know $0 \in S$ so S is not empty.

From the estimates derived in [10] and [9] we have

$$|(u^t)^2|_{C^2(\overline{\Omega})} \le C \ \ \forall\, t \in S \tag{4.11}$$

where C depends only on σ and the exterior ball condition satisfied by Ω but is independent of t and ε. This shows that S is a closed set.

Next, let $t \in S$ and denote $w^t = \sqrt{1+|Du^t|^2}$, $\mathbf{x}^t = (x, u^t(x))$. Then $w^t\mathbf{x}^t \cdot \nu^t = u^t - \sum x_k u^t_k > 0$ and therefore $\mathcal{L}^t(w^t\mathbf{x}^t \cdot \nu^t) = 0$ in Ω by Lemma 4.1. Since $\partial\Omega$ is strictly starshaped, by the maximum principle

$$w^t\mathbf{x}^t \cdot \nu^t \ge \min_{\partial\Omega} w^t\mathbf{x}^t \cdot \nu^t = \min_{\partial\Omega}(u^t - x_k u^t_k) = \min_{\partial\Omega}(\varepsilon + |\nabla u^t| x \cdot \mathbf{N}) > \varepsilon. \tag{4.12}$$

By Lemma 4.2, $\mathcal{L}^t$ has trivial kernel. This shows S is open in $[0,1]$, which is a standard consequence in elliptic theory of the implicit function theorem. Therefore $S = [0,1]$, proving the solvability of the Dirichlet problem (4.8). The uniform starshapeness estimate (4.9) follows from (4.12) and Lemma 2.3. □

Proof of Theorem 1.5. Given f satisfying (1.3)-(1.8), let $f^\theta := (1-\theta)f + \theta H_n^{\frac{1}{n}}$, $0 < \theta < 1$, which satisfies (1.12) in addition to (1.3)-(1.8). By Theorem 4.3 we obtain a

unique solution $u^{\theta,\varepsilon} \in C^\infty(\overline{\Omega})$ of the approximate problem $f^\theta(\kappa[u^\theta]) = \sigma$ with $u^{\theta,\varepsilon} = \varepsilon$ on $\partial\Omega$. Moreover, by (4.11)

$$|(u^{\theta,\varepsilon})^2|_{C^2(\overline{\Omega})} \leq C \text{ independent of } \varepsilon. \tag{4.13}$$

Letting $\varepsilon \to 0$ we obtain a solution u^θ of the asymptotic problem for $f^\theta = \sigma$. By Theorem 1.3 the principal curvatures of $\Sigma^\theta = \text{graph}(u^\theta)$ are uniformly bounded by a constant C depending only on Ω and σ. Hence as $\theta \to 0$ we obtain by passing to a subsequence a smooth locally strictly convex Σ satisfying (1.1)-(1.2) and (4.9). □

5. Uniqueness for mean convex Ω

In this section we prove Theorem 1.4. We shall assume Ω is a $C^{2,\alpha}$ domain with Euclidean mean curvature $\mathcal{H}_{\partial\Omega} \geq 0$.

The main step is to show there is always a solution $\Sigma_2 = \text{graph}(u)$ of the asymptotic problem (1.1)-(1.2) in Ω with $G_u < 0$ and moreover $u \leq v$ for any other solution $\Sigma_1 = \text{graph}(v)$. Then we show that Σ_2 is the unique solution. The proof we give is slightly circuitous in order to avoid delicate issues of boundary regularity caused by the degeneracy of the problem at the asymptotic boundary.

Proposition 5.1. *Let $0 < \sigma < 1$ and $u \in C^2(\overline{\Omega})$ be a solution of the Dirichlet problem* (4.8) *for $\varepsilon > 0$. Then $G_u < 0$ in $\overline{\Omega}$. Consequently, the linearized operator $\mathcal{L}$ satisfies the maximum principle and so has trivial kernel.*

Proof. Let $\Sigma = \text{graph}(u)$ and $\eta \equiv \frac{\sigma - \nu^{n+1}}{u}$. Since $G_u \leq \eta$ by (4.7), we only need to show $\eta < 0$ in $\overline{\Omega}$. According to Theorem 2.2, η must achieve its maximum at a boundary point $0 \in \partial\Omega$. We choose coordinates so that the x_n direction is the interior unit normal to $\partial\Omega$ at 0 where

$$\eta_n = \frac{u_n u_{nn}}{u w^3} - \eta \frac{u_n}{u} < 0, \text{ or equivalently, } \frac{u_{nn}}{w^3} < \eta. \tag{5.1}$$

On the other hand, by assumptions (1.5) and (1.8),

$$f(\kappa) \leq \sum f_i(\mathbf{1})\kappa_i = \sum \kappa_i / n.$$

That is the hyperbolic mean curvature $H(\Sigma) \geq \sigma$ and therefore, equivalently,

$$\frac{1}{w}\Big(\delta_{ij} - \frac{u_i u_j}{w^2}\Big) u_{ij} \geq n\eta. \tag{5.2}$$

Since $\sum_{\alpha<n} u_{\alpha\alpha} = -u_n(n-1)\mathcal{H}_{\partial D}$, restricting (5.2) to $\partial\Omega$ implies

$$\frac{u_{nn}}{w^3} - \frac{u_n}{w}(n-1)\mathcal{H}_{\partial\Omega} \geq n\eta \tag{5.3}$$

Combining (5.1) and (5.3) yields $w\eta(0) < -u_n\mathcal{H}_{\partial\Omega} \leq 0$. By Theorem 2.2 and the maximum principle we obtain $\eta < 0$ in $\overline{\Omega}$. □

Proposition 5.2. *Let $\sigma \in (0,1)$. There exist a solution $u \in C^\infty(\Omega) \cap C^{0,1}(\overline{\Omega})$ of the Dirchlet problem* (4.3)-(4.4) *satisfying $|u^2|_{C^2(\overline{\Omega})} \leq C$ and $G_u < 0$ in Ω.*

Proof. We first assume that f satisfies (1.12) in additon to (1.3)-(1.8). By an existence theorem in [10], for ε sufficiently small we obtain a solution $u \in C^\infty(\overline{\Omega})$ of the Dirichlet problem (4.8). By Proposition 5.1, $G_u < 0$ in $\overline{\Omega}$. Therefore the linearized operator at u satisfies the maximum principle and so has trivial kernel.

By the estimates in [10] and [9] we have $|u^2|_{C^2(\overline{\Omega})} \leq C$ independent of ε. Letting ε tend to 0 we prove Proposition 5.2 assuming (1.12).

To remove the assumption (1.12) we consider f^θ in place of f as in the proof of Theorem 1.5. From the above proof we obtain a solution u^θ of the asymptotic problem for $f^\theta = \sigma$ with $u^\theta = 0$ on $\partial\Omega$. By Theorem 1.3 the principal curvatures of $\Sigma^\theta = \text{graph}(u^\theta)$ are uniformly bounded by a constant C depending only $\partial\Omega$ and σ. Let θ tend to 0 and note that the condition $G_u \leq 0$ is preserved in the limiting process and therefore $G_u < 0$ in Ω by Theorem 2.2 and the strong maximum principle. We finish the proof of Proposition 5.2. □

Let $\hat{u}$ denote the solution of (4.3)-(4.4) constructed in Proposition 5.2. Theorem 1.4 follows from the following

Proposition 5.3. *Let $v \in C^2(\Omega) \cap C^0(\overline{\Omega})$ be a solution of the Dirchlet problem* (4.3)-(4.4)*. Then $v = \hat{u}$.*

Proof. We first prove $v \geq \hat{u}$; the strict inequality holds in Ω unless $v \equiv \hat{u}$. Let $0 < t \leq 1$, $\epsilon > 0$ and $\Omega_\epsilon = \{x \in \Omega : d(x, \partial\Omega) > \epsilon\}$. For ϵ sufficiently small, $\partial\Omega_\epsilon \in C^{2,\alpha}$ and $\mathcal{H}_{\partial\Omega_\epsilon} \geq 0$. Applying Proposition 5.2, let $\hat{u}^{\epsilon,t} \in C^\infty(\Omega_\epsilon)$ be the solution constructed in Proposition 5.2 of the Dirichlet problem (4.3)-(4.4) in Ω_ϵ with σ replaced by $\sigma_t = (1-t) + t\sigma$. Note that $\sigma_t > \sigma$ and $v > 0 = \hat{u}^{\epsilon,t}$ on $\partial\Omega_\epsilon$ for all $0 < t < 1$, and $v > \hat{u}^{\epsilon,t}$ in Ω_ϵ for t close to 0. By the maximum principle this property must continue to hold until $t = 1$. Thus as $\epsilon \to 0$ we obtain $v \geq \hat{u}$. Thus $v > \hat{u}$ in Ω or $v \equiv \hat{u}$.

Suppose now for contradiction that

$$\max_{\Omega}(v-\hat{u}) = v(x_0) - \hat{u}(x_0) > 0.$$

Set $w^t := tv + (1-t)\hat{u}$. We claim that graph(w^t) is locally strictly convex, that is, $(w^t)^2 + |x-x_0|^2$ is strictly Euclidean convex, in a small neighborhood of x_0. At x_0, $\nabla v = \nabla \hat{u}$ and $D^2 v \leq D^2 \hat{u}$. A simple computation shows

$$w^t w^t_{ij} - tvv_{ij} - (1-t)\hat{u}\hat{u}_{ij} = t(1-t)(v-\hat{u})(\hat{u}_{ij} - v_{ij}) \geq 0 \ \text{ at } x_0.$$

Hence at x_0,

$$w^t w^t_{ij} + w^t_i w^t_j + \delta_{ij} \geq t(vv_{ij} + v_i v_j + \delta_{ij}) + (1-t)(\hat{u}\hat{u}_{ij} + \hat{u}_i\hat{u}_j + \delta_{ij}) > 0$$

and the claim follows. So $G(D^2 w^t, Dw^t, w^t)$ is well defined near x_0.

Note that $\frac{d}{dt}G(D^2w^t, Dw^t, w^t) = \mathcal{L}^t w$ near x_0 where $w = v - \hat{u}$. Evaluating at $t=0$ gives

$$\frac{d}{dt}G(D^2w^t, Dw^t, w^t)(x_0)\Big|_{t=0} = G^{ij}\Big|_{\hat{u}} w_{ij}(x_0) + G_u\Big|_{\hat{u}} w(x_0) < 0.$$

Hence for $t > 0$ small enough, $\varphi(t) := G(D^2w^t, Dw^t, w^t)(x_0) < \sigma$. In particular there is a $t_0 \in (0,1]$ such that

$$\varphi(t_0) = \sigma,\ \varphi(t) < \sigma \ \text{ on } (0, t_0).$$

Using the integral form of the mean value theorem, we may write

$$0 = \varphi(t_0) - \varphi(0) = [a^{ij}w_{ij} + b^s w_s + c(x)w](x_0) := Lw(x_0) + c(x_0)w(x_0)\ ,$$

where

$$a^{ij}(x) = \int_0^{t_0} G^{ij}\Big|_{w^t} dt,\ b^s(x) = \int_0^{t_0} G^s\Big|_{w^t} dt,\ c(x) = \int_0^{t_0} G_u\Big|_{w^t} dt.$$

Since graph(w^t) is hyperbolic locally strictly convex in a small neighborhood of x_0, the operator $L = a^{ij}\frac{\partial^2}{\partial x_i \partial x_j} + b^s \frac{\partial}{\partial x_s}$ is elliptic in this neighborhood. Suppose for the moment that also $c(x_0) < 0$. Then $Lw(x_0) = -c(x_0)w(x_0) > 0$ and w has a strict interior maximum at x_0 contradicting the maximum principle.

We show $c(x_0) < 0$ to complete the proof. According to (4.7),

$$w^t G_u\Big|_{w^t}(x_0) \leq \varphi(t) - \frac{1}{\sqrt{1+|Dw^t(x_0)|^2}} < \sigma - \frac{1}{\sqrt{1+|D\hat{u}(x_0)|^2}} < 0 \ \text{ on } (0, t_0).$$

Hence $c(x_0) = \int_0^{t_0} G_u|_{w^t}(x_0)dt < 0$. □

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