MINIMAL SURFACES IN $M^n \times \mathbb{R}$

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ABSTRACT. In this paper, we investigate the problem of finding minimal surface in $M^n \times \mathbb{R}$ with general boundary conditions through an variational approach. As an application we generalize the results in [8] to $M^n \times \mathbb{R}$. We also show the long time existence and uniform convergence of the corresponding flow problem.

1. INTRODUCTION

In early 1980s', the problem of finding minimal surfaces by using calculus of variations, has been widely studied in Euclidean space. In particular, in the second part of [1], they studied the solvability of the Dirichlet problem for the minimal surface equation in the space of functions of bounded variation. They also discussed the uniqueness and regularity of the solution.

In this paper, we used a priori interior gradient estimates in $M^n \times \mathbb{R}$ (see [6]), to obtain the existence and regularity results for the solution to the Dirichlet problem of the minimal surface equation in $M^n \times \mathbb{R}$, where M^n is simply connected and complete. More specifically, we studied the solvability of Dirichlet problem

$$\begin{aligned} \mathrm{Div}\frac{Du}{w} &= 0 \text{ in } \Omega \subset M^n, \\ u &= \varphi \text{ on } \partial\Omega. \end{aligned} \tag{1.1}$$

Following [1], we applied direct methods of the calculus of variations and instead studied the functional,

$$J(u, \Omega) = \mathcal{A}(u, \Omega) + \int_{\partial\Omega} |u - \varphi| dH_{n-1}, \tag{1.2}$$

here $\mathcal{A}$ is the area functional. In this variational setting, we will easily obtain the existence of bounded local minimizers of $J(u, \Omega)$ in the class $BV(\Omega)$. However, in the space of $M^n \times \mathbb{R}$, some definitions and key ingredients used in the proof differs from the case in the Euclidean space and need to be carefully modified. Therefore, despite some of our proofs are the same as the proof in Euclidean space, we still include them here.

As applications, we first showed that the general solution u of (1.1) is the solution when φ satisfies certain conditions.

Then we considered the problem of the graphical mean curvature flow

$$\begin{cases} u_t(x,t) = nWH(u) = \triangle_M u - \frac{u^i u^j}{W^2} D^2_{ij} u & \text{in } \Omega \times (0,\infty), \\ u(x,t) = \varphi(x) & \text{on } \partial\Omega \times (0,\infty), \\ u(x,0) = u_0(x), \end{cases} \tag{1.3}$$

and showed the longtime existence and convergence of the solution under the condition that $\operatorname{Ric} M \geq 0$.

Our main results can be stated as following

Theorem 1.1. *Let $u \in BV(\Omega) \cap L^\infty(\Omega)$ be a local minimizer to* (1.2). *Then $u \in C^\infty(\Omega)$.*

By applying this result we proved

Theorem 1.2. *Let Ω be a bounded open subset of M^n with C^2 boundary $\partial\Omega$. Given $n \geq 2$, $K \in \left(0, \frac{1}{\sqrt{(n-1)\gamma}}\right)$, and $\gamma > 1$. Suppose φ is Lipschitz continuous on $\partial\Omega$ and*

$$|\varphi(x) - \varphi(y)| \leq K\|x - y\|_{M^n}, \text{ if } x, y \in \partial\Omega, \tag{1.4}$$

$$\sup_{\partial\Omega} \varphi(x) - \inf_{\partial\Omega} \varphi(x) \leq \epsilon, \tag{1.5}$$

where ϵ depends on n, K, M^n, and $\partial\Omega$. Then there exists a function $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ such that u is the solution of the equation (1.1).

Theorem 1.3. *When $\operatorname{Ric} M \geq 0$, there exists a solution $u \in C^\infty(\Omega \times (0,\infty)) \cap C^0(\bar{\Omega} \times [0,\infty))$ to* (1.3). *And u satisfies*

$$\sup_{\Omega\times(0,\infty)} |u| \leq C \tag{1.6}$$

$$\sup_{\Omega'\times(0,\infty)} |D^\alpha u| \leq C(\Omega', \alpha) \qquad \text{for all } \Omega' \subset\subset \Omega \text{ and all } \alpha. \tag{1.7}$$

Moreover, there exists a sequence $t_k \to \infty$ such that $u(\cdot, t_k) \to \bar{u}$ as $k \to \infty$. Here $\bar{u}$ is a general solution to (1.1).

2. Functions of Bounded Variation

For any vector bundle V over M, we will denote the C^1_0 sections of V over an open set $\Omega \subset M$ by $\Gamma^1_0(V)$.

Definition 2.1. Let Ω be a bounded open subset of M. The variation of a measurable function f is defined by

$$\int_\Omega |Df| \equiv \sup\{\int_\Omega f\mathrm{Div}g \ : g \in \Gamma^1_0(T\Omega) \text{ and } |g| \leq 1 \text{ pointwise } \}. \tag{2.1}$$

A function f with $\int_\Omega |Df| < \infty$ is called a function of bounded variation on Ω. We abbreviate this as $f \in BV(\Omega)$. We will also use $\|f\|_{BV(\Omega)} \equiv |f|_{L^1(\Omega)} + \int_\Omega |Df|$.

Note, that by the Riesz Representation Theorem, we have that for test functions g with support contained in a local coordinate chart, there exists $\eta \in T\Omega$ with $|\eta| = 1$ and

$$\int_\Omega f\mathrm{Div}g = \int_\Omega g_{ij}\eta^i g^j d|Df|. \tag{2.2}$$

Note, that η is actually independent of coordinates, and so gives us a well defined $\eta \in T\Omega$. For $X \in C^1(T\Omega)$ (not necessarily compactly supported) we may then define

$$\int_\Omega \langle Df, X\rangle \equiv \int_\Omega g_{ij}\eta^i X^j \, d|Df| \tag{2.3}$$

To extend the notion of the surface area of the graph of a function $f \in C^{0,1}(M)$ to functions $f \in BV(\Omega)$, we make an additional definition.

Definition 2.2. Let Ω be a bounded open subset of M and $f \in BV(\Omega)$. We define

$$\int_\Omega \sqrt{1 + |Df|^2} \equiv \sup\{\int_\Omega g_{n+1} + f\mathrm{Div}g : g \in \Gamma^1_0(T\Omega), \tag{2.4}$$

$$g_{n+1} \in C^1_0(\Omega), \text{ and } |(g, g_{n+1})| \leq 1 \text{ pointwise } \}. \tag{2.5}$$

We willl also denote this quantity by $\mathcal{A}(u, \Omega)$.

Clearly, we have that

$$\int_\Omega |Df| \leq \int_\Omega \sqrt{1 + |Df|^2} \leq \int_\Omega |Df| + |\Omega|. \tag{2.6}$$

Also, if $f \in W^{1,1}(\Omega)$, then

$$\int_\Omega \sqrt{1+|Df|^2} = \int_\Omega \sqrt{1+|\nabla f|^2}. \tag{2.7}$$

Lemma 2.3 (Lower Semi-Continuity)**.** *If $u_j \to u$ in $L^1_{\text{loc}}(\Omega)$ then*

$$\int_\Omega \sqrt{1+|Du|^2} \le \liminf_{j\to\infty} \int_\Omega \sqrt{1+|Du_j|^2}. \tag{2.8}$$

Proof. For any $(g, g_{n+1}) \in \Gamma^1_0(T\Omega \times \mathbb{R})$, we have

$$\int_\Omega g_{n+1} + u\text{Div}g = \lim_j \int_\Omega g_{n+1} + u_j\text{Div}g \le \liminf_j \int_\Omega \sqrt{1+|Du_j|^2}. \tag{2.9}$$

□

Remark 2.4. Note, that we don't get continuity. Consider the sets $S_i = [-2,2]^2 - ([-1/i, 1/i] \times [-1,1]) \subset \mathbb{R}^2$ and the functions $f_i = \chi_{S_i}$.

An argument made in [1] Theorem 1.17 shows that

Theorem 2.5. *Let Ω be a bounded open set in M and $f \in BV(\Omega)$. Then, there exists a sequence $f_j \in C^\infty(\Omega)$ such that $f_j \to f$ in $L^1(\Omega)$ and $\int_\Omega |Df_j| \to \int_\Omega |Df|$.*

From this we may show a generalization of the compactness theorem.

Theorem 2.6. *Let $\Omega \subset M$ be a bounded open subset that is sufficiently regular for the Rellich Theorem to hold. Then, sets of functions uniformly bounded in $BV(\Omega)$ are relatively compact in $L^1(\Omega)$.*

Proof. Let $f_j \in BV(\Omega)$ such that $\|f_j\|_{BV(\Omega)} \le M$. By Theorem 2.5, for each j, we may find $g_j \in C^\infty(\Omega)$ such $\int_\Omega |f_j - g_j| < j/2$ and $\int_\Omega |Dg_j| < M+2$. So, our sequence g_j is uniformly bounded in a Sobolev Space, and we may apply the Rellich Theorem to obtain a subsequence of f_j converging to a function $f \in L^1(\Omega)$.

Now, using Lemma 2.3 we can see that $f \in BV(\Omega)$.

□

Remark 2.7. A sufficient condition for the hypotheses of 2.6 to hold is that $\partial\Omega$ is Lipschitz-continuous.

By using partitions of unity, we may extend Theorem 2.16 of [1] from the Euclidean case to $M^n \times \mathbb{R}$.

Theorem 2.8 (Extension of the Boundary)**.** *Let $\Omega \subset M$ be a bounded open set with Lipschitz-continuous boundary $\partial\Omega$ and let $\varphi \in L^1(\partial\Omega)$. For every $\epsilon > 0$ there exists $f \in W^{1,1}(\Omega)$ having trace φ on $\partial\Omega$ such that*

$$\int_\Omega |f| \leq \epsilon \|\varphi\|_{L^1(\partial\Omega)} \tag{2.10}$$

$$\int_\Omega |Df| \leq A\|\varphi\|_{L^1(\partial\Omega)} \tag{2.11}$$

with A depending only on $\partial\Omega$.

Remark 2.9. In this paper, unless otherwise stated, we always assume $\partial\Omega$ is Lipschitz-continuous.

Remark 2.10. The construction of f (see [1] Chapter 2) shows that if $\partial\Omega$ is of class C^1, we may take $A = 1 + \epsilon$.

We may also generalize a result in [1] relating the minima of two different variational problems.

Theorem 2.11. *Let Ω be a bounded open set with C^1 boundary $\partial\Omega \subset M$. Also, let $\varphi \in L^1(\partial\Omega)$. We have that*

$$\inf\{\mathcal{A}(u,\Omega) : u \in BV(\Omega), u = \varphi \text{ on } \partial\Omega\} \tag{2.12}$$

$$= \inf\{\mathcal{A}(u,\Omega) + \int_{\partial\Omega} |u-\varphi| dH_{n-1} : u \in BV(\Omega)\}. \tag{2.13}$$

Proof. It is clear that the left side is " $\geq$ " to the right side. So, we will now show the opposite inequality.

Given $\epsilon > 0$ and any $u \in BV(\Omega)$, from Theorem 2.8, we have that there exists $w \in W^{1,1}(\Omega)$ such that

$$\int_\Omega |Dw| \leq (1+\epsilon)\|\varphi - u\|_{L^1(\partial\Omega)} \tag{2.14}$$

and

$$w = \varphi - u \text{ on } \partial\Omega.$$

Now, define $v \equiv u + w$. Note that $v \in BV(\Omega)$ and that $v = \varphi$ on $\partial\Omega$.

We see that

$$\int\limits_{\Omega} \sqrt{1+|Dv|^2} \leq \int\limits_{\Omega} \sqrt{1+|Du|^2} + \int\limits_{\Omega} |Dw| \tag{2.15}$$

$$\leq \int\limits_{\Omega} \sqrt{1+|Du|^2} + (1+\epsilon)\|u-\varphi\|_{L^1(\partial\Omega)}. \tag{2.16}$$

Taking $\epsilon \to 0$, we get our desired inequality and the theorem. $\square$

We also have a lemma for constructing functions of bounded variation by gluing together functions defined on adjacent domains.

Lemma 2.12. *Choosing $R > 0$ sufficiently small such that there exists local coordinates in $\mathbf{B}_{3R}$ on $M \times \{0\}$, and let $C_R = \mathcal{B}_R \times (-R, R) \subset \mathbf{B}_{3R} \subset M \times \{0\}$. Denote the upper half cylinder by $C_R^+ = \mathcal{B}_R \times (0, R)$ and the lower cylinder by $C_R^- = \mathcal{B}_R \times (-R, 0)$ so that $\mathcal{B}_R = \partial C_R^+ \cap \partial C_R^-$. Given functions $f_1 \in BV(C_R^+)$ and $f_2 \in BV(C_R^-)$, define a function $f = \begin{cases} f_1 & \text{in } C_R^+ \\ f_2 & \text{in } C_R^- \end{cases}$. Then, $f \in BV(C_R)$ and*

$$\int\limits_{\mathcal{B}_R} |Df| = \int\limits_{\mathcal{B}_R} |f^+ - f^-|, \tag{2.17}$$

where f^+ is the trace of f_1 on $\mathcal{B}_R$, and f^- is the trace of f_2 on $\mathcal{B}_R$.

Proof.

$$\begin{aligned} \int_{\mathcal{C}_R} f \mathrm{Div} g = &- \int_{\mathcal{C}_R^+} \langle g, Df\rangle + \int_{\mathcal{B}_R} f^+ \langle g, \nu\rangle d_{H_n} \\ &- \int_{\mathcal{C}_R^-} \langle g, Df\rangle - \int_{\mathcal{B}_R} f^- \langle g, \nu\rangle d_{H_n}. \end{aligned} \tag{2.18}$$

On the other hand

$$\int_{\mathcal{C}_R} f \mathrm{Div} g = - \int_{\mathcal{C}_R^+} \langle g, Df\rangle - \int_{C_R^-} \langle g, Df\rangle - \int_{\mathcal{B}_R} \langle g, Df\rangle. \tag{2.19}$$

Therefore,

$$- \int_{\mathcal{B}_R} \langle g, Df\rangle = \int_{\mathcal{B}_R} (f^+ - f^-)\langle g, \nu\rangle d_{H_n}. \tag{2.20}$$

$\square$

Remark 2.13. By using a partition of unity argument as in Remark 2.9 of [1], we have that if $A \subset\subset \Omega \subset\subset M^n$ is an open set with Lipschitz continuous boundary ∂A, then $f|_A$ and $f|_{\Omega\setminus\bar{A}}$ will have traces on ∂A which we will call f_A^- and f_A^+ respectively. Then $\int\limits_{\partial A} |f_A^+ - f_A^-| = \int\limits_{\partial A} |Df|$.

3. FUNCTIONALS

Definition 3.1. We define

$$J(u,\Omega) = \mathcal{A}(u,\Omega) + \int\limits_{\partial\Omega} |u-\varphi| dH_{n-1}. \tag{3.1}$$

Instead of directly solving the Dirichlet boundary value problem, we solve the variational problem of finding $u \in BV(\Omega)$ minimizing $J(w,\Omega)$ among all $w \in BV(\Omega)$. Note, since boundary values are not preserved by limits in L^1, we are using the boundary integral to penalize not matching the boundary values of the Dirichlet problem.

Remark 3.2. Let Ω be a bounded open set with Lipschitz continuous boundary. If $\mathfrak{B}$ is a ball containing $\bar{\Omega}$ we can use Theorem 2.8 to extend φ to a $W^{1,1}$ function in $\mathfrak{B} - \bar{\Omega}$, that we will denote again by φ. If we set for $v \in BV(\Omega)$

$$v_\varphi(x) = \begin{cases} v(x) & x \in \Omega \\ \varphi(x) & x \in \mathfrak{B} - \Omega \end{cases} \tag{3.2}$$

then $v_\varphi \in BV(\mathfrak{B})$ and

$$\int\limits_{\mathfrak{B}} \sqrt{1+|Dv_\varphi|^2} = J(v,\Omega) + \int\limits_{\mathfrak{B}-\bar{\Omega}} \sqrt{1+|D\varphi|^2} dx. \tag{3.3}$$

In the future, we denote

$$\mathcal{A}_\varphi(v,\mathfrak{B}) = \int\limits_{\mathfrak{B}} \sqrt{1+|Dv_\varphi|^2}. \tag{3.4}$$

It is clear that u is a minimizer for our Dirichlet problem for $J(v,\Omega)$ if and only if u_φ is a minimizer to the equivalent problem: Given a function $\varphi \in W^{1,1}(\mathfrak{B} - \bar{\Omega})$, find a function $u \in BV(\mathfrak{B})$, coinciding with φ in $\mathfrak{B} - \bar{\Omega}$ and minimizing the area $\mathcal{A}_\varphi(v;\mathfrak{B})$.

Theorem 3.3 (Existence of a Minimizer)**.** *Let Ω be a bounded open set in M with Lipschitz boundary $\partial\Omega$, and let φ be a function in $L^1(\partial\Omega)$. The functional $J(u,\Omega)$ attains its minimum in $BV(\Omega)$.*

Proof. Applying Theorem 2.6 and Lemma 2.3 to the equivalent problem for $A_\varphi(u, \mathfrak{B})$, we get the conclusion. □

We have a theorem relating the surface area of the graph of a function of bounded variation to the variation of the characteristic function of its subgraph.

Theorem 3.4. *Let $u \in BV(\Omega)$ and let $U = \{(x,t) \in \Omega \times \mathbb{R} : t < u(x)\}$ be the subgraph of u. We have that*

$$\int_\Omega \sqrt{1 + |Du|^2} = \int_{\Omega\times\mathbb{R}} |D\phi_U|, \tag{3.5}$$

where ϕ_U is the characteristic function of set U.

Proof. First, consider the case that u is bounded. By translating we may consider $u \geq 1$. Let $g(x) \in \Gamma_0^1(T\Omega)$ and $g_{n+1}(x) \in C_0^1(\Omega)$ such that $|(g(x), g_{n+1}(x))| \leq 1$. Let $\eta(t)$ be a function such that suppt $\eta \subset [0, 1 + \sup_\Omega u]$, $\eta \equiv 1$ on $[1, \sup_\Omega u]$, and $|\eta| \leq 1$. Let $H(x, x_{n+1}) = \eta(x_{n+1})(g(x), g_{n+1}(x))$. We have $|H| \leq 1$ in $\Omega \times \mathbb{R}$. Then, note that

$$\int_{\Omega\times\mathbb{R}} |D\phi_U| \geq \int_U \mathrm{Div} H \tag{3.6}$$

$$= \int_\Omega \int_0^{u(x)} g_{n+1}\eta'(x_{n+1}) + \eta(x_{n+1})\mathrm{Div} g \, dx_{n+1}. \tag{3.7}$$

Since

$$\int_0^{u(x)} \eta'(x_{n+1}) dx_{n+1} = 1, \tag{3.8}$$

and

$$\int_0^{u(x)} \eta(x_{n+1}) dx_{n+1} = u(x) - \int_0^1 (1 - \eta(x_{n+1})) dx_{n+1} \tag{3.9}$$

$$= u(x) - C. \tag{3.10}$$

Therefore,

$$\int_{\Omega\times\mathbb{R}} |D\phi_U| \geq \int_\Omega g_{n+1} + u\mathrm{Div} g. \tag{3.11}$$

Hence, we get that

$$\int\limits_{\Omega\times\mathbb{R}} |D\phi_U| \geq \int\limits_{\Omega} \sqrt{1+|Du|^2}. \tag{3.12}$$

To prove the opposite direction, we first note that we have equality for C^1 functions. Now, approximate $u \in BV(\Omega)$ by C^1 functions such that $u_j \to u$ in $L^1(\Omega)$ and $\int\limits_{\Omega} \sqrt{1+|Du_j|^2} \to \int\limits_{\Omega} \sqrt{1+|Du|^2}$. On the other hand we have that

$$\phi_{U_j} \to \phi_U \text{ in } L^1_{loc}(\Omega\times\mathbb{R}), \tag{3.13}$$

and therefore

$$\int\limits_{\Omega\times\mathbb{R}} |D\phi_U| \leq \liminf_{j\to\infty} \int\limits_{\Omega\times\mathbb{R}} |D\phi_{U_j}| = \lim_{j\to\infty} \int\limits_{\Omega} \sqrt{1+|Du_j|^2} \tag{3.14}$$

$$= \int\limits_{\Omega} \sqrt{1+|Du|^2}. \tag{3.15}$$

For u unbounded, we set

$$u_T(x) = \begin{cases} u(x) & \text{if } |u| < T \\ T & \text{if } u \geq T \\ -T & \text{if } u \leq -T \end{cases} \tag{3.16}$$

Letting $T \to \infty$ we get the result. $\square$

Lemma 3.5. *Let $F \subset \Omega \times \mathbb{R}$ be measurable, and suppose that for some $T > 0$ we have*

$$\Omega \times (-\infty, -T) \subset F \subset \Omega \times (-\infty, T). \tag{3.17}$$

For $x \in \Omega$ let

$$w(x) \equiv \lim_{k\to\infty} \Big(\int_{-k}^{k} \phi_F(x,t)dt - k \Big), \tag{3.18}$$

then

$$\int\limits_{\Omega} \sqrt{1+|Dw|^2} \leq \int\limits_{\Omega\times\mathbb{R}} |D\phi_F|. \tag{3.19}$$

Proof. We first note that it is clear from our conditions on F that $-T \leq w \leq T$.

Now, let $g(x) \in \Gamma_0^1(T\Omega)$ and $g_{n+1}(x) \in C_0^1(\Omega)$ such that $|(g(x), g_{n+1}(x))| \leq 1$. Also, let $\eta(t)$ be a smooth function such that $0 \leq \eta \leq 1$, $\eta(t) = 0$ if $|t| \geq T+1$, and $\eta(t) = 1$ if $|t| \leq T$.

We have that

$$\int_{-\infty}^{\infty} \eta'(t)\phi_F(x,t)dt = 1, \tag{3.20}$$

and that

$$\int_{-\infty}^{\infty} \eta(t)\phi_F(x,t)dt = w(x) + T + \int_{-T-1}^{-T} \eta(t)dt = w(x) + \alpha. \tag{3.21}$$

Hence,

$$\int_{\Omega\times\mathbb{R}} |D\phi_F| \geq \int_{\Omega\times\mathbb{R}} \phi_F(x, x_{n+1})\text{Div}(\eta(x_{n+1})(g, g_{n+1})) \tag{3.22}$$

$$= \int_{\Omega} (w + \alpha)\text{Div}g + g_{n+1} \tag{3.23}$$

$$= \int_{\Omega} w\text{Div}g + g_{n+1}. \tag{3.24}$$

Taking the supremum over $|(g(x), g_{n+1}(x))| \leq 1$, we get the result. $\square$

Theorem 3.6. *Let F be a measurable set in $Q \equiv \Omega \times \mathbb{R}$ such that*

(1) *For a.e. $x \in \Omega$ we have*

$$\lim_{t\to\infty} \phi_F(x,t) = 0 \tag{3.25}$$

$$\lim_{t\to-\infty} \phi_F(x,t) = 1. \tag{3.26}$$

(2) *The symmetric difference $F_0 = (F - Q^-) \cup (Q^- - F)$ has finite measure, where $Q^- \equiv \{(x,t) \in Q : t < 0\}$.*

Then the function $w(x) = \lim_{k\to\infty}(\int_{-k}^{k} \phi_F(x,t)dt - k)$ belongs to $L^1(\Omega)$, and

$$\int_{\Omega} \sqrt{1 + |Dw|^2} \leq \int_{Q} |D\phi_F|. \tag{3.27}$$

Remark 3.7. In the case that $\Omega \subset \mathbb{R}^n$, conditions (1) and (2) are redundant for Caccioppoli sets F, because (2) follows from (1) by an isoperimetric inequality for $\mathbb{R}^n$. Moreover, in general, the redundancy depends on the existence of an isoperimetric inequality.

Proof. We define

$$F_k \equiv F \cup [\Omega \times (-\infty, -k)] - [\Omega \times (k, \infty)]. \tag{3.28}$$

Let

$$w_k \equiv \int_{-k}^{k} \phi_F(x,t)dt - k. \tag{3.29}$$

Note that $w_k \to w$ pointwise. Denote

$$f(x) = |\{t : (x,t) \in F_0\}|. \tag{3.30}$$

By the second hypothesis we have that $f \in L^1(\Omega)$ and that $|w_k| \leq f$. Therefore, by dominated convergence, we have that $w_k \to w$ in $L^1(\Omega)$.

Note that,

$$w_k(x) = \lim_{l \to \infty} \int_{-l}^{l} \phi_{F_k}(x,t)dt - l. \tag{3.31}$$

Hence, from the preceding lemma, we get that

$$\int_{\Omega} \sqrt{1 + |Dw_k|^2} \leq \int_{Q} |D\phi_{F_k}| \tag{3.32}$$

$$\leq \int_{Q} |D\phi_F| + \int_{\Omega \times \{k\}} \phi_F dM + \int_{\Omega \times \{-k\}} 1 - \phi_F dM. \tag{3.33}$$

Taking $k \to \infty$ and using lower semi-continuity we get the result. □

Theorem 3.8. *Let $u \in BV_{loc}(\Omega)$ be a local minimum of the area functional. Then the set $U = \{(x,t) \in \Omega \times \mathbb{R} : t < u(x)\}$ minimizes locally the perimeter in $Q = \Omega \times \mathbb{R}$.*

Proof. Let $A \subset\subset \Omega$ and let F be any set in Q coinciding with U outside a compact set $K \subset A \times \mathbb{R}$. Since u is in $L^1(\Omega)$ we have that U satisfies conditions (1) and (2) of Theorem 3.6.

Now, since F differs from U on a compact set, we have that F satisfies conditions (1) and (2) of Theorem 3.6. As in Theorem 3.6, we define

$$w(x) = \lim_{k\to\infty} \int_{-k}^{k} \phi_F(x,t)dt - k. \tag{3.34}$$

Since w coincides with u outside A and u minimizes the area functional, we get that

$$\int_{A\times\mathbb{R}} |D\phi_U| = \int_A \sqrt{1+|Du|^2} \leq \int_A \sqrt{1+|Dw|^2} \leq \int_{A\times\mathbb{R}} |D\phi_F|. \tag{3.35}$$

□

4. Perimeter

Definition 4.1. For a Cacciopoli set E and Ω an open domain, we define $P(E,\Omega) = \int_\Omega |D\phi_E|$. Note, that $P(E,\Omega)$ is the perimeter of E inside Ω, but not including the perimeter coming from $\partial\Omega$.

Lemma 4.2. *For a Cacciopoli set E and $\rho < R$, we have that*
i) $P(E\cap B_\rho, B_R) = P(E, B_\rho) + H_{n-1}(\partial B_\rho \cap E)$,
ii)$P(E\backslash B_\rho, B_R) = P(E, B_R\backslash B_\rho) + H_{n-1}(\partial B_\rho \cap E)$.

Proof. From Remark 2.13, we conclude that if we define $F : \Omega \to \mathbb{R}$ by

$$F(x) = \begin{cases} f(x) & x \in A \\ 0 & x \in \Omega\backslash A \end{cases}. \tag{4.1}$$

then

$$\int_\Omega |DF| = \int_A |Df| + \int_{\partial A\cap\Omega} |f_A^-|. \tag{4.2}$$

For equation (i), we let $A = B_\rho$ and $\Omega = B_R$. Also, define

$$F = \phi_{E\cap B_\rho} = \begin{cases} \phi_E & x \in B_\rho \\ 0 & x \in \Omega\backslash B_\rho \end{cases}. \tag{4.3}$$

We then get that

$$\begin{aligned} \int_{B_R} |D\phi_{E\cap B_\rho}| &= \int_{B_\rho} |D\phi_E| + \int_{\partial B_\rho\cap B_R} |\phi_E^-| dH_{n-1} \\ &= P(E, B_\rho) + H_{n-1}(\partial B_\rho \cap E). \end{aligned} \tag{4.4}$$

Now for equation (ii), let $A = B_R \backslash \bar{B}_\rho$, $\Omega = B_R$, and

$$F = \phi_{E\backslash \bar{B}_\rho} = \begin{cases} \phi_E & x \in B_R \backslash \bar{B}_\rho \\ 0 & x \in B_\rho \end{cases}. \tag{4.5}$$

Then, we have that

$$\int\limits_{B_R} |D\phi_{E\backslash B_\rho}| = \int\limits_{B_R\backslash B_\rho} |D\phi_E| + \int\limits_{\partial(B_R\backslash B_\rho)\cap B_R} |\phi_E^-| dH_{n-1}. \tag{4.6}$$

Note that $\partial(B_R\backslash B_\rho) \cap B_R = \partial B_\rho$, and we get the lemma. □

We also recall without proof some properties of perimeter (see [2]).

Lemma 4.3. *i) (Locality) $P(E, A) = P(F, A)$ whenever $Vol((E \triangle F) \cap A) = 0$,*
ii) (Subadditivity) $P(E \cup F, A) + P(E \cap F, A) = P(E, A) + P(F, A)$,
iii) (Complementation) $P(E, A) = P(E^c, A)$.

By applying Lemma 4.3 we have

Lemma 4.4 (Absolute Continuity)**.** *If E is a set of finite perimeter in M^{n+1}, then we have that $P(E, B) = 0$ whenever $H_n(B) = 0$.*

Proof. We may assume without loss of generality that B is a compact set. Let $\mathfrak{B}_r$ be a geodsic ball cenetered at some point. Hence, there exists an R such that for all $r < R$ we have that $H_n(\partial \mathfrak{B}_r) < Cr^n$ and $H_{n+1}(\mathfrak{B}_r) < Cr^{n+1}$. Since $H_n(B) = 0$, for any $\epsilon > 0$, we can cover B in a finite number of balls $\mathfrak{B}_i^\epsilon$ of radius r_i^ϵ and center x_i^ϵ such that $\sum\limits_i (r_i^\epsilon)^n < \epsilon$. Now, let $A_\epsilon \equiv \bigcup\limits_i B_{2r_i^\epsilon}(x_i^\epsilon)$. We have $B \subset\subset A_\epsilon$ and

$$P(A_\epsilon, M^{n+1}) \le C(2r_i^\epsilon)^n \le 2^n C\epsilon. \tag{4.7}$$

So, by Lemma 4.3 we have

$$\begin{aligned} P(E \cup A_\epsilon, M^{n+1}) &= P(E \cup A_\epsilon, M^{n+1}\backslash B) \\ &\le P(E, M^{n+1}\backslash B) + P(A_\epsilon, M^{n+1}) \\ &\le P(E, M^{n+1}\backslash B) + 2^n C\epsilon. \end{aligned} \tag{4.8}$$

Since, $H_{n+1}(B) = 0$ as well, we have that $H_{n+1}(A_\epsilon) \to 0$. Therefore, upon passing to the limit as $\epsilon \searrow 0$, the lower semi-continuity of perimeter gives

$$P(E, M^{n+1}) \le P(E, M^{n+1}\backslash B). \tag{4.9}$$

Hence, $P(E, B) = 0$. □

5. Sobolev Inequality and Consequences

We first have a Sobolev inequality proven by D. Hoffman and J. Spruck.

Theorem 5.1. *(See [3] Theorem 2.2) Let $M \subset N$ be compact with boundary ∂M and assume that $K_N \leq b^2$. Then*

$$|M|^{(m-1)/m} \leq C(m)\left(Vol(\partial M) + \int\limits_M |H| dV_M \right), \tag{5.1}$$

provided that

$$b^2(1-\alpha)^{-2/m}(\omega_m^{-1} VolM)^{2/m} \leq 1 \tag{5.2}$$

and

$$2\rho_0 \leq \bar{R}(M). \tag{5.3}$$

Here, $0 < \alpha < 1$,

$$\rho_0 = \begin{cases} b^{-1}\sin^{-1} b(1-\alpha)^{-1/m}(\omega_m^{-1} VolM)^{1/m} & \text{for } b \text{ real} \\ (1-\alpha)^{-1/m}(\omega_m^{-1} VolM)^{1/m} & \text{for } b \text{ imaginary} \end{cases}, \tag{5.4}$$

and $\bar{R}(M)$ =minimum distance to the cut locus in N for all points in M.

Next, we have a proposition giving estimates for the intersection of minimal sets with balls.

Proposition 5.2. *If E is a minimal set in $W \subset M^n \times \mathbb{R}$ and $x_0 \in E \cap W$, then for every $r < dist(x_0, \partial W)$, $\rho_0 \leq \bar{R}(B_r(x_0))$, and*

$$b^2(1-\alpha)^{-2/(n+1)}(\omega_{n+1}^{-1} Vol(B_r(x_0)))^{2/(n+1)} \leq 1, \tag{5.5}$$

we have

$$|E \cap B(x_0, r)| \geq (\frac{r}{(n+1)c(n+1)})^{n+1} \tag{5.6}$$

Proof. We first claim that for any $B_\rho \subset\subset W$, we have that

$$\int\limits_{B_\rho} |D\phi_E| \leq \int\limits_W |D\phi_{E\setminus B_\rho}|. \tag{5.7}$$

From the minimality of E in W we get that

$$\int\limits_{B_\rho} |D\phi_E| \leq \int\limits_{B_{\rho+\epsilon}} |D\phi_E| \leq \int\limits_{B_{\rho+\epsilon}} |D\phi_{E\setminus B_\rho}|. \tag{5.8}$$

Since $\phi^+_{E\backslash B_\rho} = \phi^+_E$ on ∂B_ρ, $\phi^-_{E\backslash B_\rho} = 0$ on ∂B_ρ, and $\phi_{E\backslash B_\rho} \equiv 0$ on B_ρ, we have that

$$\int_{B_{\rho+\epsilon}} |D\phi_{E\backslash B_\rho}| = \int_{B_{\rho+\epsilon}\backslash \bar{B}_\rho} |D\phi_{E\backslash B_\rho}| + \int_{\partial B_\rho} |\phi^+_E| dH_n. \tag{5.9}$$

Since $B_{\rho+\epsilon}\backslash \bar{B}_\rho \to \emptyset$ as $\epsilon \to 0$, we get that

$$\int_{B_\rho} |D\phi_E| \leq \int_{\partial B_\rho} |\phi^+_E| dH_n. \tag{5.10}$$

Now, note that for every ρ, we have from Lemma 4.2 that

$$\int_W |D\phi_{E\cap B_\rho}| = \int_{B_\rho} |D\phi_E| + \int_{\partial B_\rho} |\phi^-_E| dH_n. \tag{5.11}$$

Now, define $E_\rho \equiv E \cap B_\rho$. From equation (5.10) and (5.11) we get that

$$\int_W |D\phi_{E_\rho}| \leq 2H_n(\partial B_\rho \cap E) = 2\frac{d}{d\rho}|E_\rho|. \tag{5.12}$$

By Theorem 5.1, we have that

$$\frac{d}{d\rho}|E_\rho| \geq \frac{1}{c(n+1)}|E_\rho|^{\frac{n}{n+1}}. \tag{5.13}$$

Therefore,

$$|E_r| \geq (\frac{r}{(n+1)c(n+1)})^{n+1}. \tag{5.14}$$

□

6. Minimizers of Area

Theorem 6.1. *Let $u \in BV_{loc}(\Omega)$, $\Omega \subset M^n$, minimize the area functional. Then u is locally bounded in Ω.*

Proof. Suppose that there exists a compact set $K \subset \Omega$ such that u is not bounded on K. Let $R \equiv \frac{1}{2}\text{Dist}(K, \partial\Omega)$ and

$$b^2(1-\alpha)^{-2/(n+1)}(\omega^{-1}_{n+1}\text{Vol}(B_R(x_0)))^{2/(n+1)} \leq 1. \tag{6.1}$$

For every integer $m \geq 0$ there exists a point $x_m \in K$ such that $u(x_m) > 2mR$. It follows that the points $z_i = (x_i, 2iR) \in U$. From Proposition 5.2 we have

$$|U \cap B(z_i, R)| \geq CR^{n+1}, \tag{6.2}$$

where c depends on the curvature of $M^n \times \mathbb{R}$ and on n.

Then,

$$\int_{K_R} |u| \geq \sum_{i=1}^{m} |U \cap B(z_i, R)| \geq cmR^{n+1} \tag{6.3}$$

where $K_R \equiv \{x \in \Omega : \mathrm{Dist}(x, K) < R\}$. Since m is arbitrary, this would imply that $\int\limits_{K_R} |u| = +\infty$, contrary to the hypothesis. $\square$

Remark 6.2. We can see from here that, for a general boundary data φ, in order to prove a local C^0 bound for the area functional minimizer u we have to assume some restriction on $\bar{R}$. For our later application (see Section 8), we can sacrifice the generality of boundary data φ to get rid off the restriction on $\bar{R}$.

Theorem 6.3. *Assume $\partial\Omega$ is Lipschitz-continuous and let $\varphi \in L^\infty(\partial\Omega)$. Then $\mathcal{A}_\varphi(u, \mathfrak{B})$, where $\Omega \subset \mathfrak{B}$, attains its minimum at u in $BV(\Omega)$. Moreover $u \in L^\infty(\Omega)$ and $\|u\|_\infty \leq M$ for some $M = M(\|\varphi\|_\infty)$.*

Proof. let

$$\overline{u} = c_1 > \|\varphi\|_{L^\infty(\mathfrak{B})}, \tag{6.4}$$

and

$$\underline{u} = c_2 < -\|\varphi\|_{L^\infty(\mathfrak{B})}. \tag{6.5}$$

It's easy to see that $\overline{u}$ and $\underline{u}$ both satisfies equation

$$\mathrm{Div}\frac{Dv}{W} = 0, \text{ in } \mathfrak{B}. \tag{6.6}$$

Now, let $u_j \in BV(\Omega)$ be a minimizing sequence, that is

$$\inf \mathcal{A}_\varphi(u, \mathfrak{B}) = \lim_j \mathcal{A}_\varphi(u_j, \mathfrak{B}) = I. \tag{6.7}$$

Let us approximate the u_j 's with smooth functions in $C^\infty(\Omega)$ which we still denote by u_j.

Set

$$\overline{u}_j = \min\{u_j, \overline{u}\} \tag{6.8}$$

It's easy to verify that

$$\mathcal{A}_\varphi(u_j, \mathfrak{B}) \geq \mathcal{A}_\varphi(\overline{u}_j, \mathfrak{B}). \tag{6.9}$$

Analogously, set

$$\underline{u}_j = \max\{\underline{u}, \overline{u}_j\}, \tag{6.10}$$

we have

$$\underline{u} \le \underline{u}_j \le \overline{u}. \tag{6.11}$$

Moreover,

$$\mathcal{A}_\varphi(\overline{u}_j, \mathfrak{B}) \ge \mathcal{A}_\varphi(\underline{u}_j, \mathfrak{B}) \tag{6.12}$$

Since $\underline{u}_j$'s are uniformly bounded in $BV(\mathfrak{B})$, we can extract a subsequence which converges in $L^1(\mathfrak{B})$ to some function $u \in BV(\mathfrak{B})$. Furthermore, $u \in L^\infty(\Omega)$ and $u = \varphi$ in $\mathfrak{B} - \bar{\Omega}$. Then by the lower semicontinuity of our functional we find that u is the required minimizer. □

Lemma 6.4. *On the set $L = \Omega - \mathrm{Proj}\Sigma \subset M^n$, where Σ is the singular set of U, the height function u is regular.*

Proof. It is well known that $H_{n-6}(\Sigma) = 0$ (see [4]). To see that u is regular on L, it is sufficient to show that $\nu_{n+1} > 0$ on $\partial U \backslash \Sigma$. Suppose on the contrary that at a point $x_0 \in \partial U \backslash \Sigma$, we have $\nu_{n+1} = 0$. Then, in a neighborhood V of x_0 we have $\nu_{n+1}(x) \ge 0$. Note that,

$$\triangle_S \nu_{n+1} + (|A|^2 + \mathrm{Ric}(N)) = 0. \tag{6.13}$$

Let $C^+ = (|A|^2 + \mathrm{Ric}(N))^+$ and $C^- = (|A|^2 + \mathrm{Ric}(N))^-$. We have that

$$\triangle_S \nu_{n+1} + C^- \nu_{n+1} = -C^+ \nu_{n+1} \le 0, \tag{6.14}$$

and so $\nu_{n+1} \equiv 0$.

Therefore, ν_{n+1} vanishes identically in a neighborhood of V of x_0. Let $\Gamma = \mathrm{Proj}V$. We have $H_{n-1}(\Gamma) > 0$. If $z \in \Gamma$, the vertical straight line through z contains a point $x \in \partial U \backslash \Sigma$ with $\nu_{n+1}(x) = 0$. It follows from above that if this line does not meet Σ, then it lies entirely on ∂U. This is impossible since u is locally bounded. Therefore, we must have $\Gamma \subset \mathrm{Proj}\Sigma$, but then $H_{n-1}(\Sigma) > 0$ which is a contradiciton. □

Proposition 6.5. *Let $u \in BV_{loc}(\Omega)$ minimize the area in Ω. Then $u \in W^{1,1}_{loc}(\Omega)$.*

Proof. Let $S = \mathrm{Proj}(\Sigma)$. We have seen that u is regular in $\Omega \backslash S$ and $H_{n-6}(S) = 0$. In particular $|S| = 0$. If $A \subset\subset \Omega$ is an open set, we have

$$\int\limits_{A} \sqrt{1 + |Du|^2} = \int\limits_{A \backslash S} \sqrt{1 + |Du|^2} + \int\limits_{S \cap A} \sqrt{1 + |Du|^2}. \tag{6.15}$$

On the other hand, Lemma 4.4 tells us that $P(U, S \times \mathbb{R}) = 0$, and therefore

$$\int\limits_{A} \sqrt{1 + |Du|^2} = \int\limits_{A \backslash S} \sqrt{1 + |Du|^2}. \tag{6.16}$$

Hence $\int\limits_{S \cap A} |Du| = 0$, and so $u \in W^{1,1}_{loc}(\Omega)$.

□

We have the following result from Giusti concerning the uniqueness of minimizers of the functional $J(u, \Omega)$ that we state without proof.

Proposition 6.6. *Let Ω be connected and let $\phi \in L^1(\partial\Omega)$. Suppose u, v are two minimas of the functional $J(u, \Omega)$. We have that*

$$v = u + \textit{constant}. \tag{6.17}$$

7. Regularity

Proposition 7.1. *Let $u \in BV_{loc}(\Omega)$ minimize locally the functional $\int\limits_{\Omega} \sqrt{1 + |Du|^2}$. Then, u is Lipschitz-continuous (and hence analytic) in Ω.*

Proof. Let $\mathcal{B} \equiv \mathcal{B}(x_0, R)$ be a small ball in Ω. We have

$$\int\limits_{\mathcal{B}} \sqrt{1 + |Du|^2} dx \leq \int\limits_{\mathcal{B}} \sqrt{1 + |Dw|^2} dx + \int\limits_{\partial\mathcal{B}} |w - u| dH_{n-1} \tag{7.1}$$

for every $w \in BV(\mathcal{B})$. Since the singular set S satisfies $H_{n-6}(S) = 0$, we can find a descending sequence of open sets S_n such that $S_{n+1} \subset\subset S_n$, $\bigcap\limits_n S_n = S$, and $H_{n-1}(S_j \cap \partial\mathcal{B}) \to 0$.

Now, let ϕ_j be a smooth function on $\partial\mathcal{B}$ satisfying

$$\phi_j = u \text{ in } \partial\mathcal{B} \backslash S_j$$
$$\sup_{\partial\mathcal{B}} |\phi_j| \leq 2 \sup_{\partial\mathcal{B}} |u|$$

It is well known that there exists a unique solution u_j of the Dirichlet Problem with boundary datum ϕ_j on $\partial\mathcal{B}$ (see [5]). The functions u_j are smooth in $\mathcal{B}$, and moreover $\sup_{\mathcal{B}} |u_j| \le 2 \sup_{\partial\mathcal{B}} |u|$. We have

$$\int_{\mathcal{B}} \sqrt{1+|Du_j|^2} \le \int_{\mathcal{B}} \sqrt{1+|Dw|^2} + \int_{\mathcal{B}} |w-\phi_j| dH_{n-1} \tag{7.2}$$

for every $w \in BV(\mathcal{B})$.

From the a-priori estimate of the gradient (see Theorem 1.1 in [6]), we conclude that the gradients Du_j are equibounded in every compact set $K \subset \mathcal{B}$. Using the Arzela-Ascoli Theorem and passing to a subsequence, we get uniform convergence on compact subsets of $\mathcal{B}$ to a locally Lipschitz-continuous function v. Taking $w = 0$, in equation (7.2), we get

$$\int_{\mathcal{B}} \sqrt{1+|Du_j|^2} \le |\mathcal{B}| + \int_{\partial\mathcal{B}} |\phi_j| dH_{n-1} \le C. \tag{7.3}$$

Therefore, we have $v \in W^{1,1}(\mathcal{B})$.

We want to prove that v has trace u on $\partial\mathcal{B}$. For that, let $y \in \partial\mathcal{B}$ be a regular point for u. For j sufficiently large, $y \in \partial\mathcal{B}\backslash S_j$, and therefore, for all $k > j$ $\phi_k = u$ in a neighborhood of y in $\partial\mathcal{B}$. We can therefore construct two functions ϕ^+ and ϕ^-, both of class C^2 on $\partial\mathcal{B}$ such that
(i) $\phi^\pm = u$ in a nieghborhood of y in $\partial\mathcal{B}$,
(ii) $\phi^- \le \phi_k \le \phi^+$ in $\partial\mathcal{B}$ for $k > j$.
Let $u^\pm$ be the solutions of the Dirichlet problems with boundary data $\phi^\pm$ respectively. We have $u^- \le u_k \le u^+$ for all $k > j$. Hence we get that

$$u^- \le v \le u^+. \tag{7.4}$$

Therefore, $v = u$ at every regular point $y \in \partial\mathcal{B}$. Since $H_{n-1}(S) = 0$ we have that v has trace u on $\partial\mathcal{B}$. So equation (7.2) gives us that

$$\int_{\mathcal{B}} \sqrt{1+|Dv|^2} \le \int_{\mathcal{B}} \sqrt{1+|Dw|^2} + \int_{\partial\mathcal{B}} |w-u| dH_{n-1}. \tag{7.5}$$

Since $v = u$ on $\partial\mathcal{B}$, by Proposition 6.6 we have that $u = v$. This implies that u is Lipschitz-continuous and hence, analytic in Ω.

□

8. The Dirichlet Problem

One application of Theorem 3.3 is to study the solvability of the Dirichlet problem:

$$\mathrm{Div}\frac{Du}{W} = 0 \text{ in } \Omega \subset M^n, \tag{8.1}$$

$$u = \varphi \text{ on } \partial\Omega. \tag{8.2}$$

Here, we are going to follow the argument in [7] to construct barriers on a neighborhood of the boundary and show that the general solution u (as obtained before) is the solution of (8.1) when φ satisfies certain conditions.

Theorem 8.1. *Given $n \geq 2$, $K \in (0, 1\backslash\sqrt{(n-1)\gamma})$, and $\gamma > 1$, there exists $\epsilon > 0$ depending on n, K, M^n, and $\partial\Omega$ such that if*

(1) *$x^0 \in \partial\Omega$ and $\partial\Omega$ is C^2 near x^0;*
(2) *$\varphi \in L^1(\partial\Omega)$ satisfies*

$$\varphi \leq \varphi(x^0) + \min\{K\|x - x^0\|_{M^n}, \epsilon\} \tag{8.3}$$

where $x \in \partial\Omega \cap \mathcal{N}_0$, $\mathcal{N}_0$ is a neighborhood of x^0;
(3) *u is the generalize solution of the Dirichlet problem*

then

$$\lim_{x\to x_0}\sup_{x\in\Omega} u(x) \leq \varphi(x^0). \tag{8.4}$$

Furthermore, there is a constant C depending on n, K, M^n, and $\partial\Omega$ such that

$$u(x) \leq \varphi(x^0) + C\|x - x^0\|_{M^n},\ x \in \Omega \cap \mathcal{N}_0. \tag{8.5}$$

Remark 8.2. Condition (1) can be relaxed to an exterior ball condition on $\partial\Omega$ at x^0, but for convenience we assume $\partial\Omega$ is C^2 near x^0.

Proof. Since $\partial\Omega$ is C^2 near x^0, we may assume $x^0 = 0$ and the interior unit normal to $\partial\Omega$ at x^0 is e_n. Near x^0, we have local geodesic normal coordinates $x_1, \cdots, x_n$ for M^n, and we may assume $\partial\Omega$ is given by

$$\{(x', w(x'))|x' = (x_1, \cdots, x_{n-1})\} \tag{8.6}$$

near x^0. Here w is a C^2 function with $w(0) = 0$, $Dw(0) = 0$ and $|D^2w(0)| \leq L$. Note that $d(x^0, x) = \sum_i x^i x^i$.

Now consider the function

$$\psi(x) := K^2\sum_{i=1}^{n-1} x_i x_i + 2\alpha(x_n - w(x')), \tag{8.7}$$

where α is a constant to be chosen later. The metric on M^n will be denoted by σ_{ij}. Let $v(x) := \psi^{1\backslash 2}$, then we have

$$v_i = \frac{1}{2}\psi^{-1\backslash 2}\psi_i = \frac{1}{2v}\psi_i \tag{8.8}$$

$$v_{ij} = -\frac{1}{4}\psi^{-3\backslash 2}\psi_i\psi_j + \frac{1}{2}\psi^{-1\backslash 2}\psi_{ij}. \tag{8.9}$$

Moreover,

$$\psi_i = 2K^2 x_i - 2\alpha w_i, \qquad 1 \le i \le n-1, \tag{8.10}$$

$$\psi_n = 2K^2 x_n + 2\alpha, \tag{8.11}$$

$$\psi_{ij} = 2K^2\delta_{ij} - 2\alpha w_{ij}, \qquad 1 \le i \le j \le n. \tag{8.12}$$

Now denote $Q = g^{ij}\nabla^2_{ij}$ where we are using the connection on M^n and $g^{ij} = \sigma^{ij} - \frac{v^i v^j}{1+|\nabla v|^2}$ is the inverse of the metric on the graph of v. Here $v^i = \sigma^{ij}v_j$. We have

$$\begin{aligned} Qv &= g^{ij}v_{ij} - g^{ij}\Gamma^k_{ij}v_k \\ &= g^{ij}\{-\frac{1}{4}v^{-3}\psi_i\psi_j + \frac{1}{2}v^{-1}\psi_{ij}\} - g^{ij}\Gamma^k_{ij}v_k \\ &= -\frac{1}{v}\{g^{ij}v_iv_j - K^2 g^{ii} + \alpha g^{ij}w_{ij}\} - g^{ij}\Gamma^k_{ij}v_k \\ &= -\frac{1}{v}\{g^{ij}v_iv_j - K^2 g^{ii} + \alpha g^{ij}w_{ij} + K^2\Gamma^l_{ij}g^{ij}x_l - \alpha\Gamma^l_{ij}g^{ij}w_l + \alpha\Gamma^n_{ij}g^{ij}\}. \end{aligned} \tag{8.13}$$

For brevity of notation, define $W = \sqrt{1+|\nabla v|^2}$. Since,

$$g^{ij}v_iv_j = \frac{|\nabla v|^2}{1+|\nabla v|^2} \tag{8.14}$$

and

$$g^{ii} = \sigma^{ii} - \frac{v^iv^i}{W^2}, \tag{8.15}$$

we have

$$Qv = -\frac{1}{v}\left\{\frac{W^2-1}{W^2} + K^2\left[\frac{v^iv^i}{W^2} - \sigma^{ii}\right] + \alpha g^{ij}w_{ij} + K^2\Gamma^l_{ij}g^{ij}x_l - \alpha\Gamma^l_{ij}g^{ij}w_l + \alpha\Gamma^n_{ij}g^{ij}\right\}. \tag{8.16}$$

Note that, for fixed $\alpha > 0$ that $|v_i| = O(|x|^{1/2})$ for $1 \le i \le n-1$ and that $|v_n| \ge C_\alpha |x|^{-1/2}$. Therefore, as $x \to x_0$, we have $W \to \infty$, $\frac{v^iv^j}{W^2} \to \frac{\sigma^{in}\sigma^{jn}}{\sigma^{nn}} = \delta_{in}\delta_{jn}$, and

$g^{ij} \to \sigma^{ij} - \frac{\sigma^{in}\sigma^{jn}}{\sigma^{nn}} = \delta_{ij} - \delta_{in}\delta_{jn}$. Hence, as $x \to x_0$ we have

$$\frac{W^2-1}{W^2} + K^2\left[\frac{v^iv^i}{W^2} - \sigma^{ii}\right] + \alpha g^{ij}w_{ij} + K^2\Gamma^l_{ij}g^{ij}x_l - \alpha\Gamma^l_{ij}g^{ij}w_l + \alpha\Gamma^n_{ij}g^{ij} \tag{8.17}$$

$$\to 1 + K^2(1-n) + \alpha\sigma^{ij}w_{ij}. \tag{8.18}$$

So, by first choosing $\alpha > 0$ small enough depending only on K, n, and L, we may then choose a neighborhood of x_0 where $Qv < 0$. Note that from our assumptions on φ we get that $v \geq \varphi$ on $\partial\Omega$. Therefore, v is a supersolution, and (8.4), (8.5) follows. □

Corollary 8.3. *Let Ω be a bounded open subset of M^n with C^2 boundary $\partial\Omega$. Given $n \geq 2$, $K \in \left(0, \frac{1}{\sqrt{(n-1)\gamma}}\right)$, and $\gamma > 1$. Suppose φ is Lipschitz continuous on $\partial\Omega$ and*

$$|\varphi(x) - \varphi(y)| \leq K\|x-y\|_{M^n}, \ \text{if } x, y \in \partial\Omega, \tag{8.19}$$

$$\sup_{\partial\Omega}\varphi(x) - \inf_{\partial\Omega}\varphi(x) \leq \epsilon, \tag{8.20}$$

where ϵ depends on n, K, M^n, and $\partial\Omega$. Then there exists a function $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ such that u is the solution of the equation (8.1)*.*

9. Mean Curvature Flow

During this section, for a function $u : M \to \mathbb{R}$, we will often need to make distinction between the covariant derivative Du of u as a function on M and the covariant derivative ∇u of u as a function on the surface that is the graph of u. We will also need to make distinction between the metric σ_{ij} on M and the metric g_{ij} on the graph of u. Since our calculations take place on M, we will use the convention that we only use σ_{ij} and its inverse σ^{ij} to raise and lower indices. For convenience, we will often use $W \equiv \sqrt{1 + |Du|^2}$.

Note that the upwards normal to the graph of u is $N = (1/W)(-Du, 1)$. Furthermore, by extending any function $g : M \to \mathbb{R}$ to $\bar{g} : M \times \mathbb{R} \to \mathbb{R}$ using $\bar{u}(x,t) = u(x)$, we easily see that

$$|\nabla g|^2 = |Dg|^2 - \langle Dg, N\rangle^2 \geq |Dg|^2(1 - |Du|^2/W^2) = |Dg|^2/W^2. \tag{9.1}$$

Now, we consider the problem of the graphical mean curvature flow

$$\begin{cases} u_t(x,t) = nWH(u) = \triangle_M u - \frac{u^i u^j}{W^2} D^2_{ij} u & \text{in } \Omega \times (0,\infty) \\ u(x,t) = \phi(x) & \text{on } \partial\Omega \times (0,\infty) \\ u(x,0) = u_0(x) \end{cases} \tag{9.2}$$

where $g^{ij} = \sigma^{ij} - \frac{u^i u^j}{W^2}$. Here, we assume no conditions on the mean curvature of $\partial\Omega$. We only assume that all data are C^∞ smooth, and that our compatibility condition,

$$u_0 = \phi \text{ on } \partial\Omega, \tag{9.3}$$

is of order zero.

Like Oliker and Uralt'seva [], we use solutions to the perturbed and regularized problem

$$\begin{cases} u^\epsilon_t(x,t) = \triangle_M u^\epsilon - \frac{u^{\epsilon i} u^{\epsilon j}}{W^2} D^2_{ij} u^\epsilon - \epsilon\sqrt{1+|Du^\epsilon|^2_M} \triangle_M u^\epsilon & \text{in } \Omega \times (0,\infty) \\ u^\epsilon(x,t) = \phi(x) & \text{on } \partial\Omega \times (0,\infty) \\ u^\epsilon(x,0) = u_0(x). \end{cases} \tag{9.4}$$

For convenience of notation, we will introduce the operator $L^\epsilon f = \triangle_M f - \frac{f^i f^j}{1+|Df|^2} - \epsilon\sqrt{1+|Df|^2}\triangle_M f$.Note that this regularized problem is uniformly parabolic for every $\epsilon > 0$, but note that since we only have the zeroth order compatibility condition $u_0 = \phi$ on $\partial\Omega \times \{0\}$. Therefore, we are guaranteed to have the existence of a unique solution $u^\epsilon \in C^\infty(\Omega \times [0,\infty)) \cap C^\infty(\partial\Omega \times (0,\infty)) \cap C^0(\bar{\Omega} \times [0,\infty))$ []. That is, u^ϵ is continuous everywhere but only C^∞ away from the edge of $\Omega \times [0,\infty)$. By establishing appropriate estimates (uniform in ϵ) for the perturbed problem (9.4), we show that as $\epsilon \to 0$, we get a pseudo-solution to the Mean Curvature Flow 9.2.

Just like Oliker and Uralt'seva [], we have the following estimates. Their proof is very similar to that Oliker and Uralt'seva, but we include it just for completeness.

Theorem 9.1. *The solution u^ϵ to* (9.4) *satisfies the following estimates (uniform in ϵ):*

$$\sup_{\bar{\Omega} \times [0,\infty)} |u^\epsilon| \leq C, \tag{9.5}$$

$$\sup_{\Omega \times (0,\infty)} |u^\epsilon_t| \leq C, \tag{9.6}$$

$$\sup_{t \in (0,\infty)} \int_\Omega \sqrt{1+|Du^\epsilon|^2} + \epsilon |Du^\epsilon|^2 \, dx \leq C. \tag{9.7}$$

Here $C = C(u_0, \phi, \Omega)$.

Proof. Inequality (9.5) follows easily from the maximum principle, but inequality (9.6) does not. We do not have any guarantee that u_t is continuous on the edge of the parabolic domain. In order to deal with this, we introduce a regularization of (9.4) that has a first order compatibility condition by changing the boundary conditions on $\partial\Omega \times [0,\infty)$. Let $\psi(t) \in C^\infty([0,\infty))$ such that $\psi(0) = 0$, suppt $\psi \subset [0,2]$, $\psi'(0) = 1$, and $|\psi'(0)| \leq 1$. Consider the problem

$$\begin{cases} u_t^{\epsilon\delta}(x,t) = L^\epsilon u^{\epsilon\delta} & \text{in } \Omega \times (0,\infty), \\ u^{\epsilon\delta}(x,t) = \phi(x) + \delta\psi(t/\delta)L^\epsilon u_0 & \text{on } \partial\Omega \times (0,\infty), \\ u^{\epsilon\delta}(x,0) = u_0(x). \end{cases} \tag{9.8}$$

The problem (9.8) is uniformly parabolic and satisfies a first order compatibility condition. So, we are guaranteed to have a solution $u^{\epsilon\delta}$ with $u_t^{\epsilon\delta}$ continuous. Since (9.8) has no zeroth order terms for $u^{\epsilon\delta}$ we find that $u_t^{\epsilon\delta}$ satisfies an equation of the form

$$u_{tt}^{\epsilon\delta} = a^{ij}(x,t)D_{ij}^2 u_t^{\epsilon\delta} + b^i(x,t)D_i u_t^{\epsilon\delta}, \tag{9.9}$$

where a^{ij}, b^i are smooth on $\Omega \times (0,\infty)$. On $\partial\Omega \times [0,\infty)$ we have that $|u_t^{\epsilon\delta}| \leq \sup|\psi'(t/\delta)L^\epsilon u_0(x)| \leq \sup|L^\epsilon u_0(x)|$. Therefore, by the maximum principle we have that $|u_t^{\epsilon\delta}| \leq \sup|L^\epsilon u_0|$.

Now, we may use that $u^\epsilon - u^{\epsilon\delta}$ also satisfies an equation of the form (9.9). Therefore, from the boundary conditions of (9.8) and $u^\epsilon, u^{\epsilon\delta} \in C(\bar{\Omega} \times [0,\infty))$, we see that $|u^\epsilon - u^{\epsilon\delta}| \leq \delta \sup_{\partial\Omega} |L^\epsilon u_0|$. Therefore, $u^{\epsilon\delta} \to u^\epsilon$ uniformly as $\delta \to 0$. So, therefore, for any $(x,t) \in \Omega \times (0,\infty)$, we have that

$$\left|\frac{u^\epsilon(x,t+h) - u^\epsilon(x,t)}{h}\right| \leq \sup|L^\epsilon u_0|.$$

Therefore, away from the edge of the domain, we have that $|u_t^\epsilon| \leq |L^\epsilon u_0| \leq C$. Hence, we have (9.6).

To show (9.7), we use that a cutoff function $\eta \in C_0^1(\Omega)$, the fact that $\frac{u_t^\epsilon}{W} - \mathrm{Div}_M \frac{Du^\epsilon}{W} - \epsilon\triangle_M u^\epsilon = 0$ in $\Omega \times (0,\infty)$, and integration by parts to get

$$\int_{\Omega\times\{t\}} \frac{u_t^\epsilon}{W}\eta + \frac{\langle Du^\epsilon, D\eta\rangle}{W} + \epsilon\langle Du^\epsilon, D\eta\rangle = 0 \tag{9.10}$$

We then use the choice of test function $\eta(x) = u^\epsilon(x,t) - u_0(x)$ in (9.10) to get

(9.11)
$$\int\limits_{\Omega\times\{t\}} \sqrt{1+|Du^\epsilon|^2} + \epsilon|Du^\epsilon|^2 = \int\limits_{\Omega\times\{t\}} \frac{1+\langle Du^\epsilon, Du_0\rangle - (u^\epsilon - u_0)u_t^\epsilon}{W} + \epsilon\langle Du^\epsilon, Du_0\rangle.$$

Using a Cauchy-Schwarz inequality we then get

(9.12)
$$\int\limits_{\Omega\times\{t\}} \sqrt{1+|Du^\epsilon|^2} + (\epsilon/2)|Du^\epsilon|^2 \leq \int\limits_{\Omega\times\{t\}} \frac{1+\langle Du^\epsilon, Du_0\rangle - (u^\epsilon - u_0)u_t^\epsilon}{W} + (\epsilon/2)|Du_0|^2$$

$$\leq C. \tag{9.13}$$

From this, (9.7) follows. □

Now, to guarantee the convergence of u^ϵ as $\epsilon \to 0$ we need uniform (in ϵ) estimates on the spatial derivatives $|Du^\epsilon|$. Oliker and Uralt'seva [] use variations of an iteration scheme to construct estimates for $|Du^\epsilon|$. This iteration scheme depends on the use of the Sobolev inequality. First they make use of their scheme and the Sobolev inequality $|u|_{L^{p^*}(\mathbb{R}^n)} \leq C_n(\int\limits_{\mathbb{R}^n} |Du|^p)^{1/p}$ for functions on $\mathbb{R}^n$ to put estimates on $\epsilon|Du^\epsilon|$.

From this estimate, they may use a form of the Sobolev inequality by Uralt'seva [] for functions on graphs realized as surfaces that obey elliptic equations of a certain type. Letting S_u being the graph of u, this Sobolev Inequality takes the form $\int\limits_{S_u} f^2 \leq DH_n(S_u)^{2/n} \int\limits_{S_u} |\nabla f|^2$ where D depends on certain estimates of the coefficients of the elliptic equation. From here, they are able to obtain uniform bounds for $|Du^\epsilon|$.

We will use a similar procedure, but first we must be careful of the fact that our base is M so we must be careful to use quantities in a tensorial manner. Following Oliker and Uralt'seva [], we define the quantities

$$F : TM \to \mathbb{R} \tag{9.14}$$

$$F^\epsilon : TM \to \mathbb{R} \tag{9.15}$$

by

$$F(x,v) = \sqrt{1+|v|_M^2}, \tag{9.16}$$

$$F^\epsilon(x,v) = F(x,v) + (\epsilon/2)|v|_M^2. \tag{9.17}$$

When needed we will use the notation that any frame $\{e_i\}$ on TM is extended in a natural way to $\{v_i\}$ on TTM. Also, we will need to make use of a certain type of pull-back of tensor on TM. Note that the map $x \in M \to (x, \nabla u) \in TM$ gives us a section of TM. We use this to construct a sort of identity map $f : T_xM \to T_{(x,\nabla u)}TM$ by $(x, v) \to (x, \nabla u, v)$. Note that $f(e_i) = v_i$. If T is a tensor on TM, then we use f to define a pull-back by $f^*T_p(X) = T_{(p,\nabla u)}(f(X))$. For brevity and clarity of notation we also use $T^* = f^*T$ and $D^*T^* = f^*(DT)$. We will also make use of the notation $T_{i^*} = f^*(T)_i$ and $T_{j^*,i^*} = f^*(D_iT_j)$. We will often use subscripts to denote the appropriate covariant derivatives of scalar functions.

First we write down a simple computational lemma that we will need. It is only written for one tensors T on TM, but its generalization is very clear.

Lemma 9.2. *If T is a tensor on TM such that T is independent of the spatial variables on TM (i.e. $D_{e_i}T = 0$ where e_i is clearly identified with $(e_i, 0) \in TTM$), then*

$$D_iT^*_j = (D^2_{ik}u)D^*_kT^*_j = D^2_{ik}uT_{j^*,k^*}. \tag{9.18}$$

Proof. Let $\{e_i\}$ be a geodesic orthonormal frame on M. So $\{e_i, v_i\}$ is a geodesic orthonormal frame on TM. We compute that

$$T^*_j = f^*T_j = T_{v_j} \circ (x, Du).$$

Therefore,

$$D_iT^*_j = D_if^*T_j = D_{e_i}T_{v_j} \circ (x, Du) + D_{v_k}T_{v_j} \circ (x, Du)(D^2_{ki}u).$$

From $D_{e_i}T = 0$ we get the claim. □

Note that $F_{i^*} = D^*_iF^* = \frac{D_iu}{W}$ and that as tensors on TM, $D_{e_i}F = D_{e_i}F^\epsilon = 0$. We immediately see that (9.4) becomes

$$\frac{u_t}{W} - D_iD^*_iF^{\epsilon *} = 0. \tag{9.19}$$

We rewrite as $u_t - WD_iF^\epsilon_{i^*} = 0$. Noting that $(F_{j^*})D_jF^\epsilon_{i^*} = W_kF^\epsilon_{k^*i^*}$, we apply the operator $(F_{j^*})D_j$ to get the tensorial equation

$$W_t - WD_i(D_kWF^\epsilon_{k^*i^*}) + \Lambda W = W_jF_{j^*}D_iF^\epsilon_{i^*} - (\frac{1}{W} + \epsilon)\,\mathrm{Ric}_M(Du, Du), \tag{9.20}$$

where $\Lambda = D^2_{ik}uD^2_{jl}uF_{k^*j^*}F^\epsilon_{l^*i^*} \geq 0$.

Consider the case that $\mathrm{Ric}_M \geq 0$. Now, taking a cut-off function $\eta \in H^1_0(\Omega)$ and $\eta \geq 0$, we multiply by $W^{-1}\eta$, integrate by parts, use that $|W_j F_{j^*}| = (1/W)|\langle Du, DW\rangle| \leq |DW|$, use inequality (9.1), and use equation (9.19) to get

$$\int\limits_{\Omega} \frac{W_t}{W}\eta + W_k F^{\epsilon}_{k^* i^*} D_i \eta + \Lambda \eta \leq C \int\limits_{\Omega} \frac{|\nabla W|}{W}\eta. \tag{9.21}$$

NEED TO DEFINE CLASS OF FUNCTIONS G(,)!!!!!!!!!!!!
For a given $\zeta \in G(\rho, \sigma)$ and for a constant M we use the test function $\eta = (W - M)_+ \zeta^2$. Now, we choose another constant M^* such that for $M \geq M^*$, we are guaranteed that $\eta = 0$ on $\Omega \times \{0\}$. So, we let

$$M^* = \begin{cases} \sup\limits_{\Omega \times \{0\}} (1 + |Du_0|^2)^{1/2} & \sigma = 0 \\ 1 & \sigma > 0. \end{cases} \tag{9.22}$$

Now we use the test function $\eta = (W - M)_+ \zeta^2$ for some $\zeta \in G(\rho, \sigma)$ and $M \geq M^*$. Letting $A_k(t) = \{x \in \Omega \cap B(\rho) | W(x,t) > M\}$. We get

$$\int\limits_{A_M(t)} \frac{W_t (W - M)}{W}\zeta^2 + F^{\epsilon}_{i^* j^*} W_i W_j \zeta^2 + \Lambda (W - M)\zeta^2 \leq \tag{9.23}$$

$$\int\limits_{A_M(t)} C\frac{|\nabla W|}{W}(W - M)\zeta^2 - \int\limits_{A_M(t)} 2 W_k F^{\epsilon}_{k^* i^*}(W - M)\zeta\zeta_i. \tag{9.24}$$

Let $\mathbf{\Phi}(w, k) = \int\limits_k^w \frac{(y-k)_+}{y}\, dy$. From $F^{\epsilon}_{i^* j^*} W_i W_j = |\nabla W|^2/W + \epsilon |DW|^2$ and $\Lambda \geq 0$, we get

$$\int\limits_{A_M(t)} \frac{\partial}{\partial t}\left[\mathbf{\Phi}(W, t)\zeta^2\right] + (|\nabla W|^2/W + \epsilon|DW|^2)\zeta^2 \leq \int\limits_{A_M(t)} C\frac{|\nabla W|}{W}(W - M)\zeta^2 \tag{9.25}$$

$$- \int\limits_{A_M(t)} 2 W_k F^{\epsilon}_{k^* i^*}(W - M)\zeta\zeta_i - 2\mathbf{\Phi}(W, t)\zeta\zeta_t. \tag{9.26}$$

Now, we estimate $C|\nabla W|(W-M) \leq (1/4)|\nabla W|^2 + C^2(W-M)^2$, and also $|2W_k F^{\epsilon}_{k^*i^*}(W-M)\zeta\zeta_i| \leq (1/4)F^{\epsilon}_{k^*i^*}W_kW_i\zeta^2 + 4|D\zeta|^2(W^{-1}+\epsilon)(W-M)^2$. Therefore,

$$\int\limits_{A_M(t)} \frac{\partial}{\partial t}\left[\mathbf{\Phi}(W,t)\zeta^2\right] + \frac{1}{2}(|\nabla W|^2/W + \epsilon|DW|^2)\zeta^2 \leq \int\limits_{A_M(t)} C^2\frac{(W-M)^2}{W}\zeta^2 \tag{9.27}$$

$$+ \int\limits_{A_M(t)} 4|D\zeta|^2(W^{-1}+\epsilon)(W-M)^2 + 2\mathbf{\Phi}(W,t)\zeta\zeta_t. \tag{9.28}$$

For now, let ϵ_0 be such that $\epsilon_0^{-1} \geq M^*$, and consider $\epsilon < \epsilon_0$. Also, set $M_0 = \epsilon_0^{-1}$. We have that $(\epsilon W)^{-1} \leq (\epsilon M_0)^{-1} = 1$ on the set $A_M(t)$. Dividing (9.27) by ϵ, integrating over $[t_0, t]$, and individually estimating terms on the left hand side, we get

$$M_0 \sup_{[t_0,T]} \int\limits_{A_M(t)} \mathbf{\Phi}(W,M)\zeta^2 + \frac{1}{2}\int\limits_{t_0}^{T}\int\limits_{A_M(t)} |DW|^2\zeta^2\,dx\,dt \tag{9.29}$$

$$\leq C^2 \int\limits_{A_M(t)} (W-M)^2\zeta^2\,dx + \int\limits_{A_M(t)} 8|D\zeta|^2(W-M)^2 + 2M_0\mathbf{\Phi}(W,M)\zeta\zeta_t. \tag{9.30}$$

Therefore, setting $\mathbf{\Psi}(W,M) = M\mathbf{\Phi}(W,M)$, we have the following:

Lemma 9.3. *Consider any $\eta \in G(\rho,\sigma)$. Let*

$$M^* = \begin{cases} \sup\limits_{\Omega\times\{0\}} (1+|Du_0|^2)^{1/2} & \sigma = 0, \\ 1 & \sigma > 0. \end{cases}$$

Let $\epsilon_0 = 1/M^*$. For $\epsilon < \epsilon_0$, $M_0(\epsilon) = \epsilon^{-1}$, and $M \in [M_0, 2M_0]$ we have that

$$\sup_{[t_0,T]} \int\limits_{A_M(t)} \mathbf{\Psi}(W,M)\zeta^2 + \int\limits_{t_0}^{T}\int\limits_{A_M(t)} |DW|^2\zeta^2\,dx\,dt \tag{9.31}$$

$$\leq D\int\limits_{t_0}^{T}\int\limits_{A_M(t)} (W-M)^2(\zeta^2+|D\zeta|^2)\,dx + \mathbf{\Psi}(W,M)|\zeta\zeta_t|. \tag{9.32}$$

Here $D = D(C)$ where C comes from inequality (9.6)*.*

Furthermore, by estimating $\sqrt{1+(t_0+r)^2} \leq \sqrt{1+t_0^2} + t_0(1+t_0^2)^{-1/2}r + (1/2)(1+t_0^2)^{-3/2}r^2$, we find that

$$\|(W-M_0)_+\|_{2,\tilde{Q}} \leq [C_0(T-t_0)\epsilon]^{1/2}M_0. \tag{9.33}$$

Therefore, given any Θ, we may find $\epsilon_0 = \epsilon_0(T - t_0, \Theta, C_0)$ such that

$$\|(W - M_0)_+\|_{2,\tilde{Q}} \leq \Theta M_0. \tag{9.34}$$

We would like to apply something similar to Lemma 4.1 from [], which gives pointwise bounds for functions W that satisfy integration bounds of the form 9.3. Oliker-Uralt'seva [] derive it using an integration iteration estimate and a Sobolev Inequality of the form $\int_\Omega g^2\,dx \leq \beta|\text{suppt } g|^{2/n}\int_\Omega |Dg|^2\,dx$. Unfortunately, in the setting of $M \times \mathbb{R}$, for $\Omega' \subset\subset \Omega$, we need to cover Ω' by geodesic balls B of small enough size such that the in local coordinates $\sigma_{ij} = \delta_{ij} + O(r^2)$ and $|\sigma_{ij,k}| = O(r)$. By choosing the geodesic balls small enough to get uniform estimates of this nature, we may apply the euclidean Sobolev inequality on each geodesic neighborhood. Since Ω' is compact we can recover the result of Oliker-Uralt-seva[], but the dependencies of the constants are a little weaker. Using this, (9.34), Lemma 9.3, and Lemma 4.1 from [], we have

Lemma 9.4. *Let* $\operatorname{Ric} M \geq 0$. *Let* $\Omega \subset\subset \Omega$ *and* $T > 0$, $0 \leq \sigma < T$. *There exists*

$$\epsilon_0 = \begin{cases} \epsilon_0(T, \sigma, \Omega', u_0, \phi) & \sigma > 0, \\ \epsilon_0(T, \max_\Omega |Du_0|, \Omega', u_0, \phi) & \sigma = 0, \end{cases} \tag{9.35}$$

such that for $\epsilon \leq \epsilon_0$ *and in* $\Omega' \times [\sigma, T]$ *we have*

$$W\epsilon \leq C \tag{9.36}$$

where

$$C = \begin{cases} C(T, \sigma, \Omega', u_0, \phi) & \sigma > 0 \\ C(T, \max_\Omega |Du_0|, \Omega', u_0, \phi) & \sigma = 0. \end{cases} \tag{9.37}$$

Following the approach of Oliker-Uralt'seva[] and using the estimate $\epsilon|Du^\epsilon| \leq C$, it is now possible to obtain a Sobolev inequality of surfaces for functions of compact support on the graph of u^ϵ. From $\frac{u_t}{W} = D_i F^\epsilon_{i^*}$ we have $|F^\epsilon_{i^*}| \leq 1 + C$. This and the estimate $|\frac{u_t}{W}| \leq C$ allow us, on small geodesic balls B, to apply the Sobolev Inequality of Ladyzhenskaya-Ural'tseva[] to get

$$\int_{S^\epsilon(t)} f^2\,dH_n \leq \beta H_n^{2/n}(\text{suppt } f \cap S^\epsilon(t)) \int_{S^\epsilon(t)} |\nabla f|^2\,dH_n \tag{9.38}$$

where $f \in C^1_0(B)$ and $\beta = \beta(C)$. The proof of Theorem 2.5 in Oliker-Ural'tseva[] may now be carried out on small geodesic balls to obtain

Theorem 9.5. *Let $\Omega' \subset\subset \Omega$ and $T > 0$. There exists a constant $C = C(\Omega', T, u_0, \phi)$ such that*

$$\sup_{\Omega' \times [0,T]} |Du^\epsilon| \leq C \tag{9.39}$$

for $\epsilon < \epsilon_0 = \epsilon(\Omega', T, u_0, \phi)$.

From here, by standard parabolic theory, we get estimates on the other spatial derivatives of u^ϵ.

Corollary 9.6. *Let $\Omega' \subset\subset \Omega$ and $T > 0$. There exists $\epsilon_0 = \epsilon_0(\Omega', T, u_0, \phi)$ such that for every $\epsilon < \epsilon_0$ we have*

$$\sup_{\Omega' \times [0,T]} |D^\alpha u^\epsilon| \leq C \tag{9.40}$$

where $C = C(\Omega', T, u_0, \phi, \alpha)$.

From these estimates we see that there is a sequence $\epsilon_i \to 0$ such that on compact sets $\Omega' \subset\subset \Omega$ we have that u^ϵ and its derivatives converge uniformly to a function u and its derivatives.

PUT UNIQUENESS STATEMENT HERE From the estimates, we have that there exists a sequence $t_i \to \infty$ such $u(\cdot, t_i)$ and its derivatives converge uniformly on compact subset $\Omega' \subset\subset \Omega$ to a function $\bar{u}_1$. Consider another such s_i with limit $\bar{u}_2$. We show that $\bar{u}_1 = \bar{u}_2$. Note, Oliker-Ural'tseva[] show this for $\Omega \subset \mathbb{R}^n$ by working directly with the function u, using that minimizers of the functional J differ by a constant, and the part of the boundary that is mean convex still has suitable barriers so we do get uniqueness. For the case of general M, we may have that the boundary has no mean convex part. For example, consider a large enough ball in a sphere S^n.

We show that there is uniqueness depending on the choice of the sequence ϵ_i, but not depending on the choice of sequence in time t_i. To do this, we use the uniqueness of the limit for u^ϵ that comes from the uniqueness of minimizers for a perturbed functional. First, a discussion of this uniqueness.

Similar to Oliker-Ural'tseva[], we have that u^ϵ satisfies an estimate uniform in ϵ:

$$\int_0^\infty \int_\Omega \frac{|u_t^\epsilon|^2}{\sqrt{1 + |Du^\epsilon|^2}} \, dx \, dt \leq C, \tag{9.41}$$

where $C = C(u_0, \Omega, \phi)$.

Consider the functional

$$E^{\epsilon,t}(w) = \int\limits_{\Omega} F^{\epsilon}(x, Dw) + f^{\epsilon,t} w \, dx \tag{9.42}$$

where $f^{\epsilon,t} = \frac{u^{\epsilon}_t(t)}{\sqrt{1+|Du^{\epsilon}(t)|^2}}$. Note, by the convexity of $E^{\epsilon,t}$, we have that $u^{\epsilon}(\cdot,t)$ is a minimizer of $E^{\epsilon,t}$ for functions $w \in W^{1,2}(\Omega)$ with $w = \phi$ on $\partial\Omega$. Note, also that from (9.41), we have for some sequence t_k that $\int\limits_{\Omega} |f^{\epsilon,t_k}|^2 \, dx \to 0$.

Since the function u^{ϵ} satisfies a uniformly parabolic equation, we know that it's derivatives are bounded uniformly on $\Omega \times (0,\infty)$ (but of course the bound may depend on ϵ). Therefore, for any sequence t_i, we may pass to a subsequence such that $u^{\epsilon}(\cdot,t_i) \to \bar{u}^{\epsilon}(\cdot,t_i)$ uniformly on Ω. Note, also that from (9.41), we have for some sequence t_k that $\int\limits_{\Omega} |f^{\epsilon,t_k}|^2 \, dx \to 0$.

Now consider the convex functional

$$E^{\epsilon}(w) = \int\limits_{\Omega} F^{\epsilon}(x, Dw) \, dx. \tag{9.43}$$

First, we again pass to a subsequence such that $Du^{\epsilon}(\cdot,t_k) \to D\bar{u}^{\epsilon}$ uniformly. Since we have uniform convergence, we have that for any $w \in W^{1,2}(\Omega) \cap L^{\infty}(\Omega)$ with $w = \phi$ on $\partial\Omega$, we have that

$$E^{\epsilon}(\bar{u}^{\epsilon}) = \lim_{k\to\infty} E^{\epsilon}(u^{\epsilon}(\cdot,t_k)) = \lim_{k\to\infty} E^{\epsilon,t_k}(u^{\epsilon}(\cdot,t_k)) - \int\limits_{\Omega} f^{\epsilon,t_k} u^{\epsilon} \tag{9.44}$$

$$= \lim_{k\to\infty} E^{\epsilon,t_k}(u^{\epsilon}(\cdot,t_k)) \leq \lim_{k\to\infty} E^{\epsilon,t_k}(w) \to E^{\epsilon}(w). \tag{9.45}$$

Therefore, the limit $\bar{u}^{\epsilon}$ is a minimizer of E^{ϵ} which implies it is unique. Now, we may apply this to discuss the uniqueness of u.

Theorem 9.7. *Given a sequence $\epsilon_i \to 0$ such that the solutions u^{ϵ_i} to the perturbed functional and their spatial derivatives converge uniformly on compact subsets of $\Omega \times (0,\infty)$ to pseudo-solution u and sequence of times t_i such that $u(\cdot,t_i)$ and their spatial derivatives converges uniformly on compact subsets $\Omega' \subset\subset \Omega$ to a function $\bar{u}$, then $\bar{u}$ is a pseudo-solution to the Dirichlet problem and $\bar{u}$ depends only on the sequence ϵ_i. That is, $\bar{u}$ is independent of the sequence t_i.*

Proof. Consider another sequence s_i such that $u^\epsilon(\cdot, s_i) \to \bar{v}^\epsilon$ and their spatial derivatives converge uniformly on compact subsets $\Omega' \subset\subset \Omega$. For any $x \in \Omega$ and $\delta > 0$, we have for some ϵ_j that

$$|\bar{u}(x) - \bar{v}(x)| \leq 2\delta + |u(x, t_i) - u(x, s_i)| \leq 4\delta + |u^{\epsilon_j}(x, t_i) - u^{\epsilon_j}(x, s_i)|. \tag{9.46}$$

We may pass to a subsequence such that $u^{\epsilon_j}(\cdot, t_i) \to \bar{u}^{\epsilon_j}$ and $u^{\epsilon_j}(\cdot, s_i) \to \bar{v}^{\epsilon_j}$ uniformly on compact subsets of $\Omega' \subset\subset \Omega$. From our discussion above, we know that $\bar{u}^{\epsilon_j} = \bar{v}^{\epsilon_j}$, therefore we get $|\bar{u}(x) - \bar{v}(x)| \leq 4\delta$. Hence, $u = v$.

□

OLD STUFF

Theorem 9.8. *There exists a solution $u \in C^\infty(\Omega \times (0, \infty)) \cap C^0(\bar{\Omega} \times [0, \infty))$ to 9.2. Moreover, u satisfies*

$$\sup_{\Omega \times (0,\infty)} |u| \leq C \tag{9.47}$$

$$\sup_{\Omega' \times (0,\infty)} |D^\alpha u| \leq C(\Omega', \alpha) \qquad \textit{for all } \Omega' \subset\subset \Omega \textit{ and all } \alpha. \tag{9.48}$$

Proof. First, note that from the maximum principle we have

$$|u(x, t)| \leq \max_{x \in \bar{\Omega}} |u_0(x)|. \tag{9.49}$$

From (SPRUCK ??, Theorem 1.3) we have a C^1 estimate, and so the theorem follows.

□

(OLIKER AND URALTSEVA USE A SOBOLEV INEQUALITY FOR SHOWING THE ESTIMATE ON ANY SUBDOMAIN. THIS MAY NEED TO BE RELOOKED AT).

Next, we wish to study the convergence of the time slices $u(t, \cdot)$ of our solution as $t \to \infty$. First, we establish an estimate on u_t. Note, that a bound for u_t is not a direct consequence of the maximum principle since we have no guarantee that u_t is continuous in our boundary data.

Lemma 9.9. *For a solution u to 9.2 we have that*

$$\sup_{\Omega \times (0,\infty)} |u_t| \leq M \tag{9.50}$$

Proof. We solve the mean curvature flow for boundary data that is perturbed from 9.2. First, for $\delta > 0$ let $\phi^\delta(x,t) \equiv \phi(x) + \delta\psi(t/\delta)nH(u_0)W(u_0)$, where ψ is a smooth non-negative function satsifying $\psi(0) = 0$, $\psi'(0) = 1 = \sup|\psi'|$, and suppt $\psi = [0,2]$. Then, let u^δ be the solution to

$$\begin{cases} u_t^\delta(x,t) = g^{ij}D_iD_ju^\delta & \text{in } \Omega \times (0,\infty) \\ u^\delta(x,t) = \phi^\delta(x,t) & \text{on } \partial\Omega \times (0,\infty) \\ u^\delta(x,0) = u_0(x). \end{cases} \tag{9.51}$$

Note that $w^\delta \equiv u_t^\delta$ belongs to $C(\bar{\Omega} \times [0,\infty)) \cap C^\infty(\Omega \times (0,\infty))$ and satisfies a parabolic equation of the form

$$w_t^\delta = a_{ij}(x,t)D_iD_jw^\delta + b_i(x,t)D_iw^\delta \tag{9.52}$$

in $\Omega \times [0,\infty)$. Also, the boundary data for w^δ satisfies an estimate of the form $|w^\delta| \le M$, where M is independent of δ. Similaryl, $u^\delta - u$ satisfies a parabolic equation, and we get an estimate $|u^\delta - u| \le \delta M$ with M independent of δ. Therefore, $u^\delta \to u$ uniformly $\bar{\Omega} \times [0,\infty)$ and we get an estimate $\sup\limits_{\Omega\times(0,\infty)} |u_t| \le M$. $\square$

Next, we make an estimate on $\int\limits_{\Omega\times\{t\}} \sqrt{1+|Du|^2}$.

Lemma 9.10. *For any $t \in (0,\infty)$, we have $\int\limits_{\Omega\times\{t\}} \sqrt{1+|Du|^2} \le M$ where M is independent of t.*

Proof. Since $\frac{u_t}{W} - \mathrm{Div}\frac{Du}{W} = 0$ in $\Omega \times (0,\infty)$, we have for any $t \in (0,\infty)$ and $\eta \in C_0^1(\Omega)$ that

$$\int\limits_{\Omega\times\{t\}} \frac{u_t}{W} + \frac{\langle Du, D\eta\rangle}{W} = 0 \tag{9.53}$$

Now, using $\eta = u(x,t) - u_0(x)$ we get

$$\int\limits_{\Omega\times\{t\}} \sqrt{1+|Du|^2} = \int\limits_{\Omega\times\{t\}} \frac{1 + \langle Du, Du_0\rangle - (u-u_0)u_t}{W} \le M \tag{9.54}$$

$\square$

Now we make an estimate for $\int\limits_0^\infty \int\limits_\Omega \frac{|u_t|^2}{\sqrt{1+|Du|^2}}\,dxdt$.

Lemma 9.11.

$$\int_0^\infty \int_\Omega \frac{|u_t|^2}{\sqrt{1+|Du|^2}}\, dxdt \leq M \tag{9.55}$$

Proof. We use $\frac{u_t}{W} = \mathrm{Div}\frac{Du}{W}$ to get

$$\int_{t_1}^{t_2} \int_\Omega \frac{u_t^2}{W} = \int_{t_1}^{t_2} \int_\Omega u_t \mathrm{Div}\frac{Du}{W} \tag{9.56}$$

$$= \int_{t_1}^{t_2} \frac{Du}{W} u_t \Big|_{\partial\Omega} - \int_{t_1}^{t_2} \int_\Omega \langle \frac{Du}{W}, Du_t \rangle \tag{9.57}$$

$$= -\int_\Omega \int_{t_1}^{t_2} \langle \frac{Du}{W}, Du_t \rangle \tag{9.58}$$

$$= -\int_\Omega \sqrt{1+|Du|^2}\Big|_{t_1}^{t_2} \tag{9.59}$$

Therefore, $\int_{t_1}^{t_2} \int_\Omega \frac{u_t^2}{W}\, dxdt \leq M$, where M is independent of t_1 and t_2. Therefore, let $t_1 \to 0$ and $t_2 \to \infty$ to obtain the lemma. □

Corollary 9.12. *There exists a subsequence $t_k \to \infty$ such that $u(\cdot, t_k) \to \bar{u}$ as $k \to \infty$ where the convergence is in $L^q(\Omega)$ for any $q < \infty$.*

Theorem 9.13. *$u(\cdot, t_k) \to \bar{u}$ is a general solution to*

$$H(u) = 0 \qquad \text{in } \Omega \tag{9.60}$$

$$u(x) = \phi(x) \qquad \text{on } \partial\Omega \tag{9.61}$$

Proof. It is sufficient to show that u is a minimizer of the functional

$$J(u,\Omega) = A(u,\Omega) + \int_{\partial\Omega} |u-\phi|\, dH_{n-1} \tag{9.62}$$

In general, we may infact let

$$J(v,\Omega) = \int_\Omega \sqrt{1+|Dv|^2} + fv + \int_{\partial\Omega} |v-\phi|\, dH_{n-1} \tag{9.63}$$

for $v \in W^{1,1}(\Omega)$, the minimizer for $J(V,\Omega)$ is the general solution to the Dirichlet Problem

$$H(v) = f \qquad \text{in } \Omega \tag{9.64}$$

$$v = \phi \qquad \text{in } \partial\Omega \tag{9.65}$$

For each k, we denote

$$J_k(v,\Omega) = \int_\Omega \sqrt{1+|Dv|^2} + nH(u(\cdot,t_k))v\,dx + \int_{\partial\Omega} |v-\phi|\,dH_{n-1} \tag{9.66}$$

Then, we have that

$$J_k(u_k,\Omega) \le J_k(u,\Omega)\text{for all } v \in W^{1,1}(\Omega) \tag{9.67}$$

Now, let $r(t_k) \equiv \int_\Omega \frac{u_t^2}{\sqrt{1+|Du|^2}}\,dx\Big|_{t=t_k}$. Then, we have $r(t_k) \to 0$ as $t_k \to \infty$. Since,

$$\int_\Omega |nH(u(\cdot,t_k))|\,dx = \int_\Omega |\frac{u_t}{W}|\,dx \tag{9.68}$$

$$\le r(t_k)^{1/2}|\Omega|^{1/2} \qquad \to 0\text{as } t_k \to \infty, \tag{9.69}$$

we have that for any $v \in W^{1,1}(\Omega) \cap L^\infty(\Omega)$ that

(9.70)

$$J(u,\Omega) \le \liminf_{k\to\infty} J(u_k,\Omega) \quad = \liminf_{k\to\infty}(J_k(u,\Omega) - \int_\Omega nH(u(\cdot,t_k))u_k\,dx)$$

$$= \liminf_{k\to\infty} J_k(u_k,\Omega) \qquad \le \liminf_{k\to\infty} J_k(v,\Omega) = J(v,\Omega) \tag{9.71}$$

Now, for any $v \in W^{1,1}(\Omega)$ and $M = \sup_{\partial\Omega}\phi$, put

$$w = \begin{cases} v & \text{if } |v| < M \\ M & \text{if } v \ge M \\ -M & \text{if } v \le -M \end{cases} \tag{9.72}$$

Then, we have

$$J(u,\Omega) \le J(w,\Omega) \le J(v,\Omega). \tag{9.73}$$

Thus, u is a minimizer of $J(\cdot,\Omega)$. $\square$

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