

CONVEX SPACELIKE HYPERSURFACES OF CONSTANT CURVATURE IN DE SITTER SPACE

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Dedicated to Avner Friedman on the occasion of his 80th birthday

1. INTRODUCTION

Let $\mathbb{R}^{n+1,1}$ be the $(n+2)$ dimensional Minkowski space, that is, the real vector space \mathbb{R}^{n+2} endowed with the Lorentz metric

$$(1.1) \quad \langle u, v \rangle = u_0 v_0 + \dots + u_n v_n - u_{n+1} v_{n+1}$$

for all $u, v \in \mathbb{R}^{n+2}$. The one sheeted hyperboloid

$$dS_{n+1} = \{p \in \mathbb{R}^{n+2} | \langle p, p \rangle = 1\}$$

consisting of all unit spacelike vectors and equipped with the induced metric is called de Sitter space. It is a geodesically complete simply connected Lorentzian manifold with constant curvature one. De Sitter space corresponds to a vacuum solution of the Einstein equations with a positive cosmological constant.

Choose a non-zero null vector $\mathbf{a} \in \mathbb{R}^{n+1,1}$ in the past half of the null cone with vertex at the origin, i.e. $\langle \mathbf{a}, \mathbf{a} \rangle = 0$, $\langle \mathbf{a}, \mathbf{e} \rangle > 0$ where $\mathbf{e} = (0, \dots, 0, 1)$. Then the open region of de Sitter space defined by

$$(1.2) \quad \mathcal{H} = \{p \in dS_{n+1} | \langle p, \mathbf{a} \rangle > 0\}$$

is called the *steady state space*. Since the steady state space is only half the de Sitter space, it is incomplete. Its boundary as a subset of dS_{n+1} is the null hypersurface

$$(1.3) \quad L_0 = \{p \in dS_{n+1} | \langle p, \mathbf{a} \rangle = 0\}$$

which represents the past infinity of \mathcal{H} . The spacelike hypersurfaces

$$(1.4) \quad L_\tau = \{p \in dS_{n+1} | \langle p, \mathbf{a} \rangle = \tau, 0 < \tau < \infty\}$$

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are umbilic hypersurfaces of de Sitter space with constant mean curvature one with respect to the past oriented unit normal $N_\tau(p) = -p + \frac{1}{\tau}\mathbf{a}$ and foliate the steady state space. The limit boundary L_∞ represents a spacelike future infinity for timelike and null lines of de Sitter space [8].

In this paper we are interested in finding complete spacelike (i.e. the induced metric is Riemannian) strictly locally convex immersions $\psi : \Sigma^n \rightarrow \mathcal{H}^{n+1}$ with constant curvature and with prescribed (compact) future asymptotic boundary Γ . That is we want to find Σ satisfying

$$(1.5) \quad f(\kappa[\Sigma]) = \sigma$$

$$(1.6) \quad \partial\Sigma = \Gamma$$

where $\kappa[\Sigma] = (\kappa_1, \dots, \kappa_n)$ denote the positive principal curvatures of Σ (in the induced de Sitter metric with respect to the future oriented unit normal N) and $\sigma > 1$ is a constant. This is the exact analogue of the problem considered by Guan and Spruck [7] in hyperbolic space and as we shall make precise later, the two problems are essentially dual equivalent problems. Our study was motivated by the beautiful paper of Montiel [9] who treated the case of mean curvature.

As in our earlier work [10, 12, 4, 5, 6, 7], we prefer to use the half space model because we find it has great advantages. Following Montiel [9] we can define a half space model for the steady state space \mathcal{H} in the following way. Define the map $\phi : \mathbb{R}^{n+1,1} \setminus \{p \in \mathbb{R}^{n+1,1} | \langle p, \mathbf{a} \rangle = 0\} \rightarrow \mathbb{R}^n \times \mathbb{R} = \mathbb{R}^{n+1}$ given by

$$(1.7) \quad \phi(p) = \frac{1}{\langle p, \mathbf{a} \rangle} (p - \langle p, \mathbf{a} \rangle \mathbf{b} - \langle p, \mathbf{b} \rangle \mathbf{a}, 1)$$

where $\mathbf{b} \in \mathbb{R}^{n+2}$ is a null vector such that $\langle \mathbf{a}, \mathbf{b} \rangle = 1$ (\mathbf{b} is in the future directed null cone and \mathbb{R}^n stands for the orthogonal complement of the Lorentz plane spanned by \mathbf{a} and \mathbf{b}). Then the image of \mathcal{H} by the map ϕ lies in the half space $\mathbb{R}_+^{n+1} = \mathbb{R}^n \times \mathbb{R}_+$ and ϕ restricted to \mathcal{H} is a diffeomorphism to \mathbb{R}_+^{n+1} . Moreover for $v \in T_p \mathcal{H}^{n+1} = T_p dS_{n+1}$,

$$(1.8) \quad (d\phi)_p(v) = \frac{1}{\langle p, \mathbf{a} \rangle} (v - \langle v, \mathbf{a} \rangle \mathbf{b} - \langle v, \mathbf{b} \rangle \mathbf{a}, 0) - \frac{\langle v, \mathbf{a} \rangle}{\langle p, \mathbf{a} \rangle^2} (p - \langle p, \mathbf{a} \rangle \mathbf{b} - \langle p, \mathbf{b} \rangle \mathbf{a}, 1).$$

It follows that

$$\langle (d\phi)_p(v), (d\phi)_p(v) \rangle = \frac{1}{\langle p, \mathbf{a} \rangle^2} \langle v, v \rangle.$$

Hence the map $\phi : \mathcal{H}^{n+1} \rightarrow \mathbb{R}_+^{n+1} = \mathbb{R}^n \times \mathbb{R}_+$ is an isometry if \mathbb{R}_+^{n+1} is endowed with the Lorentz metric

$$(1.9) \quad g_{(x, x_{n+1})} = \frac{1}{x_{n+1}^2} (dx^2 - dx_{n+1}^2),$$

which is called the half space model for \mathcal{H}^{n+1} . It is important to note that *the isometry ϕ reverses the time orientation*.

Thus $\partial_\infty \mathcal{H}^{n+1}$ is naturally identified with $\mathbb{R}^n = \mathbb{R}^n \times \{0\} \subset \mathbb{R}^{n+1}$ and (1.6) may be understood in the Euclidean sense. For convenience we say Σ has compact asymptotic boundary if $\partial\Sigma \subset \partial_\infty \mathcal{H}^{n+1}$ is compact with respect the Euclidean metric in \mathbb{R}^n .

The curvature function $f(\lambda)$ in (1.5) is assumed to satisfy the fundamental structure conditions in the convex cone

$$(1.10) \quad K := K_n^+ := \{ \lambda \in \mathbb{R}^n : \text{each component } \lambda_i > 0 \} :$$

$$(1.11) \quad f \text{ is symmetric,}$$

$$(1.12) \quad f_i(\lambda) \equiv \frac{\partial f(\lambda)}{\partial \lambda_i} > 0 \text{ in } K, \quad 1 \leq i \leq n,$$

$$(1.13) \quad f \text{ is a concave function in } K,$$

$$(1.14) \quad \text{the dual function } f^*(\lambda) = (f(\frac{1}{\lambda_1}, \dots, \frac{1}{\lambda_n}))^{-1} \text{ is also concave in } K,$$

$$(1.15) \quad f > 0 \text{ in } K, \quad f = 0 \text{ on } \partial K$$

In addition, we shall assume that f is normalized

$$(1.16) \quad f(1, \dots, 1) = 1,$$

$$(1.17) \quad f \text{ is homogeneous of degree one}$$

and satisfies the following more technical assumption

$$(1.18) \quad \lim_{R \rightarrow +\infty} f(\lambda_1, \dots, \lambda_{n-1}, \lambda_n + R) \geq 1 + \varepsilon_0 \quad \text{uniformly in } B_{\delta_0}(\mathbf{1})$$

for some fixed $\varepsilon_0 > 0$ and $\delta_0 > 0$, where $B_{\delta_0}(\mathbf{1})$ is the ball of radius δ_0 centered at $\mathbf{1} = (1, \dots, 1) \in \mathbb{R}^n$.

The assumption (1.14) is closely related to the well-known fact [11], [3], [13] that the Gauss map \mathbf{n} of a spacelike locally strictly convex hypersurface Σ^n in de Sitter space is an embedding into hyperbolic space H^{n+1} which inverts principal curvatures. We shall formulate a precise global version of this correspondence (see Theorem 2.2 and Corollary 2.3 in Section 2) which will be important for our deliberations. For the moment note that if $f = (\sigma_n/\sigma_l)^{\frac{1}{n-l}}$, $0 \leq l < n$, defined in K where σ_l is the normalized l -th elementary symmetric polynomial ($\sigma_0 = 1$), then $f^*(\lambda) = (\sigma_{n-l}(\lambda))^{\frac{1}{n-l}}$. Also one easily computes that

$$\lim_{R \rightarrow +\infty} f(\lambda_1, \dots, \lambda_{n-1}, \lambda_n + R) = \left(\frac{n}{l}\right)^{\frac{1}{n-l}}.$$

Since f is symmetric, by (1.13), (1.16) and (1.17) we have

$$(1.19) \quad f(\lambda) \leq f(\mathbf{1}) + \sum f_i(\mathbf{1})(\lambda_i - 1) = \sum f_i(\mathbf{1})\lambda_i = \frac{1}{n} \sum \lambda_i \text{ in } K \subset K_1$$

and

$$(1.20) \quad \sum f_i(\lambda) = f(\lambda) + \sum f_i(\lambda)(1 - \lambda_i) \geq f(\mathbf{1}) = 1 \text{ in } K.$$

Using (1.14), we see that $\sum \lambda_i^2 f_i^*(\lambda) = (f^*)^2 \sum f_i(\frac{1}{\lambda}) \geq (f^*)^2$. Since $(f^*)^* = f$, it follows that

$$(1.21) \quad \sum \lambda_i^2 f_i \geq f^2 \text{ in } K.$$

In this paper all hypersurfaces in \mathcal{H}^{n+1} we consider are assumed to be connected and orientable. If Σ is a complete spacelike hypersurface in \mathcal{H}^{n+1} with compact asymptotic boundary at infinity, then the normal vector field of Σ is chosen to be the one pointing to the unique unbounded region in $\mathbb{R}_+^{n+1} \setminus \Sigma$, and the (both de Sitter and Minkowski) principal curvatures of Σ are calculated with respect to this normal vector field.

Because Σ is strictly locally convex and strictly spacelike, we are forced to take $\Gamma = \partial\Omega$ where $\Omega \subset \mathbb{R}^n$ is a smooth domain and seek Σ as the graph of a “spacelike” function $u(x)$ over Ω , i.e.

$$(1.22) \quad \Sigma = \{(x, x_{n+1}) : x \in \Omega, x_{n+1} = u(x)\}, \quad |\nabla u| < 1, \text{ in } \overline{\Omega}.$$

We will compute the first and second fundamental forms g_{ij}, h_{ij} with respect to the induced de Sitter metric as well as $\tilde{g}_{ij}, \tilde{h}_{ij}$ the corresponding forms in the induced

Minkowski metric viewing Σ as a graph in the Minkowski space $\mathbb{R}^{n,1}$ with unit normal ν . We use

$$X_i = e_i + u_i e_{n+1}, \quad \mathbf{n} = u\nu = u \frac{u_i e_i + e_{n+1}}{w},$$

where $w = \sqrt{1 - |\nabla u|^2}$. The first fundamental form g_{ij} is then given by

$$(1.23) \quad g_{ij} = \langle X_i, X_j \rangle_D = \frac{1}{u^2}(\delta_{ij} - u_i u_j) = \frac{\tilde{g}_{ij}}{u^2}.$$

For computing the second fundamental form we use

$$(1.24) \quad \Gamma_{ij}^{n+1} = -\frac{1}{x_{n+1}}\delta_{ij}, \quad \Gamma_{in+1}^k = -\frac{1}{x_{n+1}}\delta_{ik}$$

to obtain

$$(1.25) \quad \nabla_{X_i} X_j = \left(-\frac{\delta_{ij}}{x_{n+1}} + u_{ij} - \frac{u_i u_j}{x_{n+1}} \right) e_{n+1} - \frac{u_j e_i + u_i e_j}{x_{n+1}}.$$

Then

$$(1.26) \quad \begin{aligned} h_{ij} &= \langle \nabla_{X_i} X_j, u\nu \rangle_D = \frac{1}{uw} \left(\frac{\delta_{ij}}{u} - u_{ij} + \frac{u_i u_j}{u} - 2 \frac{u_i u_j}{u} \right) \\ &= \frac{1}{u^2 w} (\delta_{ij} - u_i u_j - u u_{ij}) = \frac{\tilde{h}_{ij}}{u} + \frac{\tilde{g}_{ij}}{u^2 w}. \end{aligned}$$

The principal curvature κ_i of Σ in de Sitter space are the roots of the characteristic equation

$$\det(h_{ij} - \kappa g_{ij}) = u^{-n} \det \left(\tilde{h}_{ij} - \frac{1}{u} \left(\kappa - \frac{1}{w} \right) \tilde{g}_{ij} \right) = 0.$$

Therefore,

$$(1.27) \quad \kappa_i = u \tilde{\kappa}_i + \frac{1}{w}, \quad i = 1, \dots, n.$$

Note that from (1.26), Σ is locally strictly convex if and only if

$$(1.28) \quad x^2 - u^2 \text{ is (Euclidean) locally strictly convex.}$$

As in our earlier work, we write the Minkowski principal curvatures $\tilde{\kappa}[\Sigma]$ as the eigenvalues of the symmetric matrix $\tilde{A}[u] = \{\tilde{a}_{ij}\}$:

$$(1.29) \quad \tilde{a}_{ij} := -\frac{1}{w} \gamma^{ik} u_{kl} \gamma^{lj}$$

where

$$(1.30) \quad \gamma^{ij} = \delta_{ij} + \frac{u_i u_j}{w(1+w)}.$$

By (1.27) the de Sitter principal curvatures $\kappa[u]$ of Σ are the eigenvalues of the symmetric matrix $A[u] = \{a_{ij}[u]\}$:

$$(1.31) \quad a_{ij}[u] := \frac{1}{w} (\delta_{ij} - u\gamma^{ik}u_{kl}\gamma^{lj}).$$

Define

$$(1.32) \quad F(A) := f(\kappa[A]) \quad \text{and} \quad G(D^2u, Du, u) := F(A[u])$$

where $A[u] = \{a_{ij}[u]\}$ is given by (1.31). Problem (1.5)-(1.6) then reduces to a Dirichlet problem for a fully nonlinear second order equation

$$(1.33) \quad G(D^2u, Du, u) = \sigma > 1, \quad u > 0 \quad \text{in } \Omega \subset \mathbb{R}^n$$

with the boundary condition

$$(1.34) \quad u = 0 \quad \text{on } \partial\Omega.$$

We seek solutions of equation (1.33) satisfying the spacelike condition (1.11) and (1.28). Following the literature we call such solutions *admissible*. By [2] condition (1.28) implies that equation (1.33) is elliptic for admissible solutions. Our goal is to show that the Dirichlet problem (1.33)-(1.34) admits smooth admissible solutions for all $\sigma > 1$ which is optimal.

Our main result of the paper may be stated as follows.

Theorem 1.1. *Let $\Gamma = \partial\Omega \times \{0\} \subset \partial_\infty \mathcal{H}^{n+1}$ where Ω is a bounded smooth domain in \mathbb{R}^n . Suppose that $\sigma > 1$ and that f satisfies conditions (1.11)-(1.18) with $K = K_n^+$. Then there exists a complete locally strictly convex spacelike hypersurface Σ in \mathcal{H}^{n+1} satisfying (1.5)-(1.6) with uniformly bounded principal curvatures*

$$(1.35) \quad |\kappa[\Sigma]| \leq C \quad \text{on } \Sigma.$$

Moreover, Σ is the graph of an admissible solution $u \in C^\infty(\Omega) \cap C^1(\bar{\Omega})$ of the Dirichlet problem (1.33)-(1.34). Furthermore, $u^2 \in C^\infty(\Omega) \cap C^{1,1}(\bar{\Omega})$ and

$$(1.36) \quad \begin{aligned} u|D^2u| &\leq C \quad \text{in } \Omega, \\ \sqrt{1 - |Du|^2} &= \frac{1}{\sigma} \quad \text{on } \partial\Omega \end{aligned}$$

As a concrete application we have existence for the canonical curvature functions.

Corollary 1.2. *Let $\Gamma = \partial\Omega \times \{0\} \subset \partial_\infty \mathcal{H}^{n+1}$ where Ω is a bounded smooth domain in \mathbb{R}^n . Then there exists a complete locally strictly convex spacelike hypersurface Σ in \mathcal{H}^{n+1} satisfying*

$$(\sigma_n/\sigma_l)^{\frac{1}{n-l}} = \sigma > 1, \quad 0 \leq l < n$$

with $\partial\Sigma = \Gamma$ and uniformly bounded principal curvatures

$$(1.37) \quad |\kappa[\Sigma]| \leq C \quad \text{on } \Sigma.$$

Moreover, Σ is the graph of an admissible solution $u \in C^\infty(\Omega) \cap C^1(\bar{\Omega})$ of the Dirichlet problem (1.33)-(1.34). Furthermore, $u^2 \in C^\infty(\Omega) \cap C^{1,1}(\bar{\Omega})$ and

$$(1.38) \quad \begin{aligned} u|D^2u| &\leq C \quad \text{in } \Omega, \\ \sqrt{1 - |Du|^2} &= \frac{1}{\sigma} \quad \text{on } \partial\Omega \end{aligned}$$

As we mentioned earlier, in Section 2 we prove strong duality theorems (see Theorem 2.2 and Corollary 2.3) which allows us to transfer our existence results for \mathbb{H}^{n+1} in [7] to \mathcal{H}^{n+1} and conversely. In particular we have

Corollary 1.3. *Let $\Gamma = \partial\Omega \times \{0\} \subset \partial_\infty \mathcal{H}^{n+1}$ where Ω is a bounded smooth domain in \mathbb{R}^n . Then there exists a complete locally strictly convex spacelike hypersurface Σ in \mathcal{H}^{n+1} satisfying*

$$(\sigma_l)^{\frac{1}{l}} = \sigma > 1, \quad 1 \leq l \leq n$$

with $\partial\Sigma = \Gamma$ and uniformly bounded principal curvatures

$$(1.39) \quad |\kappa[\Sigma]| \leq C \quad \text{on } \Sigma.$$

Moreover, Σ is the graph of an admissible solution $u \in C^\infty(\Omega) \cap C^1(\bar{\Omega})$ of the Dirichlet problem (1.33)-(1.34). Furthermore, $u^2 \in C^\infty(\Omega) \cap C^{1,1}(\bar{\Omega})$ and

$$(1.40) \quad \begin{aligned} u|D^2u| &\leq C \quad \text{in } \Omega, \\ \sqrt{1 - |Du|^2} &= \frac{1}{\sigma} \quad \text{on } \partial\Omega \end{aligned}$$

Further, if $l = 1$ or $l = 2$ (mean curvature and normalized scalar curvature) we have uniqueness among convex solutions and even among all solutions convex or not if Ω is simple

The uniqueness part of Corollary 1.3 follows from the uniqueness Theorem 1.6 of [7] and a continuity and deformation argument like that used in [12]. Note that Montiel [9] proved existence for $H = \sigma > 1$ assuming $\partial\Omega$ is mean convex. Our result shows

that for arbitrary Ω there is always a locally strictly convex solution. If Ω is simple and mean convex the solutions constructed by Montiel must agree with the ones we construct.

Transferring by duality the results of Corollary 1.2 to \mathbb{H}^{n+1} gives the mildly surprising

Corollary 1.4. *Let $\Gamma = \partial\Omega \times \{0\} \subset \partial_\infty \mathbb{H}^{n+1}$ where Ω is a bounded smooth domain in \mathbb{R}^n . Then there exists a complete locally strictly convex hypersurface Σ in \mathbb{H}^{n+1} satisfying*

$$(\sigma_l)^{\frac{1}{l}} = \sigma^{-1} \in (0, 1), \quad 1 \leq l < n$$

with $\partial\Sigma = \Gamma$ and uniformly bounded principal curvatures

$$(1.41) \quad |\kappa[\Sigma]| \leq C \quad \text{on } \Sigma.$$

Moreover, Σ is the graph of an admissible solution $v \in C^\infty(\Omega) \cap C^1(\bar{\Omega})$ of the Dirichlet problem dual to (1.33)-(1.34). Furthermore, $u^2 \in C^\infty(\Omega) \cap C^{1,1}(\bar{\Omega})$ and

$$(1.42) \quad \begin{aligned} u|D^2u| &\leq C \quad \text{in } \Omega, \\ \frac{1}{\sqrt{1+|Du|^2}} &= \frac{1}{\sigma} \quad \text{on } \partial\Omega \end{aligned}$$

Equation (1.33) is singular where $u = 0$. It is therefore natural to approximate the boundary condition (1.34) by

$$(1.43) \quad u = \epsilon > 0 \quad \text{on } \partial\Omega.$$

When ϵ is sufficiently small, we shall show that the Dirichlet problem (1.33), (1.43) is solvable for all $\sigma > 1$.

Theorem 1.5. *Let Ω be a bounded smooth domain in \mathbb{R}^n and $\sigma \in (1, \infty)$. Suppose f satisfies conditions (1.11)-(1.18) with $K = K_n^+$. Then for any $\epsilon > 0$ sufficiently small, there exists an admissible solution $u^\epsilon \in C^\infty(\bar{\Omega})$ of the Dirichlet problem (1.33), (1.43). Moreover, u^ϵ satisfies the a priori estimates*

$$(1.44) \quad \sqrt{1 - |Du^\epsilon|^2} = \frac{1}{\sigma} + O(\epsilon) \quad \text{on } \partial\Omega$$

and

$$(1.45) \quad u^\epsilon |D^2u^\epsilon| \leq C \quad \text{in } \Omega$$

where C is independent of ϵ .

In the proof of Theorem 1.5 we mostly follow the method of [5], [7] except that we will appeal to the duality results of Section 2 to use the global maximal principle of [7] to control the principal curvatures and prove the first inequality in (1.36). Because there is no a priori uniqueness (i.e $G_u \geq 0$ need not hold) it is not sufficient to derive estimates just for solutions of (1.33) with constant right hand side σ , one must also consider perturbations. *However to avoid undue length and tedious repetition, we will prove the estimates for solutions of constant curvature σ as the necessary modifications are straightforward* (see [5]).

By Theorem 1.5, the hyperbolic principal curvatures of the admissible solution u^ϵ of the Dirichlet problem (1.33),(1.43) are uniformly bounded above independent of ϵ . Since $f(\kappa[u^\epsilon]) = \sigma$ and $f = 0$ on ∂K_n^+ , the hyperbolic principal curvatures admit a uniform positive lower bound independent of ϵ and therefore (1.33) is uniformly elliptic on compact subsets of Ω for the solution u^ϵ . By the interior estimates of Evans and Krylov, we obtain uniform $C^{2,\alpha}$ estimates for any compact subdomain of Ω . The proof of Theorem 1.1 is now routine.

An outline of the contents of the paper are as follows. Section 2 contains the important duality results, Theorem 2.2 and Corollary 2.3. Section 3 contains preliminary formulas and computations that are used in Section 4 to prove the asymptotic angle result Theorem 4.2. In Section 5 we use the linearized operator to bound the principal curvatures of a solution on the boundary. Here is where the condition (1.18) comes into play. Finally in Section 6 we use duality to establish global curvature bounds and complete the proof of Theorem 1.5. The use of duality to prove this global estimate is unusual but seems to be necessary since $F(A)$ is a concave function of A but G is a convex function of $\{u_{ij}\}$.

2. THE GAUSS MAP AND LEGENDRE TRANSFORM

Let $\psi : \Sigma^n \rightarrow \mathcal{H}^{n+1}$ be a strictly locally convex spacelike immersion with prescribed (compact) boundary Γ either in the timeslice L_τ or in $L_\infty = \partial_\infty \mathcal{H}$. We are constructing such Σ^n as the graph of an admissible function u :

$$S = \{(x, u(x)) \in \mathbb{R}_+^{n+1} : u \in C^\infty(\overline{\Omega}), u(x) > 0, |\nabla u(x)| < 1\} ,$$

where Ω is a smooth bounded domain in \mathbb{R}^n . We know that the Gauss map

$$\mathbf{n} : \Sigma^n \rightarrow \mathbb{H}^{n+1}$$

takes values in hyperbolic space. Using the map ϕ defined in (1.7) that was used to identify the de Sitter and upper halfspace models of the steady state space \mathcal{H}^{n+1} , Montiel [9] showed that if we use the upper halfspace representation for both \mathcal{H} and \mathbb{H}^{n+1} , then the Gauss map \mathbf{n} corresponds to the map

$$L : S \rightarrow \mathbb{H}^{n+1}$$

defined by

$$(2.1) \quad L(x, u(x)) = (x - u(x)\nabla u(x), u(x)\sqrt{1 - |\nabla u|^2}) \quad x \in \Omega .$$

We now identify the map L in terms of a hodograph and associated Legendre transform. Define the map $y = \nabla p(x) : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}^n$ by

$$(2.2) \quad y = \nabla p(x), \quad x \in \Omega \quad \text{where } p(x) = \frac{1}{2}(x^2 - u(x)^2).$$

Note that p is strictly convex in the Euclidean sense by (1.28) and hence the map y is globally one to one. Therefore $v(y) := u(x)\sqrt{1 - |\nabla u(x)|^2}$ is well defined in $\Omega^* := y(\Omega)$. The associated Legendre transform is the function $q(y)$ defined in Ω^* by $p(x) + q(y) = x \cdot y$ or $q(y) = -p(x) + x \cdot \nabla p(x)$.

Lemma 2.1. *The Legendre transform $q(y)$ is given by*

$$q(y) = \frac{1}{2}(y^2 + v(y)^2) \quad \text{where } v(y) := u(x)\sqrt{1 - |\nabla u(x)|^2} .$$

Moreover, $\sqrt{1 + |\nabla v(y)|^2} = (1 - |\nabla u|^2)^{-\frac{1}{2}}$ and $u(x) = v(y)\sqrt{1 + |\nabla v(y)|^2}$. Therefore $x = \nabla q(y)$, $(q_{ij}(y)) = (p_{ij}(x))^{-1}$ and the inverse map L^* of L is given by $L^*(y, v(y)) = (x, u(x))$.

Proof. We calculate

$$\begin{aligned} p(x) + q(y) &= \frac{1}{2}(x^2 - u(x)^2) + \frac{1}{2}(y^2 + v(y)^2) \\ &= \frac{1}{2}(x^2 - u(x)^2) + \frac{1}{2}(x^2 - 2u(x)x \cdot \nabla u(x) + u^2|\nabla u|^2) + \frac{1}{2}(u^2(1 - |\nabla u(x)|^2)) \\ &= x^2 - u(x)x \cdot \nabla u(x) = x \cdot y , \end{aligned}$$

as required. It is then standard that $x = \nabla q(y)$ and $(q_{ij}(y)) = (p_{ij}(x))^{-1}$. Then $x = \nabla q(y) = y + v\nabla v(y)$ and $y = x - u(x)\nabla u(x)$ implies $u\nabla u = v\nabla v$ so $u^2|\nabla u|^2 =$

$v^2|\nabla v|^2 = u^2(1 - |\nabla u|^2)|\nabla v|^2$ and so $|\nabla v(y)|^2 = \frac{|\nabla u(x)|^2}{1 - |\nabla u(x)|^2}$. Therefore,

$$\sqrt{1 + |\nabla v(y)|^2} = (1 - |\nabla u|^2)^{-\frac{1}{2}} \quad \text{and} \quad u(x) = v(y)\sqrt{1 + |\nabla v(y)|^2}.$$

□

Theorem 2.2. *Let L be defined by (2.1) and let y be defined by (2.2). Then the image of S by L is the locally strictly convex graph (with respect to the induced hyperbolic metric)*

$$S^* = \{(y, v(y)) \in \mathbb{R}_+^{n+1} : v \in C^\infty(\overline{\Omega^*}), v(y) > 0\},$$

with principal curvatures $\kappa_i^* = (\kappa_i)^{-1}$. Here $\kappa_i > 0, i = 1, \dots, n$ are the principal curvatures of S with respect to the induced de Sitter metric. Moreover the inverse map $L^* : S^* \rightarrow S$ defined by

$$L^*(y, v(y)) = (y + v(y)\nabla v(y), v(y)\sqrt{1 + |\nabla v(y)|^2}) \quad y \in \Omega^*$$

is the dual Legendre transform and hodograph map $x = \nabla q(y)$.

Proof. By Lemma 2.1 it remains only to show $\kappa_i^* = (\kappa_i)^{-1}$. The principal curvatures of S, S^* are respectively the eigenvalues of the matrices

$$A[u] = (\gamma^{ij})(h_{ij})(\gamma^{ij}), \quad A[v] = (\gamma^{*ij})(h_{ij}^*)(\gamma^{*ij}),$$

where

$$g_{ij}^* = \frac{\delta_{ij} + v_i v_j}{v^2}, \quad (\gamma^{*ij}) = (g_{ij}^*)^{-\frac{1}{2}}, \quad h_{ij}^* = \frac{\delta_{ij} + v v_{ij} + v_i v_j}{v^2 \sqrt{1 + |\nabla v|^2}}.$$

By Lemma 2.1,

$$h_{ij}^* = \frac{q_{ij}}{v^2 \sqrt{1 + |\nabla v|^2}} = \frac{u^2 \sqrt{1 - |\nabla u|^2}}{v^2 u^2} q_{ij} = \frac{1}{u^2 v^2} (h_{ij})^{-1},$$

$$g_{ij}^* = \frac{\delta_{ij} + v_i v_j}{v^2} = \frac{\delta_{ij} + \frac{u_i u_j}{1 - |\nabla u|^2}}{v^2} = \frac{g^{ij}}{u^2 v^2}, \quad (\gamma^{*ij}) = uv(\gamma^{ij})^{-1},$$

and therefore $A[v] = (A[u])^{-1}$ completing the proof. □

Corollary 2.3. *If the graph $S = \{(x, u(x)) : x \in \Omega\}$ is a strictly locally convex space-like graph with constant curvature $f(\kappa) = \sigma > 1$ in the steady state space \mathcal{H}^{n+1} with $\partial_\infty \mathcal{H}^{n+1} = \Gamma = \Omega$, then the dual graph $S^* = \{(x - u(x)\nabla u(x), u(x)\sqrt{1 + |\nabla u|^2}) = (y, v(y)) : y \in \Omega^*\}$ is a strictly locally convex graph of constant curvature $f^*(\kappa^*) = \sigma^{-1}$ in \mathbb{H}^{n+1} with $\partial_\infty \mathbb{H}^{n+1} = \Gamma = \partial\Omega$. The converse holds as well.*

3. FORMULAS ON HYPERSURFACES

In this section we will derive some basic identities on a hypersurface by comparing the induced metric in steady state space $\mathcal{H}^{n+1} \subset dS_{n+1}$ and Minkowski space.

Let Σ be a hypersurface in \mathcal{H}^{n+1} . We shall use g and ∇ to denote the induced metric and Levi-Civita connection on Σ , respectively. As Σ is also a submanifold of $\mathbb{R}^{n,1}$, we shall usually distinguish a geometric quantity with respect to the Minkowski metric by adding a ‘tilde’ over the corresponding quantity. For instance, \tilde{g} denotes the induced metric on Σ from $\mathbb{R}^{n,1}$, and $\tilde{\nabla}$ is its Levi-Civita connection.

Let \mathbf{x} be the position vector of Σ in $\mathbb{R}^{n,1}$ and set

$$u = \mathbf{x} \cdot \mathbf{e}$$

where $\mathbf{e} = (0, \dots, 0, 1)$ is the unit vector in the positive x_{n+1} direction in \mathbb{R}^{n+1} , and ‘ \cdot ’ denotes the Euclidean inner product in \mathbb{R}^{n+1} . We refer u as the *height function* of Σ .

Throughout the paper we assume Σ is orientable and let \mathbf{n} be a (global) unit normal vector field to Σ with respect to the de Sitter metric. This also determines a unit normal ν to Σ with respect to the Minkowski metric by the relation

$$\nu = \frac{\mathbf{n}}{u}.$$

We denote $\nu^{n+1} = \mathbf{e} \cdot \nu$.

Let (z_1, \dots, z_n) be local coordinates and

$$\tau_i = \frac{\partial}{\partial z_i}, \quad i = 1, \dots, n.$$

The de Sitter and Minkowski metrics of Σ are given by

$$g_{ij} = \langle \tau_i, \tau_j \rangle_D, \quad \tilde{g}_{ij} = \langle \tau_i, \tau_j \rangle_M = u^2 g_{ij},$$

while the second fundamental forms are

$$(3.1) \quad \begin{aligned} h_{ij} &= \langle D_{\tau_i} \tau_j, \mathbf{n} \rangle_D = -\langle D_{\tau_i} \mathbf{n}, \tau_j \rangle_D, \\ \tilde{h}_{ij} &= \langle \nu, \tilde{D}_{\tau_i} \tau_j \rangle_M = -\langle \tau_j, \tilde{D}_{\tau_i} \nu \rangle_M, \end{aligned}$$

where D and \tilde{D} denote the Levi-Civita connection of \mathcal{H}^{n+1} and $\mathbb{R}^{n,1}$, respectively, and $\langle \cdot, \cdot \rangle_D, \langle \cdot, \cdot \rangle_M$ denote the corresponding inner product.

The Christoffel symbols Γ_{ij}^k and $\tilde{\Gamma}_{ij}^k$ are related by the formula

$$(3.2) \quad \Gamma_{ij}^k = \tilde{\Gamma}_{ij}^k - \frac{1}{u}(u_i \delta_{kj} + u_j \delta_{ik} - \tilde{g}^{kl} u_l \tilde{g}_{ij}).$$

It follows that for $v \in C^2(\Sigma)$

$$(3.3) \quad \nabla_{ij} v = v_{ij} - \Gamma_{ij}^k v_k = \tilde{\nabla}_{ij} v + \frac{1}{u}(u_i v_j + u_j v_i - \tilde{g}^{kl} u_k v_l \tilde{g}_{ij})$$

where (and in sequel)

$$v_i = \frac{\partial v}{\partial z_i}, \quad v_{ij} = \frac{\partial^2 v}{\partial z_i \partial z_j}, \quad \text{etc.}$$

In particular,

$$(3.4) \quad \nabla_{ij} u = \tilde{\nabla}_{ij} u + \frac{2u_i u_j}{u} - \frac{1}{u} \tilde{g}^{kl} u_k u_l \tilde{g}_{ij}$$

and

$$(3.5) \quad \nabla_{ij} \frac{1}{u} = -\frac{1}{u^2} \tilde{\nabla}_{ij} u + \frac{1}{u^3} \tilde{g}^{kl} u_k u_l \tilde{g}_{ij}.$$

Moreover,

$$(3.6) \quad \nabla_{ij} \frac{v}{u} = v \nabla_{ij} \frac{1}{u} + \frac{1}{u} \tilde{\nabla}_{ij} v - \frac{1}{u^2} \tilde{g}^{kl} u_k v_l \tilde{g}_{ij}.$$

In $\mathbb{R}^{n,1}$,

$$(3.7) \quad \begin{aligned} \tilde{g}^{kl} u_k u_l &= |\tilde{\nabla} u|^2 = (\nu^{n+1})^2 - 1 \\ \tilde{\nabla}_{ij} u &= -\tilde{h}_{ij} \nu^{n+1}. \end{aligned}$$

Therefore, by (1.26) and (3.5),

$$(3.8) \quad \begin{aligned} \nabla_{ij} \frac{1}{u} &= -\frac{1}{u^2} \tilde{\nabla}_{ij} u + \frac{1}{u^3} \tilde{g}_{ij} [(\nu^{n+1})^2 - 1] \\ &= \frac{1}{u} (h_{ij} \nu^{n+1} - g_{ij}). \end{aligned}$$

We note that (3.6) and (3.8) still hold for general local frames τ_1, \dots, τ_n . In particular, if τ_1, \dots, τ_n are orthonormal in the de Sitter metric, then $g_{ij} = \delta_{ij}$ and $\tilde{g}_{ij} = u^2 \delta_{ij}$.

We now consider equation (1.1) on Σ . Let \mathcal{A} be the vector space of $n \times n$ matrices and

$$\mathcal{A}^+ = \{A = \{a_{ij}\} \in \mathcal{A} : \lambda(A) \in K_n^+\},$$

where $\lambda(A) = (\lambda_1, \dots, \lambda_n)$ denotes the eigenvalues of A . Let F be the function defined by

$$(3.9) \quad F(A) = f(\lambda(A)), \quad A \in \mathcal{A}^+$$

and denote

$$(3.10) \quad F^{ij}(A) = \frac{\partial F}{\partial a_{ij}}(A), \quad F^{ij,kl}(A) = \frac{\partial^2 F}{\partial a_{ij} \partial a_{kl}}(A).$$

Since $F(A)$ depends only on the eigenvalues of A , if A is symmetric then so is the matrix $\{F^{ij}(A)\}$. Moreover,

$$F^{ij}(A) = f_i \delta_{ij}$$

when A is diagonal, and

$$(3.11) \quad F^{ij}(A) a_{ij} = \sum f_i(\lambda(A)) \lambda_i = F(A),$$

$$(3.12) \quad F^{ij}(A) a_{ik} a_{jk} = \sum f_i(\lambda(A)) \lambda_i^2.$$

Equation (1.1) can therefore be rewritten in a local frame τ_1, \dots, τ_n in the form

$$(3.13) \quad F(A[\Sigma]) = \sigma$$

where $A[\Sigma] = \{g^{ik} h_{kj}\}$. Let $F^{ij} = F^{ij}(A[\Sigma])$, $F^{ij,kl} = F^{ij,kl}(A[\Sigma])$.

Lemma 3.1. *Let Σ be a smooth hypersurface in \mathcal{H}^{n+1} satisfying equation (1.1). Then in a local orthonormal frame,*

$$(3.14) \quad F^{ij} \nabla_{ij} \frac{1}{u} = \frac{\sigma \nu^{n+1}}{u} - \frac{1}{u} \sum f_i.$$

and

$$(3.15) \quad F^{ij} \nabla_{ij} \frac{\nu^{n+1}}{u} = -\frac{\sigma}{u} + \frac{\nu^{n+1}}{u} \sum f_i \kappa_i^2.$$

Proof. The first identity follows immediately from (3.8), (3.11) and assumption (1.17).

To prove (3.15) we recall the identities in $\mathbb{R}^{n,1}$

$$(3.16) \quad \begin{aligned} (\nu^{n+1})_i &= -\tilde{h}_{il} \tilde{g}^{lk} u_k, \\ \tilde{\nabla}_{ij} \nu^{n+1} &= -\tilde{g}^{kl} (-\nu^{n+1} \tilde{h}_{il} \tilde{h}_{kj} + u_l \tilde{\nabla}_k \tilde{h}_{ij}). \end{aligned}$$

By (3.10), (3.11), (3.12), and $\tilde{g}^{ik} = \delta_{jk}/u^2$ we see that

$$(3.17) \quad \begin{aligned} F^{ij} \tilde{g}^{kl} \tilde{h}_{il} \tilde{h}_{kj} &= \frac{1}{u^2} F^{ij} \tilde{h}_{ik} \tilde{h}_{kj} \\ &= F^{ij} (h_{ik} h_{kj} - 2\nu^{n+1} h_{ij} + (\nu^{n+1})^2 \delta_{ij}) \\ &= f_i \kappa_i^2 - 2\nu^{n+1} \sigma + (\nu^{n+1})^2 \sum f_i. \end{aligned}$$

As a hypersurface in $\mathbb{R}^{n,1}$, it follows from (1.27) that Σ satisfies

$$f(u\tilde{\kappa}_1 + \nu^{n+1}, \dots, u\tilde{\kappa}_n + \nu^{n+1}) = \sigma,$$

or equivalently,

$$(3.18) \quad F(\{\tilde{g}^{ik}(u\tilde{h}_{kj} + \nu^{n+1}\tilde{g}_{kj})\}) = \sigma.$$

Differentiating equation (3.18) and using $\tilde{g}_{ik} = u^2\delta_{ik}$, $\tilde{g}^{ik} = \delta_{ik}/u^2$, we obtain

$$(3.19) \quad F^{ij}(u\tilde{\nabla}_k\tilde{h}_{ij} + u_k\tilde{h}_{ij} + (\nu^{n+1})_k u^2\delta_{ij}) = 0.$$

That is,

$$(3.20) \quad \begin{aligned} F^{ij}\tilde{\nabla}_k\tilde{h}_{ij} + (\nu^{n+1})_k u \sum F^{ii} &= -\frac{u_k}{u} F^{ij}\tilde{h}_{ij} \\ &= -u_k F^{ij}(h_{ij} - \nu^{n+1}\delta_{ij}) \\ &= -u_k \left(\sigma - \nu^{n+1} \sum f_i \right). \end{aligned}$$

Finally, combining (3.6), (3.14), (3.16), (3.17), (3.20), and the first identity in (3.7), we derive

$$(3.21) \quad \begin{aligned} F^{ij}\nabla_{ij}\frac{\nu^{n+1}}{u} &= \nu^{n+1}F^{ij}\nabla_{ij}\frac{1}{u} + \frac{|\tilde{\nabla}u|^2}{u}F^{ij}\tilde{h}_{ij} - \frac{\nu^{n+1}}{u^3}F^{ij}\tilde{h}_{ik}\tilde{h}_{kj} \\ &= \frac{\nu^{n+1}}{u} \left(\nu^{n+1}\sigma - \sum f_i \right) + \frac{|\tilde{\nabla}u|^2}{u} \left(\sigma - \nu^{n+1} \sum f_i \right) \\ &\quad + \frac{\nu^{n+1}}{u} \left(f_i \kappa_i^2 - 2\nu^{n+1}\sigma + (\nu^{n+1})^2 \sum f_i \right) \\ &= -\frac{\sigma}{u} + \frac{\nu^{n+1}}{u} \sum f_i \kappa_i^2. \end{aligned}$$

This proves (3.15). □

4. THE ASYMPTOTIC ANGLE MAXIMUM PRINCIPLE AND GRADIENT ESTIMATES

In this section we show that the upward unit normal of a solution tends to a fixed asymptotic angle on approach to the asymptotic boundary and that this holds approximately for the solutions of the approximate problem.. This implies a global (spacelike) gradient bound on solutions.

Our estimates are all based on the use of special barriers (see section 3 of [9]). These correspond to horospheres for the dual problem in hyperbolic space and our argument follows that of section 3 of [6]. Let

$$Q(r, c) = \{x \in \mathbb{R}^{n,1} \mid \langle x - c, x - c \rangle_M \leq -r^2\}$$

be a ball of radius r centered at c in Minkowski space, where $c \in \mathbb{R}^{n+1}$. Moreover, let $Q_+(r, c)$ denote the region above the upper hyperboloid and $Q_-(r, c)$ denote the region below the lower hyperboloid. If we choose $a = (a', -r\sigma)$, then $S_+(r, a) = \partial Q_+(r, a) \cap \mathcal{H}^{n+1}$ is an umbilical hypersurface in \mathcal{H}^{n+1} with constant curvature σ with respect to its upward normal vector. For convenience we sometimes call $S_+(r, a)$ an upper hyperboloid of constant curvature σ in \mathcal{H}^{n+1} . Similarly, when we choose $b = (b', r\sigma)$, then $S_-(r, b) = \partial Q_-(r, b) \cap \mathcal{H}^{n+1}$ is the lower hyperboloid of constant curvature σ with respect to its upward normal vector. These hyperboloids serve as useful barriers.

Now let Σ be a hypersurface in \mathcal{H}^{n+1} with $\partial\Sigma \subset P(\varepsilon) := \{x_{n+1} = \varepsilon\}$ so Σ separates $\{x_{n+1} \geq \varepsilon\}$ into an inside (bounded) region and outside (unbounded) one. Let Ω be the region in $\mathbb{R}^n \times \{0\}$ such that its vertical lift Ω^ε to $P(\varepsilon)$ is bounded by $\partial\Sigma$ (and $\mathbb{R}^n \setminus \Omega$ is connected and unbounded). (It is allowable that Ω have several connected components.) Suppose $\kappa[\Sigma] \in K_n^+$ and $f(\kappa) = \sigma \in (1, \infty)$ with respect to its outer normal.

Lemma 4.1.

- (i) $\Sigma \cap \{x_{n+1} < \varepsilon\} = \emptyset$.
- (ii) If $\partial\Sigma \subset Q_-(r, b)$, then $\Sigma \subset Q_-(r, b)$.
- (iii) If $Q_-(r, b) \cap P(\varepsilon) \subset \Omega^\varepsilon$, then $Q_-(r, b) \cap \Sigma = \emptyset$.
- (iv) If $Q_+(r, a) \cap \Omega^\varepsilon = \emptyset$, then $Q_+(r, a) \cap \Sigma = \emptyset$.

Proof. For (i) let $c = \min_{x \in \Sigma} x_{n+1}$ and suppose $0 < c < \varepsilon$. Then the horizontal plane $P(c)$ satisfies $f(\kappa) = 1$ with respect to the upward normal, lies below Σ , and has an interior contact point. Then $f(\kappa[\Sigma]) \leq 1$ at this point, which leads to a contradiction (notice that in the Euclidean case we have the reverse inequality).

For (ii), (iii), (iv) we consider the family $\{h_s\}_{s \in R}$ of isometries of \mathcal{H}^{n+1} consisting of Euclidean homotheties. We perform homothetic dilations from $(a', 0)$ and $(b', 0)$ respectively, and then use the maximum principle. For (ii), choose s_0 big enough so that $h_{s_0}(Q_-(r, b))$ contains Σ and then decrease s . Since the curvature of Σ and $S_-(r, b)$ are calculated with respect to their outward normals and both hypersurfaces satisfy $f(\kappa) = \sigma$, there cannot be a first contact.

For (iii) and (iv) we shrink $Q_+(r, a)$ and $Q_-(r, b)$ until they are respectively inside and outside Σ . When we expand $Q_-(r, b)$ there cannot be a first contact as above.

Now decrease s to a certain value $s_1 \in R$ such that $h_{s_1}(Q_+(r, a))$ is disjoint from Σ (outside of). Then we increase s_1 and suppose there is a first interior contact. The outward normal to Σ at this contact point is the upward normal to $S_+(r, a)$. Since the curvatures of $S_+(r, a)$ are calculated with respect to the upward normal and $S_+(r, a)$ satisfies $f(\kappa) = \sigma$, we have a contradiction of the maximum principle. \square

Theorem 4.2. *Let Σ be a smooth strictly locally convex spacelike hypersurface in \mathcal{H}^{n+1} satisfying equation (1.1). Suppose Σ is globally a graph:*

$$\Sigma = \{(x, u(x)) : x \in \Omega\}$$

where Ω is a domain in $\mathbb{R}^n \equiv \partial\mathcal{H}^{n+1}$. Then

$$(4.1) \quad F^{ij} \nabla_{ij} \frac{\sigma - \nu^{n+1}}{u} = \frac{\sigma}{u} \left(1 - \sum f_i\right) + \frac{\nu^{n+1}}{u} \left(\sigma^2 - \sum f_i \kappa_i^2\right) \leq 0$$

and so,

$$(4.2) \quad \frac{\sigma - \nu^{n+1}}{u} \geq \inf_{\partial\Sigma} \frac{\sigma - \nu^{n+1}}{u} \quad \text{on } \Sigma.$$

Moreover, if $u = \epsilon > 0$ on $\partial\Omega$, then there exists $\epsilon_0 > 0$ depending only on $\partial\Omega$, such that for all $\epsilon \leq \epsilon_0$,

$$(4.3) \quad \frac{r_1 \sqrt{\sigma^2 - 1}}{r_1^2 - \epsilon^2} + \frac{\epsilon(\sigma - 1)}{r_1^2 - \epsilon^2} > \frac{\sigma - \nu^{n+1}}{u} > -\frac{r_2 \sqrt{\sigma^2 - 1}}{r_2^2 - \epsilon^2} - \frac{\epsilon(1 + \sigma)}{r_2^2 - \epsilon^2} \quad \text{on } \partial\Sigma$$

where r_2, r_1 are the maximal radius of exterior and interior spheres to $\partial\Omega$, respectively. In particular, $\nu^{n+1} \rightarrow \sigma$ on $\partial\Sigma$ as $\epsilon \rightarrow 0$.

Proof. It's easy to see that (4.1) follows from equations (3.14), (3.15) and (1.20), (1.21). Thus, (4.2) follows from the maximum principle.

In order to prove (4.3), we first assume $r_2 < \infty$. Fix a point $x_0 \in \partial\Omega$ and let e_1 be the outward pointing unit normal to $\partial\Omega$ at x_0 . Let $S_+(R_2, a)$, $S_-(R_1, b)$ be the upper and lower hyperboloid with centers $a = (x_0 + r_2 e_1, -R_2 \sigma)$, $b = (x_0 - r_1 e_1, R_1 \sigma)$ and radii R_2, R_1 respectively satisfying

$$r_2^2 - (R_2 \sigma + \epsilon)^2 = -R_2^2, \quad r_1^2 - (R_1 \sigma - \epsilon)^2 = -R_1^2.$$

Then $Q_-(R_1, b) \cap P(\epsilon)$ is an n -ball of radius r_1 internally tangent to $\partial\Omega^\epsilon$ at x_0 while $Q_+(R_2, a) \cap P(\epsilon)$ is an n -ball of radius r_2 externally tangent to $\partial\Omega^\epsilon$ at x_0 . By Lemma 4.1 (iii) and (iv), $Q_\pm \cap \Sigma = \emptyset$. Hence

$$(4.4) \quad \frac{\sigma R_1 - u}{R_1} < \nu^{n+1} < \frac{\sigma R_2 + u}{R_2}.$$

Moreover, by a simple calculation we have

$$(4.5) \quad \frac{1}{R_1} = \frac{-\varepsilon\sigma + \sqrt{r_1^2(\sigma^2 - 1) + \varepsilon^2}}{r_1^2 - \varepsilon^2} < \frac{r_1\sqrt{\sigma^2 - 1}}{r_1^2 - \varepsilon^2} + \frac{\varepsilon(\sigma - 1)}{r_1^2 - \varepsilon^2},$$

$$(4.6) \quad \frac{1}{R_2} = \frac{\varepsilon\sigma + \sqrt{(\sigma^2 - 1)r_2^2 + \varepsilon^2}}{r_2^2 - \varepsilon^2} < \frac{r_2\sqrt{\sigma^2 - 1}}{r_2^2 - \varepsilon^2} + \frac{\varepsilon(1 + \sigma)}{r_2^2 - \varepsilon^2}.$$

Finally (4.3) follows from (4.4), (4.5) and (4.6).

If $r_2 = \infty$, in the above argument one can replace r_2 by any $r > 0$ and then let $r \rightarrow \infty$. \square

From Theorem 4.2 we conclude

Corollary 4.3. *Let Ω be a bounded smooth domain in \mathbb{R}^n and $\sigma > 1$. Suppose f satisfies conditions (1.11)-(1.18) with $K = K_n^+$. Then for any $\epsilon > 0$ sufficiently small, any admissible solution $u^\epsilon \in C^\infty(\bar{\Omega})$ of the Dirichlet problem (1.33), (1.43) satisfies the apriori estimate*

$$(4.7) \quad |\nabla u^\epsilon| \leq C < 1 \quad \text{in } \Omega$$

where C is independent of ϵ .

5. THE LINEARIZED OPERATOR AND BOUNDARY ESTIMATES FOR SECOND DERIVATIVES.

In this section we establish boundary estimates for second derivatives of admissible solutions.

Theorem 5.1. *Suppose that f satisfies conditions (1.11)-(1.18) with $K = K_n^+$. If ε is sufficiently small, then*

$$(5.1) \quad u|D^2u| \leq C \quad \text{on } \partial\Omega$$

where C is independent of ε .

Define the linearized operator of G at u (recall (1.32))

$$(5.2) \quad \mathcal{L} = G^{st}\partial_s\partial_t + G^s\partial_s + G_u$$

where

$$(5.3) \quad G^{st} = \frac{\partial G}{\partial u_{st}}, \quad G^s = \frac{\partial G}{\partial u_s}, \quad G_u = \frac{\partial G}{\partial u}.$$

Note that

$$(5.4) \quad G^{st} = -\frac{u}{w} F^{ij} \gamma^{is} \gamma^{jt}, \quad G^{st} u_{st} = u G_u = \sigma - \frac{\sum f_i}{w}.$$

After some straightforward but tedious calculations we derive

$$(5.5) \quad G^s = \frac{u_s}{w^2} \sigma + 2 \frac{F^{ij} a_{ik}}{w(1+w)} (u_k \gamma^{sj} w + u_j \gamma^{ks}) - 2 \frac{F^{ij} u_i \gamma^{sj}}{w^2}.$$

It follows that

Lemma 5.2. *Suppose that f satisfies (1.7), (1.8), (1.10) and (1.12). Then*

$$(5.6) \quad |G^s| \leq C_0 (1 + \sum f_i),$$

where C_0 denotes a controlled constant independent of ε .

Since $\gamma^{sj} u_s = u_j/w$,

$$(5.7) \quad G^s u_s = \frac{1-w^2}{w^2} \sigma + 2 \frac{F^{ij} a_{ik} u_k u_j}{w^2} - 2 \frac{F^{ij} u_i u_j}{w^3}.$$

Let

$$(5.8) \quad \mathcal{L}' = -\mathcal{L} + G_u = -G^{st} \partial_s \partial_t - G^s \partial_s.$$

Then from (5.4) and (5.7) we obtain

$$(5.9) \quad \begin{aligned} \mathcal{L}' u &= \frac{1}{w} \sum f_i - \frac{\sigma}{w^2} - 2 \frac{F^{ij} a_{ik} u_k u_j}{w^2} + 2 \frac{F^{ij} u_i u_j}{w^3} \\ &\leq C_1 + C_2 \sum f_i. \end{aligned}$$

In the following we denote by C_1, C_2, \dots controlled constants independent of ε .

We will employ a barrier function of the form

$$(5.10) \quad v = u - \varepsilon + td - Nd^2$$

where d is the distance function from $\partial\Omega$, and t, N are positive constants to be determined. We may take $\delta > 0$ small enough so that d is smooth in $\Omega_\delta = \Omega \cap B_\delta(0)$.

Lemma 5.3. *For $\delta = c_0 \varepsilon$, $N = \frac{C_4}{\varepsilon}$, $t = c_0 C_4$ with C_4 sufficiently large and c_0 sufficiently small independent of ε ,*

$$\mathcal{L}' v \leq -(1 + \sum f_i) \text{ in } \Omega_\delta, \quad v \geq 0 \text{ on } \partial\Omega_\delta.$$

Proof. Since $|Dd| = 1$ and $-CI \leq D^2d \leq CI$, we have

$$(5.11) \quad \begin{aligned} |\mathcal{L}'d| &= |-G^{st}d_{st} - G^s d_s| \\ &\leq C_3(1 + \sum f_i). \end{aligned}$$

Furthermore, since $d_n(0) = 1$, $d_\beta(0) = 0$ for all $\beta < n$, we have, when $\delta > 0$ sufficiently small,

$$(5.12) \quad \begin{aligned} -G^{st}d_s d_t &\geq -G^{nn}d_n^2 - 2 \sum_{\beta < n} G^{n\beta} d_n d_\beta \\ &\geq \frac{-1}{2} \sum G^{nn} = \frac{u}{2w} \sum F^{ij} \gamma^{in} \gamma^{jn} \\ &\geq \frac{u}{2nw} \sum F^{ii}. \end{aligned}$$

Therefore,

$$(5.13) \quad \begin{aligned} \mathcal{L}'v &= \mathcal{L}'u + (t - 2Nd)\mathcal{L}'d - 2NG^{st}d_s d_t \\ &\leq C_1 + C_2 \sum f_i + C_3(t + 2N\delta)(1 + \sum f_i) - \frac{N\varepsilon}{nw} \sum f_i \\ &\leq (C_1 + tC_3 + 2N\delta C_3 - \frac{N\varepsilon}{2n}) + (C_2 + tC_3 + 2N\delta C_3 - \frac{N\varepsilon}{2n}) \sum f_i. \end{aligned}$$

Now if we require $0 < c_0 < \frac{1}{18nC_3}$ and $C_4 \gg 3n \max\{C_1, C_2\} + 3n$. let $N = C_4/\varepsilon$, $t = C_4 c_0$, $\delta = c_0 \varepsilon$, then Lemma 5.3 is proved. \square

The following lemma is proven in [6]; it applies to our situation since horizontal rotations are isometries for \mathcal{H}^{n+1} .

Lemma 5.4. *Suppose that f satisfies (1.7), (1.8), (1.10) and (1.12). Then*

$$(5.14) \quad \mathcal{L}(x_i u_j - x_j u_i) = 0, \quad \mathcal{L}u_i = 0, \quad 1 \leq i, j \leq n.$$

Proof of Theorem 5.1. Consider an arbitrary point on $\partial\Omega$, which we may assume to be the origin of \mathbb{R}^n and choose the coordinates so that the positive x_n axis is the interior normal to $\partial\Omega$ at the origin. There exists a uniform constant $r > 0$ such that $\partial\Omega \cap B_r(0)$ can be represented as a graph

$$x_n = \rho(x') = \frac{1}{2} \sum_{\alpha, \beta < n} B_{\alpha\beta} x_\alpha x_\beta + O(|x'|^3), \quad x' = (x_1, \dots, x_{n-1}).$$

Since $u = \varepsilon$ on $\partial\Omega$, we see that $u(x', \rho(x')) = \varepsilon$ and

$$u_{\alpha\beta}(0) = -u_n \rho_{\alpha\beta}, \quad \alpha, \beta < n.$$

Consequently,

$$|u_{\alpha\beta}(0)| \leq C|Du(0)|, \quad \alpha, \beta < n,$$

where C depends only on the (Euclidean maximal principal) curvature of $\partial\Omega$.

Next, following [1] we consider for fixed $\alpha < n$ the approximate tangential operator

$$(5.15) \quad T = \partial_\alpha + \sum_{\beta < n} B_{\beta\alpha}(x_\beta \partial_n - x_n \partial_\beta).$$

We have

$$(5.16) \quad \begin{aligned} |Tu| &\leq C, \text{ in } \Omega \cap B_\delta(0) \\ |Tu| &\leq C|x|^2, \text{ on } \partial\Omega \cap B_\delta(0) \end{aligned}$$

since $u = \varepsilon$ on $\partial\Omega$. By Lemma 5.4 and (5.4), (5.16),

$$(5.17) \quad \begin{aligned} |\mathcal{L}'(Tu)| &= |-\mathcal{L}Tu + G_u Tu| \\ &= |G_u Tu| \\ &\leq \frac{C_5}{\varepsilon}(1 + \sum f_i). \end{aligned}$$

A straightforward calculation gives

$$(5.18) \quad \begin{aligned} |\mathcal{L}'|x|^2| &= \left| -2 \sum G^{ss} - 2 \sum x_s G^s \right| \\ &\leq C_6 \varepsilon (1 + \sum f_i). \end{aligned}$$

Now let

$$\Phi = \frac{A}{\varepsilon}v + \frac{C}{\delta^2}|x|^2 \pm Tu.$$

By Lemma 5.3 and (5.17), (5.18),

$$(5.19) \quad \mathcal{L}'\Phi \leq -\frac{A}{\varepsilon}(1 + \sum f_i) + \frac{C_6 C}{c_0^2 \varepsilon}(1 + \sum f_i) + \frac{C_5}{\varepsilon}(1 + \sum f_i) \text{ in } \Omega \cap B_\delta$$

Choosing $A \gg C_5 + \frac{C_6 C}{c_0^2}$ makes $\mathcal{L}'\Phi \leq 0$ in $\Omega \cap B_\delta$. It is also easy to see that $\Phi \geq 0$ on $\partial(\Omega \cap B_\delta)$.

By the maximum principle $\Phi \geq 0$ in $\Omega \cap B_\delta$. Since $\Phi(0) = 0$, we have $\Phi_n(0) \geq 0$ which gives

$$(5.20) \quad |u_{\alpha n}(0)| \leq \frac{A(u_n(0) + C_4 c_0)}{\varepsilon} \leq \frac{C}{\varepsilon}.$$

Finally to estimate $|u_{nn}(0)|$ we use our hypothesis (1.18) and Theorem 4.2. We may assume $[u_{\alpha\beta}(0)]$, $1 \leq \alpha, \beta < n$, to be diagonal. Note that $u_\alpha(0) = 0$ for $\alpha < n$.

We have at $x = 0$

$$A[u] = \frac{1}{w} \begin{bmatrix} 1 - uu_{11} & 0 & \cdots & -\frac{uu_{1n}}{w} \\ 0 & 1 - uu_{22} & \cdots & -\frac{uu_{2n}}{w} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{uu_{n1}}{w} & -\frac{uu_{n2}}{w} & \cdots & 1 - \frac{uu_{nn}}{w^2} \end{bmatrix}.$$

By Lemma 1.2 in [2], if $|\varepsilon u_{nn}(0)|$ is very large, the eigenvalues $\lambda_1, \dots, \lambda_n$ of $A[u]$ are asymptotically given by

$$(5.21) \quad \begin{aligned} \lambda_\alpha &= \frac{1}{w}(1 + |\varepsilon u_{nn}(0)|) + o(1), \quad \alpha < n \\ \lambda_n &= \frac{|\varepsilon u_{nn}(0)|}{w^3} \left(1 + O\left(\frac{1}{|\varepsilon u_{nn}(0)|}\right) \right). \end{aligned}$$

If $|\varepsilon u_{nn}(0)| \geq R$ where R is a controlled constant only depends on σ . By the hypothesis (1.18) and Theorem 4.2,

$$\sigma = \frac{1}{w} F(wA[u](0)) \geq (\sigma - C\varepsilon) F(wA[u](0)) \geq (\sigma - C\varepsilon)(1 + \varepsilon_0) \geq \sigma(1 + \frac{\varepsilon_0}{2})$$

leads to a contradiction. Therefore

$$|u_{nn}(0)| \leq \frac{R}{\varepsilon}$$

and the proof is complete. \square

6. COMPLETION OF THE PROOF OF THEOREM 1.5

As we emphasized in the introduction, we will derive a global curvature estimate for solutions of the Dirichlet problem (1.33), (1.43). In Theorem 5.1 of the previous section we have shown that the principal curvatures satisfy $0 < \kappa_i \leq C$, $i = 1, \dots, n$ on $\Gamma = \partial\Omega$, hence lie in a compact set E of the cone K . Since $f(\kappa) = \sigma$ and $f(\kappa) \rightarrow 0$ uniformly on E when any $\kappa_i \rightarrow 0$, it follows that

$$(6.1) \quad \frac{1}{C} < \kappa_i \leq C \quad \text{on } \Gamma.$$

We now appeal to duality. By Corollary 2.3, the dual graph S^* satisfies $f^*(\kappa^*) = \frac{1}{\sigma}$ and by (6.1)

$$(6.2) \quad \frac{1}{C} < \kappa_i^* \leq C \quad \text{on } \Gamma^* = L(\Gamma).$$

Hence by the global maximum principal for principal curvatures proved in Theorem 4.1 of [7],

$$(6.3) \quad \frac{1}{\overline{C}} < \kappa_i^* \leq \overline{C} \quad \text{on } S^* .$$

Once more using duality to return to the graph S , we obtain the desired global estimate

$$(6.4) \quad \frac{1}{\overline{C}} < \kappa_i \leq \overline{C} \quad \text{on } S .$$

The proof of Theorem 1.1 follows by letting $\varepsilon \rightarrow 0$ as mentioned in the introduction.

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