INFINITE BOUNDARY VALUE PROBLEM FOR TRANSLATING SOLITONS

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ABSTRACT. In this paper, we study the Dirichlet problem for translating solitons in Euclidean space. We prove the existence of corresponding infinite boundary value problem.

1. INTRODUCTION

In [1], Jenkins and Serrin considered bounded domain $D \subset \mathbb{R}^2$, with $\partial D$ composed with straight lines and convex arcs. They found the necessary and sufficient conditions that guarantee the existence of a Scherk type minimal graph over $D$. Later, Spruck in [6] studied the infinite boundary value problem for constant mean curvature surface. He constructed Scherk type constant mean curvature graph in $\mathbb{R}^3$, and found the necessary and sufficient conditions for the existence of such graph. In recent years, Dirichlet problems for the minimal surface and constant mean curvature surface with infinite boundary data are studied actively in Riemannian surface, see [4, 5, 3]. Here, we are going to study the Dirichlet problem to translating solition equation, where infinite boundary data are allowed. More specifically we are going to study the following equation.

$$
\text{div} \frac{Du}{W} = \frac{1}{W} \text{ in } D
$$

where $W = \sqrt{1 + |Du|^2}$. Our main results are the followings.

Theorem 1.1. Let $D \subset \mathbb{R}^2$ be a bounded domain with $C^2$ boundary whose inward curvature satisfies $\kappa \geq 0$, moreover, diam$(D) < 2$. Then the Dirichlet problem

$$
\begin{cases}
\text{div} \frac{Du}{W} = \frac{1}{W} \text{ in } D \\
u = \phi \text{ on } \partial D
\end{cases}
$$

is uniquely solvable for arbitrary continuous boundary data $\phi$.

Remark 1.2. We want to point out here that our method of proving Theorem 1.1 works in $\mathbb{R}^n$.

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Using Theorem 1.1 we go on and study the Dirichlet Problem with infinite boundary data. Let $D$ be a domain satisfies $\text{diam}(D) < 2$. Moreover, the boundary of $D$ contains two sets of open arcs $\{B_i\}$ and $\{C_i\}$, satisfying $\kappa(B_i) = 0$ and $\kappa(C_i) \geq 0$. We suppose that no two of the arcs $B_i$ have a common endpoint. We are then to find a solution of (1.1) in $D$ which assumes the value $-\infty$ on $B_i$ and assigned continuous data on each of the open arcs $C_i$.

**Theorem 1.3.** Consider the Dirichlet problem stated above. There exists a solution to the Dirichlet problem described above if $2\beta \leq \gamma - A(P)$ for each simple closed polygon $P$, whose vertices are chosen from among the endpoints $B_i$. Here $\beta$ is the total length of the segments $B_i$ which are part of the polygon $P$, and $\gamma$ is the perimeter of $P$.

Similarly, when $u \to +\infty$ on part of boundary $\partial D$, we have the following theorem.

Let $D$ be a domain satisfies $\text{diam}(D) < 2$. Moreover, the boundary of $D$ contains two sets of open arcs $\{A_i\}$ and $\{C_i\}$, satisfying $\kappa(A_i) = 0$ and $\kappa(C_i) \geq 0$. We suppose that no two of the arcs $A_i$ have a common endpoint. We are then to find a solution of (3.1) in $D$ which assumes the value $+\infty$ on $A_i$ and assigned continuous data on each of the open arcs $C_i$.

**Theorem 1.4.** Consider the Dirichlet problem stated above. Then there exists a solution to the Dirichlet problem if $2\alpha < \gamma$ for each simple closed polygon $P$, whose vertices are chosen from among the endpoints $A_i$. Here $\alpha$ is the total length of the segments $A_i$ which are part of the polygon $P$, and $\gamma$ is the perimeter of $P$.

**Remark 1.5.** Recall that in minimal surface case. If we denote Type 1: $B_i$ is empty and assigned data on $C_i$ is 0, $A_i$ is $\infty$, there exists a solution if and only if $2\alpha < \gamma$; Type 2: $A_i$ is empty, and assigned data on $C_i$ is 0, $B_i$ is $-\infty$, there exists a solution if and only if $2\beta < \gamma$; Scherk type: $C_i$ is empty, and assigned data on $A_i$ is $\infty$ $B_i$ is $-\infty$, there exists a solution if and only if $2\alpha = 2\beta$, which is equivalent to $2\alpha = 2\beta = \gamma$. Then a necessary condition for the existence of Scherk type minimal surface in domain $D$ is the nonexistence of Type 1 and Type 2 surface in the domain $D$. However, here we don’t have necessary and sufficient conditions for the existence of corresponding Type 1 and Type 2 solitons, so we can’t construct Scherk type soliton here. We also think that Scherk type soliton doesn’t exist. We will come back to this question later.
2. The Dirichlet Problem

In this section we will consider the solvability of the following equation:

\[
\begin{cases}
\text{div} \frac{D u}{W} = \frac{1}{W} \text{ in } D \\
u = \phi \text{ on } \partial D,
\end{cases}
\]

where \( W = \sqrt{1 + |D u|^2} \).

**Theorem 2.1.** Let \( u \in C^3(D) \) be a non-negative solution of (2.1). Then we have

\[
W(P) \leq 32 \max \left\{ 1, \frac{u(P)^2}{\rho^2} \right\} e^{16C u(P)} e^{16C \left( \frac{u(P)}{\rho} \right)^2}
\]

for a constant \( C \) independent of \( u \), but depending on an upper bound of \( \Delta d^2 \) on \( D \).

**Proof.** Since

\[
\langle \nabla \tau_i \tau_j, \nu \rangle = h_{ij} = \langle \tau_i, -\nabla \tau_j, \nu \rangle
\]

we have

\[
\nabla_i \langle \nu, e^{n+1} \rangle = \langle -h^j_i \tau_j, e^{n+1} \rangle
\]

and

\[
\Delta \langle \nu, e^{n+1} \rangle = \langle -\nabla_i h_i^j \tau_j - |A|^2 \nu, e^{n+1} \rangle
\]

\[
= \langle -\nabla H, e^{n+1} \rangle - |A|^2 \frac{1}{W},
\]

which gives us

\[
\Delta \frac{1}{W} + \nabla_i \frac{1}{W}, e^{n+1} \rangle = -|A|^2 \frac{1}{W}.
\]

Therefore,

\[
\Delta W = -W^2 \Delta \frac{1}{W} + 2W^3 \left| \nabla \frac{1}{W} \right|^2
\]

\[
= W^2 \nabla_i \frac{1}{W}, e^{n+1} \rangle + |A|^2 W + \frac{2}{W} g^{ij} W_i W_j.
\]

Moreover,

\[
W^2 \nabla_i \frac{1}{W}, e^{n+1} \rangle = -g^{ij} W_i u_j.
\]

Thus we have,

\[
\Delta W + g^{ij} u_j W_i - \frac{2}{W} g^{ij} W_i W_j = |A|^2 W \geq 0.
\]
We will derive a maximum principle for the function $h = \eta(x)W$.

\[
\mathcal{L}h = \Delta h + g^{ij}u_jh_i - \frac{2}{W}g^{ij}W_iW_jh_j \\
= \eta \left( \Delta W + g^{ij}u_jW_i - \frac{2}{W}g^{ij}W_iW_j \right) + W\Delta \eta + Wg^{ij}u_j\eta_i \\
\geq W(\Delta \eta + \langle \nabla u, \nabla \eta \rangle).
\]

Now let $\eta(x) \equiv g(\phi(x)); g(\phi) = e^{K\phi} - 1$, with $K > 0$ to be determined and

\[
\phi(x) = \left(-\frac{u(x)}{2u_0} + \left(1 - \left(\frac{d(x)}{\rho}\right)^2\right)\right)^+.
\]

Here $d(x)$ is the distance function from $P, u_0 = u(P)$, the geodesic ball $B_{\rho}(P) \subset D$, and we will bound $W(P)$. By a straight forward calculation we have

\[
\nabla \eta = \nabla g = Ke^{K\phi}\nabla \phi,
\]

\[
\Delta \eta = K^2 e^{K\phi} |\nabla \phi|^2 + Ke^{K\phi} \Delta \phi,
\]

and

\[
\Delta \phi = -\frac{\Delta u}{2u_0} + \Delta \left(1 - \left(\frac{d(x)}{\rho}\right)^2\right) \\
\geq -\frac{1}{2u_0W^2} - \frac{C}{\rho^2};
\]

where $|\Delta d^2| \leq C$. Moreover,

\[
|\nabla \phi|^2 = \langle \nabla \phi, \nabla \phi \rangle \\
= \left(-\frac{\nabla u}{2u_0} - \frac{2d\nabla d}{\rho^2}, -\frac{\nabla u}{2u_0} - \frac{2d\nabla d}{\rho^2}\right) \\
= \frac{|\nabla u|^2}{4u_0^2} + \frac{2d}{u_0\rho^2} \langle \nabla d, \nabla u \rangle + \frac{4d^2|\nabla d|^2}{\rho^4} \\
\geq \frac{|Du|^2}{4u_0^2W^2} - \frac{2|Du|}{u_0\rho^2W^2},
\]

(2.13)
and

\[
\langle \nabla u, \nabla \eta \rangle = Ke^{K\phi} \langle \nabla u, \nabla \phi \rangle \\
= Ke^{K\phi} \left( \nabla u, \frac{\nabla u}{2u_0} - \frac{2d\nabla d}{\rho^2} \right) \\
\geq -Ke^{K\phi} \left( \frac{|Du|^2}{2u_0 W^2} + 2d \langle \nabla u, \nabla d \rangle \right) \\
\geq -Ke^{K\phi} \left( \frac{|Du|^2}{2u_0 W^2} + \frac{2}{\rho^2 W^2} |Du| \right).
\]

(2.14)

Therefore,

\[
\Delta \eta + \langle \nabla u, \nabla \eta \rangle \\
\geq K^2 e^{K\phi} \left\{ \frac{|Du|^2}{4u_0 W^2} - 2\frac{|Du|}{\rho^2 u_0 W^2} \right\} \\
+ Ke^{K\phi} \left\{ -\frac{1}{2u_0} - \frac{C}{\rho^2} - 2\frac{|Du|}{\rho^2 W^2} \right\} \\
\geq e^{K\phi} \left\{ K^2 \left( \frac{|Du|^2}{4u_0^2 W^2} - 2\frac{|Du|}{\rho^2 u_0 W^2} \right) - CK \left( \frac{1}{u_0} + \frac{1}{\rho^2} \right) \right\}. 
\]

(2.15)

On the set where \( h > 0 \) and \( W > 16 \max\{1, \left( \frac{u_0}{\rho} \right)^2 \} \), we have

\[
\Delta \eta + \langle \nabla u, \nabla \eta \rangle \\
\geq e^{K\phi} \left\{ \frac{K^2}{8u_0^2} - CK \left( \frac{1}{u_0} + \frac{1}{\rho^2} \right) \right\}. 
\]

(2.16)

We now choose \( K = Mu_0 \left( 1 + \frac{u_0}{\rho^2} \right) \), where \( M \) is large but independent of \( u_0 \) and \( \rho \), then

\[
\Delta \eta + \langle \nabla u, \nabla \eta \rangle \geq e^{K\phi} \left\{ \left( \frac{M^2}{8} - CM \right) \left( 1 + \frac{u_0}{\rho^2} \right)^2 \right\} > 0
\]

for \( M = 16C \). We conclude that \( W \leq 16 \max\{1, \left( \frac{u_0}{\rho} \right)^2 \} \) at the point where \( h \) achieves its maximum, which gives

\[
(e^{K/2} - 1) W(P) \leq (e^K - 1) 16 \max\{1, \left( \frac{u_0}{\rho} \right)^2 \},
\]

(2.17)

thus the equation (2.2).
Theorem 2.2. Let $D \subset \mathbb{R}^2$ be a bounded domain with $C^2$ boundary whose inward curvature satisfies $\kappa \geq 0$, moreover, $\text{diam}(D) < 2$. Then the Dirichlet problem
\[
\begin{cases}
\text{div} \frac{Du}{W} = \frac{1}{W} \text{ in } D \\
u = \phi \text{ on } \partial D
\end{cases}
\] is uniquely solvable for arbitrary continuous boundary data $\phi$.

Remark 2.3. Our proof works for dimension $n \geq 2$.

Before we start our proof, let’s recall two Lemmas that will be used later.

Lemma 2.4. (YY.Li and Nirenberg [2]) Assume $\partial D \in C^2$ and let $D^0$ be the largest open subset of points $x \in D$ which have a unique closest point $y \in \partial D$. Then the distance function $d(x)$ to $\partial D$ is $C^2(D^0)$.

Lemma 2.5. Assume $\kappa|_{\partial D} \geq 0$, for $x_0 \in D^0$, let $\kappa(x_0)$ be the inward curvature of the level set of $d(x)$ passing through $x_0$. Then $\kappa(x_0) \geq 0$.

Proof. (proof of Theorem 2.2) Step 1. Estimation of $\sup_D |u|$. We now construct an upper barrier for $-u$ of the form
\[
w = \sup_D -\phi + h(d(x))
\] where $d(x)$ is the distance function to $\partial D$. Then if $x \in D^0$ we have
\[
Lw = \frac{1}{w} \left( \delta_{ij} - \frac{w_i w_j}{W^2} \right) \left( h'd_{ij} + h''d_i d_j \right)
= \frac{h''}{(1 + h'^2)^{3/2}} - \frac{h'}{\sqrt{1 + h'^2}} \kappa(x).
\]
Choose $h = \log \frac{A}{A-d} - d$, where $A = \max_D d + \epsilon_0 < 1$ for a fixed $\epsilon_0 > 0$. Then we get
\[
h' = \frac{1}{A-d} - 1 > 0
\]
and
\[
h'' = \frac{-1}{(A-d)^2} = -(h' + 1)^2 < -h'^2 - 1.
\]
Thus,
\[
Lw < -\frac{1}{\sqrt{1 + |Dw|^2}}.
\]
Let $v = -u$, now we claim $v \leq w$. If not, suppose $M = \sup(v - w) > 0$ is achieved at $x_0$ and let $y_0 \in \partial D$ be the closest point to $x_0$. It is easy to see that

\begin{equation}
(2.24) \quad v(x) - v(x_0) \leq w(x) - w(x_0)
\end{equation}

and

\begin{equation}
(2.25) \quad |Dv(x_0)| = h'(d(x_0)) > 0.
\end{equation}

Thus the local level set

$$
\Gamma = \{x \in D : v(x) = w(x_0) + M\}
$$

is $C^2$ near $x_0$, and $w(x) \geq w(x_0)$ on $\Gamma$. Since $h$ is increasing we have $d(x) \geq d(x_0)$ on $\Gamma$. we can find a small ball $B_c(z_0)$ tangent to $\Gamma$ at $x_0$ such that

$$
w(x) + M \geq v(x) \geq w(x_0) + M \text{ on } B_c(z_0).
$$

This gives us $w(x) \geq w(x_0)$, i.e. $d(x) \geq d(x_0)$ on $B_c(z_0)$. Therefore, the ball of radius $d(x_0) + \epsilon$ centered at $z_0$ is contained in $\bar{D}$ and $z_0$ is on the extension of geodesic from $y_0$ to $x_0$. Therefore $x_0 \in D^0$ which violates the maximum principle.

Step 2. Estimation of $\sup_{\partial D} |Du|$. First, let’s assume $\phi(x) \in C^2(\partial D)$ and let

\begin{equation}
(2.26) \quad h(d(x)) = \frac{1}{C} \log \frac{2\delta_0}{2\delta_0 - Cd(x)}, \text{ on } \{0 < d(x) < \delta_0\}
\end{equation}

Denote

$$
U_\delta = \{x \in D, 0 < d(x) < \delta\}
$$

and

$$
D_{d_0} = \{x \in D, d(x) > d_0(x)\}.
$$

Let $\phi(x) \in C^2(\bar{U}_{2\delta_0})$ be an extension of $\phi(x)$ on $U_{2\delta_0}$. Choose a proper $d_0 < \delta_0$ such that $Cd_0 < 2\delta_0$ and

$$
\frac{1}{C} \log \frac{2\delta_0}{2\delta_0 - Cd_0} + \phi(x) \geq \max_D u(x) = \max_{\partial D} \phi(x)
$$
for any \( x \in \partial D \). Consider \( w(x) = \varphi(x) + h(d(x)) \), by construction we have \( w(x) \geq u(x) \) on \( \partial U_{d_0} \). Moreover, \( h' = \frac{1}{2\delta_0 - C d} \) and \( h'' = \frac{-C}{(2\delta_0 - C d)^2} = -Ch^2 \). Therefore,

\[
WLw = \left( \delta_{ij} - \frac{w_i w_j}{1 + |Dw|^2} \right) (\varphi_{ij} + h'd_{ij} + h''d_i d_j)
\]

(2.27)

\[
\leq \left( \delta_{ij} - \frac{w_i w_j}{1 + |Dw|^2} \right) \varphi_{ij} + \left( \delta_{ij} - \frac{w_i w_j}{1 + |Dw|^2} \right) h''d_i d_j
\]

\[
\leq C_0 - \frac{Ch'^2}{1 + |D\varphi + Dh|^2}
\]

where \( C_0 = |\varphi|_{C^2(\partial D)} \). Choose \( C \) large \( d_0 > 0 \) small we have \( WLw \leq 1 \) in \( U_{d_0} \). By maximum principle we have \( w(x) \geq u(x) \) in \( U_{d_0} \). By a similar argument we can also find a lower barrier for \( u(x) \) in a small neighborhood of \( \partial D \). Therefore, we get

\[
|Du| \leq C_1 \text{ on } \partial D.
\]

Combine with (2.8) we conclude that

\[
|Du| \leq C_1 \text{ in } D.
\]

Thus, we prove the existence result for (2.18) when \( \phi(x) \in C^2(\partial D) \). A standard approximation argument then yields Theorem 2.2.

Using Theorem 2.2, the classical compactness principle, and an approximation argument we can show

**Theorem 2.6.** Let \( D \) be a bounded convex domain in the plane with \( \text{diam}(D) < 2 \). Consider a finite set of points on the boundary of \( D \), and let \( \gamma \) denote the remaining boundary of \( D \). Then there exists a translating soliton in \( D \), taking on the preassigned bounded continuous date on the arcs \( \gamma \).

### 3. Qualitative Properties for Infinite Boundary Values

In the following, we consider the solution to equation

\[
(3.1) \quad \text{div} \left( \frac{Du}{W} \right) = \frac{1}{W}.
\]

First of all, let’s recall the following General Maximum Principle.
Theorem 3.1. Let \( u_1 \) and \( u_2 \) satisfy \( Lu_1 \geq Lu_2 \) in a bounded domain \( D \). Suppose that 
\[
\lim \inf (u_2 - u_1) \geq 0
\]
for any approach to the boundary of \( D \) with the possible exception of a finite number of points on \( \partial D \). Then \( u_2 \geq u_1 \) in \( D \). Here \( Lu := \text{div} \left( \frac{Du}{W} \right) \).

We also have the following weak form Maximum Principle.

**Lemma 3.2.** (See [6]) Let \( D \) be a bounded domain whose boundary is the union of two closed sets \( \gamma^1 \) and \( \gamma^2 \) with \( \gamma^2 \in C^1 \). Let \( u_1 \in C^2(D) \cap C^1(\gamma^2) \), \( u_2 \in C^2(D) \cap C^0(D) \) satisfy \( Mu_1 \geq Mu_2 \) in \( D \). Suppose that \( \partial u_2/\partial \nu = +\infty \) (\( \nu \) is the outer normal) at every point of \( \gamma^2 \). Then if \( \lim \inf (u_2 - u_1) \geq 0 \) for any approach to points of \( \gamma^1 \) we have \( u_2 \geq u_1 \) in \( D \). Here \( Mu := (\delta_{ij} - \frac{u_i u_j}{W^2}) u_{ij} \).

**Theorem 3.3.** Let \( u \) be a solution of (3.1) in a domain \( D \). Let \( P \in \partial D \), then if \( \kappa(P) < 0 \), \( u \) cannot tend to \( +\infty \) at \( P \). If \( \kappa(P) < 0 \), \( u \) cannot tend to \( -\infty \) at \( P \).

**Proof.** If \( \kappa(P) < 0 \) we can find a straight line \( l \), internally tangent to \( \partial D \) at \( P \). Then in the neighborhood of \( P \), we can find a right isosceles triangle \( \Delta_l \subset D \), whose bottom edge is \( l \).

By pushing \( l \) inside slightly, \( u \) will be a solution in \( \bar{\Delta}_l \) and under proper coordinates, 
\[
\chi = M + \frac{\delta}{2} \left( \log \cos \frac{2}{\delta} x - \log \cos \frac{2}{\delta} y \right)
\]
satisfies \( \partial \chi/\partial \nu = +\infty \) on \( l \), and \( \chi = M \) on \( \partial \bar{\Delta}_l \setminus l \). Choose \( M = \sup_{\partial \bar{\Delta}_l \setminus l} u \), then by Lemma 3.2 we have \( u \leq \chi \) in \( \Delta_l \). Letting \( l \) tend to its original value gives \( u(P) < \infty \).

Choosing \( \chi_- = m + \frac{1}{a} \log \cos ay - \frac{1}{b} \log \cos bx \), where \( b > a + 1 \), \( m = \inf_{\partial \Delta_l \setminus l} \chi_- \), and \( \Delta_l \) is an equilateral triangle, by a similar argument we can show \( u(P) > -\infty \). \( \square \)

Following the argument of [6] we can also prove

**Lemma 3.4.** Let \( D \) be a domain contained in the annulus \( R_1 < r < \frac{1}{H} - R_1 \) and suppose that \( \text{div} \left( \frac{Du}{W} \right) < 2H \) in \( D \), where \( H > 0 \) is constant. If \( \lim \inf (u(x,y) - \psi(R_1, r)) \geq M \) holds for any sequence of points tending to a boundary point of \( D \) not on the circle \( R = R_1 \), then \( u \geq \psi + M \) in \( D \).

**Theorem 3.5.** Let \( D \) be a domain bounded in part by arc \( \gamma \). Let \( u(x,y) \) be a solution of (3.1) in \( D \) such that \( u(x,y) \to \pm \infty \) for any approach to interior points of \( \gamma \). Then \( \kappa(\gamma) = 0 \).

**Proof.** Now suppose \( u(x,y) \to \infty \) on \( \gamma \) and \( \kappa(\gamma) > 2H > 0 \) on some subarc \( \gamma' \subset \gamma \). By choosing \( \gamma' \) sufficiently small we can find a domain \( \Delta \subset D \) bounded by \( \gamma' \) and a circular
arc $C_{R_1}$ of radius $R_1 < \frac{1}{2H}$ such that $\triangle$ is contained in the annulus $R_1 < r < \frac{1}{H} - R_1$ and \( \frac{1}{Wu} \mid_{\triangle} < 2H \). Let $\psi(R_1, r)$ denote the rotationally symmetric hypersurface with constant mean curvature equals $H$, and $\psi(R_1, r)$ is defined in the annulus $R_1 < r < \frac{1}{H} - R_1$ (see pg. 5 [Spruck] for details.) It is easy to see if $u \to +\infty$ on $\gamma$ we would have
\[
\lim \inf (u(x, y) - \psi(R_1, r)) \geq M
\]
for any sequence of points that approaches $\gamma'$. Therefore, by Lemma 3.4 $u \geq \psi + M$ in $\triangle$. Since $M$ is arbitrary, we have a contradiction.

When $u(x, y) \to -\infty$ on $\gamma$ and $\kappa(\gamma) > 0$ on some subarc $\gamma' \subset \gamma$. We can find a straight line $l$, such that the region bounded by $l$ and $\gamma'$ lies in $D$, we denote this region by $\triangle_l$. Under suitable coordinates, let
\[
\chi = M + \frac{\delta}{2} \left( \log \cos \frac{2}{\delta} x - \log \cos \frac{2}{\delta} y \right),
\]
such that $\chi(x, y) = -\infty$ on $l$, and $\chi(x, y) > -\infty$ on $\gamma'$. By Lemma 3.2 , we have $u > \chi(x, y) + M$ in $\triangle_l$ for arbitrary $M$, which leads to a contradiction. \( \square \)

4. ASYMPTOTIC BEHAVIOR FOR SOLUTIONS WITH INFINITE BOUNDARY VALUES

Lemma 4.1. Let $u$ be a solution of (3.1) in a domain $D$ and let $\Gamma$ be a piecewise differentiable curve lying in $\bar{D}$. Then
\[
\int\int_D \frac{1}{W} dxdy = \int_{\partial D} \frac{Du \cdot \nu}{W} ds
\]
and
\[
\left| \int_{\Gamma} \frac{Du \cdot \nu}{W} ds \right| \leq |\Gamma|.
\]

Lemma 4.2. Let $D$ be a domain bounded in part by an arc $\gamma$ and suppose $u$ is a solution of (3.1) in $D$. Then
(1) if $u \to +\infty$ on $\gamma$, we have $\int_{\gamma} \frac{Du \cdot \nu}{W} ds = |\gamma|$;
(2) if $u \to -\infty$ on $\gamma$, we have $\int_{\gamma} \frac{Du \cdot \nu}{W} ds = -|\gamma|$.

Proof. By Theorem 3.5 we have $\kappa(\gamma) = 0$. Let’s assume $\gamma \subset \{(x, y) | y = \frac{\pi}{2}\}$. It’s sufficient to prove it for any subarc $\gamma' \subset \gamma$. Wlog, we also assume $|\gamma'| < \frac{\pi}{2}$ and $(0, \pi/2) \in \gamma'$. Let
Γ be a simple smooth arc joining the endpoints of γ′ such that the domain △ bounded by γ′ ∪ Γ is contained in D. Let \( u_1 = u \) and

\[
u_2 = \frac{a}{2} \left( \log \cos \frac{2}{a} x - \log \cos \frac{2}{a} y \right),
\]

where \( a > 2 \) is chosen such that \( \frac{|D\nu_2|^2}{W_2^2} \geq 1 - \epsilon \) on γ′ for \( \epsilon > 0 \) small.

\[
\begin{diagram}
\gamma'
\end{diagram}
\]

Now, set \( \varphi = u_1 - u_2 \), wlog, we may assume \( \varphi > 0 \). Let \( M_0 \) be chosen so large that the linear measure of \( \Gamma \cap \{ \varphi > M_0 \} \) is less than \( \epsilon/8 \). Let \( \Gamma_0 \) be the subarc of \( \Gamma \) with endpoints at distance \( \epsilon/16 \) from the endpoints of \( \Gamma \). For \( M > \sup_{\Gamma_0} \varphi \), let \( \triangle_M \) be the component of the set \( \triangle \cap \{ \varphi < M \} \), containing \( \Gamma_0 \) in its closure. Then \( \triangle_M \) will be bounded by a curve \( \gamma_M \) with endpoints on \( \Gamma \) close to \( \gamma' \), by subarcs \( \Gamma_M \subset \Gamma \), and possibly by other curves \( \tilde{\gamma}_M \).

Since

\[
\text{div} \varphi \left( \frac{Du_1}{W_1} - \frac{Du_2}{W_2} \right) = D\varphi \cdot \left( \frac{Du_1}{W_1} - \frac{Du_2}{W_2} \right) + \varphi \cdot \left( \text{div} \left( \frac{Du_1}{W_1} \right) - \text{div} \left( \frac{Du_2}{W_2} \right) \right),
\]

we have

\[
(4.3) \quad I = \iint_{\triangle_M} D\varphi \cdot \left( \frac{Du_1}{W_1} - \frac{Du_2}{W_2} \right) dxdy
\]

\[
= M \left\{ \int_{\gamma_M} \left( \frac{Du_1}{W_1} - \frac{Du_2}{W_2} \right) \cdot \nu \, ds + \int_{\tilde{\gamma}_M} \left( \frac{Du_1}{W_1} - \frac{Du_2}{W_2} \right) \cdot \nu \, ds \right\}
\]

\[
+ \int_{\Gamma_M} \varphi \left( \frac{Du_1}{W_1} - \frac{Du_2}{W_2} \right) \cdot \nu \, ds - \iint_{\triangle_M} \varphi \cdot \left( \text{div} \left( \frac{Du_1}{W_1} \right) - \text{div} \left( \frac{Du_2}{W_2} \right) \right) dxdy.
\]

Moreover,

\[
(4.4) \quad \int_{\Gamma_M} \varphi \left( \frac{Du_1}{W_1} - \frac{Du_2}{W_2} \right) \cdot \nu \, ds \leq 2|\Gamma|M_0 + \frac{\epsilon}{4}M,
\]
and
\[
M \int_{\tilde{\gamma}_M} \left( \frac{Du_1 \cdot \nu}{W_1} - \frac{Du_2 \cdot \nu}{W_2} \right) ds
\]
\[
= M \int_{\tilde{\gamma}_M + \bar{\Gamma}_M} \left( \frac{Du_1 \cdot \nu}{W_1} - \frac{Du_2 \cdot \nu}{W_2} \right) ds - M \int_{\bar{\Gamma}_M} \left( \frac{Du_1 \cdot \nu}{W_1} - \frac{Du_2 \cdot \nu}{W_2} \right) ds
\]
\[
\leq \frac{\epsilon}{4} M + \iint_{\tilde{\Delta}_M} \frac{1}{W_1} dxdy,
\]
where \( \tilde{\Delta}_M \) is a domain bounded by \( \bar{\Gamma}_M \) and \( \tilde{\gamma}_M \). Therefore we have
\[
0 \leq \int_{\gamma_M} \left( \frac{Du_1 \cdot \nu}{W_1} - \frac{Du_2 \cdot \nu}{W_2} \right) ds + \epsilon \frac{1}{4}
\]
\[
+ \frac{1}{M} \iint_{\tilde{\Delta}_M} \frac{1}{W_1} dxdy + 2|\bar{\Gamma}|M_0 + \frac{\epsilon}{4}.
\]
Now, let's consider the domain \( \Delta'_M \) bounded by \( \gamma_M, \gamma', \) and parts of \( \Gamma \) which we denote by \( \bar{\Gamma} \). Since
\[
\int_{\gamma'} \left( \frac{Du_1 \cdot \nu}{W_1} - \frac{Du_2 \cdot \nu}{W_2} \right) ds = \iint_{\Delta'_M} \frac{1}{W_1} dxdy,
\]
we have
\[
\int_{\gamma'} \left( \frac{Du_1 \cdot \nu}{W_1} - \frac{Du_2 \cdot \nu}{W_2} \right) ds \geq -\int_{-\gamma_M} \left( \frac{Du_1 \cdot \nu}{W_1} - \frac{Du_2 \cdot \nu}{W_2} \right) ds - \frac{\epsilon}{2}.
\]
Thus by (4.6) we get
\[
\int_{\gamma'} \left( \frac{Du_1 \cdot \nu}{W_1} - \frac{Du_2 \cdot \nu}{W_2} \right) ds \geq -\epsilon - \frac{1}{M}(2|\bar{\Gamma}|M_0 + A(D)).
\]
Since \( M \) is arbitrary we have
\[
|\gamma'| \geq \int_{\gamma'} \frac{Du_1 \cdot \nu}{W_1} ds \geq \int_{\gamma'} \frac{Du_2 \cdot \nu}{W_2} ds - \epsilon \geq |\gamma'| - (1 + |\gamma'|)\epsilon.
\]
Since \( \epsilon \) is arbitrary we must have
\[
\int_{\gamma} \frac{Du \cdot \nu}{W} ds = |\gamma|.
\]
The proof of part (2) is essentially the same, we omit here. \( \square \)

The following lemma is a simple extension of Lemma 4.2.
**Lemma 4.3.** Let $D$ be a domain bounded in part by an oriented straight segment $T$. Let \( \{u_n\} \) be a sequence of solution of (3.1) in $D$, each $u_n$ being continuous in $D \cap T$. Assume that the sequence divergence uniformly to infinity on compact subsets of $T$, while remaining uniformly bounded on compact subsets of $D$. Then

\[
\lim_{n \to \infty} \int_T \frac{\nabla u_n \cdot \nu}{W_n} ds = |T|.
\]

On the other hand, if the sequence diverges uniformly to infinity on compact subsets of $D$, while remaining uniformly bounded on compact subsets of $T$, then

\[
\lim_{n \to \infty} \int_T \frac{\nabla u_n \cdot \nu}{W_n} ds = -|T|.
\]

## 5. LIMITING BEHAVIOR OF MONOTONE SEQUENCES OF SOLUTIONS

First, let’s recall Serrin’s result[20].

**Theorem 5.1.** (Monotone convergence theorem) Let \( \{u_n\} \) be a monotonically increasing or decreasing sequence of solutions of (2.1) in a domain $D$. If the sequence is bounded at a single point of $D$, there exists a non-empty open set $U \subset D$ such that \( \{u_n\} \) converges to a solution in $U$. The convergence is uniform on compact subsets of $U$, and divergence is uniform in compact subsets of $V = D - U$.

As in the minimal surface case we have the following Lemma.

**Lemma 5.2.** (Straight line lemma) Let $D$ be a domain bounded in part by a straight line $L$, and lying entirely on one side of $L$. Let the remaining part of $D$ consist of a bounded arc $\gamma$. Suppose that $u$ is a solution of

\[
\text{div} \left( \frac{Du}{W} \right) = \frac{1}{W}, \text{ in } D,
\]

such that $m \leq u \leq M$ on $\gamma$. Then for any compact subarc $\gamma' \subset \gamma$, there is a neighborhood $\Delta$ of $\gamma'$, such that $m - c \leq u \leq M + c$ in $\Delta$, where $c$ depends on $D$ and $\Delta$.

**Proof.** By [1] we know that there is a solution of

\[
\begin{cases}
\text{div} \frac{\nabla u}{W} = 0 \\
u \to +\infty \text{ on } c \\
u = 0 \text{ on } a \cup b.
\end{cases}
\]
By the general maximum principle we have 
\[ u - M \leq u^+ \], where \( u^+ \) is the solution of (5.2). Now let’s consider

\[ u^- = \frac{1}{a} \log \cos ay - \frac{1}{b} \log \cos bx + \frac{1}{b} \log \sqrt{\frac{2}{2}}, \]

where \( a, b > 0 \) and \( b > a + 1 \). Again, by Lemma 3.2 we have \( u - m \geq u^- \). Note that we can always cover \( \gamma' \) by finitely many such rectangle, thus we prove this lemma. \( \square \)

Next, we are going to study the structure of the divergence set \( V \).

**Lemma 5.3.** *(Divergence set lemma)* Let \( D \) be a convex domain bounded by straight lines and arcs \( C \) with \( \kappa(C) \geq 0 \). Suppose the divergence set \( V \) is non-empty. Then the boundary of \( V \) consists of straight line chords of \( D \) and parts of the boundary of \( D \). Moreover, no two interior chords forming part of the boundary of \( V \) can have a common end point at a convex corner of \( V \), nor can any component of \( V \) consist only an interior chord of \( D \).

**Proof.** The first statement is a direct consequence of the straight line Lemma. We show next that no component of \( V \) can consist simply of an interior chord of \( D \). Assuming this were the case, let \( U_1 \) and \( U_2 \) denote the components of \( U \) on either side of the chord \( T \). Choosing an appropriate orientation for \( T \) in domain \( U_1 \) we have

\[ \lim_{n \to \infty} \int_T \frac{\nabla u_n \cdot \nu}{\sqrt{1 + |\nabla u|^2}} ds = |T|. \]

While in the domain \( U_2 \) we have

\[ \lim_{n \to \infty} \int_T \frac{\nabla u_n \cdot \nu}{\sqrt{1 + |\nabla u|^2}} ds = -|T|, \]
leads to a contradiction.

Now suppose that two interior chords $T_1$ and $T_2$ forming part of the boundary $V$ have a common endpoint $Q$. Obviously, $Q \in \partial D$. Let $Q_1, Q_2$ be points of $T_1, T_2$ respectively, chosen so that the open triangle $\triangle$ with vertices $Q, Q_1, Q_2$ lies in $D$. Then we have

\[(5.6) \quad \int_{Q_1 Q_2} \frac{\nabla u_n \cdot \nu}{W_n} ds + \int_{Q_1 Q_2} \frac{\nabla u_n \cdot \nu}{W_n} ds + \int_{Q_2 Q} \frac{\nabla u_n \cdot \nu}{W_n} ds = \iint_{\triangle} \frac{1}{W_n} dxdy.\]

Therefore

\[(5.7) \quad A(\triangle) \geq \iint_{\triangle} \frac{1}{W_n} dxdy \geq |QQ_1| + |QQ_2| - |Q_1 Q_2|,\]

and

\[(5.8) \quad \frac{A(\triangle)}{|Q_1 Q_2|} \geq \frac{|QQ_1| + |QQ_2|}{|Q_1 Q_2|} - 1.\]

If we shrink $\triangle$ down by similarity, then we will see that the left hand side goes to 0, while the right hand side $> 0$. This leads to a contradiction. \(\square\)

Concerning the boundary behavior of \{u_n\}, we obtain following stronger conclusion.

**Lemma 5.4.** Let $D$ be a domain bounded in part by a convex arc $C$. Let \{u_n\} be an increasing or decreasing sequence of solutions of (5.1) in $D$, each $u_n$ is continuous in $D \cup C$. Let $T$ be an interior chord of $D$ forming part of boundary of $V$. Then $T$ cannot terminate at an interior point of $C$ if \{u_n\} either diverges on $C$ to $\pm \infty$ or remains uniformly bounded on compact subsets of $C$.

**Proof.** If $C$ is not a straight segment, by the straight line lemma, we know that the interior of the convex hull of $C$ either in $V$ or in $U$.

We may thus assume that $C$ is a straight segment, and \{u_n\} is decreasing. Assume $T$ terminates at an interior point $Q$ of $C$. Suppose first that the sequence diverges on $C$. Let $P$ be a point of $T$, choose $R$ on $C$ so that $RP$ lies in $U$. This is possible by the monotone converges theorem. Therefore we have

\[(5.9) \quad \int_{QP} \frac{\nabla u_n \cdot \nu}{W_n} ds + \int_{PR} \frac{\nabla u_n \cdot \nu}{W_n} ds + \int_{RQ} \frac{\nabla u_n \cdot \nu}{W_n} ds = \iint_{\triangle} \frac{1}{W_n} dxdy.\]

As $n \to \infty$ we have

\[(5.10) \quad A(\triangle) \geq |QP| + |RQ| - |PR|,\]
which yields,

\[(5.11) \quad \frac{A(\triangle)}{|PR|} \geq \frac{|QP| + |RQ|}{|PR|} - 1.\]

Shrinking \( \triangle \) down by similarity leads to a contradiction. When the sequence remains uniformly bounded on the compact subsets of \( C \), we can choose \( R \) such that \( \triangle_{RQP} \) lies in \( V \). In this case we would have

\[(5.12) \quad -|QR| + |PR| - |PQ| \geq 0,
leads to a contradiction. \qed

**Lemma 5.5.** Let \( D \) be a domain bounded in part by a convex arc \( C \). Let \( \{u_n\} \) be a sequence of \((5.1)\) in \( D \), which converges uniformly on compact subsets of \( D \) to a solution \( u \) in \( D \). Suppose also that each \( u_n \) is continuous in \( D \cup C \) and that the boundary values converges uniformly on compact subsets of \( C \) to the limit function \( f(s) \). Then \( u \) is continuous in \( D \cup C \) and takes on the boundary values \( f(s) \) on \( C \).

**Proof.** We first show that the sequence \( \{u_n\} \) is uniformly bounded in the neighborhood of any interior point \( Q \) of \( C \). In this regard, we note that if \( C \) is not straight, then the result is immediate from the straight line lemma. We may thus assume that \( C \) is a straight segment. Introduce rectangular coordinates \((x, y)\), so that \( C \) lies along the line \( y = \frac{\pi}{4b} \) and \( Q = (0, \frac{\pi}{4b}) \). We also assume \( E = \{|x| < \frac{a}{2}, |y| < \frac{\pi}{4b}\} \) forms a compact subset of \( D \cap C \). Then there exists \( M' \) such that \( u_n \leq M' \) on \( \bar{E} \cap C \); and \( M'' \) such that \( u_n \leq M'' \) on \( \bar{E} \cap \{y = -\frac{\pi}{4b}\} \). Now, by [JS] we know that there exists a solution \( v^+ \) of the minimal surface equation such that

\[(5.13) \quad v^+ = +\infty \quad \text{on} \quad \bar{E} \cap \{x = \pm\frac{\pi}{2a}\}\]

and

\[(5.14) \quad v^+ = \max(M', M'') \quad \text{on} \quad \bar{E} \cap \{y = \pm\frac{\pi}{4b}\}.\]

By the general maximum principle we have

\[u_n \leq v^+ \quad \text{in} \quad E.\]

Now consider

\[(5.15) \quad v_- = \frac{1}{a} \log \cos ax - \frac{1}{b} \log \cos by + \frac{1}{b} \log \frac{\sqrt{2}}{2},\]
where \( a, b > 0 \) and \( b \geq a + 1 \). It’s easy to see that \( v_- = -\infty \) on \( \{x = \pm \frac{\pi}{2a}\} \cap \bar{E} \). Since there exists \( m \) such that \( u_n \geq m \) on \( \{y = \pm \frac{\pi}{3b}\} \cap \bar{E} \), by the maximum principle we have
\[
u_n \geq v_- + m.
\]
Therefore \( \{u_n\} \) is uniformly bounded in a neighborhood of \( Q \). With this being shown, the rest of the argument is standard. \( \square \)

**Lemma 5.6.** Let \( D \) be a domain bounded in part by a straight segment \( A \). Let \( \{u_n\} \) be a sequence of solutions of (5.1) in \( D \), which converges uniformly on compact subsets of \( D \) to a solution \( u \) in \( D \). Suppose also that each \( u_n \) is continuous in \( D \cap A \), and that the boundary values diverges uniformly to \( +\infty \) on \( A \). Then \( u \) takes on the boundary value \( +\infty \) on \( A \).

**Proof.** Just as in the proof of Lemma 5.5, we can assume \( Q = (0, \frac{\pi}{4a}) \). Let \( D^* \) denote the set \( \{|x| < \frac{\pi}{2a}, 0 < y < \frac{\pi}{4b}\} \), where \( a, b > 0 \) and \( b \geq a + 1 \). We can see that \( \{u_n\} \) is bounded below in \( D^* \). Assume \( u_n \geq -M \) in \( D^* \), and let
\[
u^*_m = \frac{1}{a} \log \cos ax - \frac{1}{b_m} \log \cos b_my - M,
\]
where \( 2a + 1 < b_m \) and \( \{b_m\} \to 2b \) as \( m \to \infty \), such that \( u^*_m(0, \frac{\pi}{4b}) = m \). It’s easy to see that when \( n \) is suitably large we have \( u_n \geq u^*_m \) in \( D^* \). This implies \( u \geq u^*_m \) in \( D^* \). Since
\[
u^*_m \to \frac{1}{a} \log \cos ax - \frac{1}{2b} \log \cos 2by, \text{ as } m \to \infty,
\]
we conclude that \( u \) takes \( +\infty \) at \( Q \). This completes the proof. \( \square \)

### 6. Existence Theorems

#### 6.1. \(-\infty\) boundary data
Let \( D \) be a domain satisfies \( \text{diam}(D) < 2 \). Moreover, the boundary of \( D \) contains two sets of open arcs \( \{B_i\} \) and \( \{C_i\} \), satisfying \( \kappa(B_i) = 0 \) and \( \kappa(C_i) \geq 0 \). We suppose that no two of the arcs \( B_i \) have a common endpoint. We are then to find a solution of (5.1) in \( D \) which assumes the value \(-\infty\) on \( B_i \) and assigned continuous data on each of the open arcs \( C_i \).

**Theorem 6.1.** Consider the Dirichlet problem stated above. There exists a solution to the Dirichlet problem described above if \( 2\beta \leq \gamma - A(\mathcal{P}) \) for each simple closed polygon \( \mathcal{P} \), whose vertices are chosen from among the endpoints \( B_i \). Here \( \beta \) is the total length of the segments \( B_i \) which are part of the polygon \( \mathcal{P} \), and \( \gamma \) is the perimeter of \( \mathcal{P} \).
Proof. For convenience, we will assume the case when the assigned data on \( C_i \) is 0. For a general continuous function, the proof is the same. Let \( \{f_n\}_{n \in \mathbb{N}} \) be a sequence of decreasing continuous function defined on \( \partial D \). \( f_n = 0 \) on \( C_i \) and \( f_n = -n \) on \( B_i \), except on the corner of length \( \frac{1}{n} \), where it’s linear with slope \( 2n^2 \). Let \( \{u_n\} \) be a solution of (5.1) such that \( u_n = f_n \) on \( D \). This solution exists and is unique. Moreover, \( \{u_n\} \) is monotonically decreasing in \( D \). Thus, the monotone convergence Theorem applies. Assume now that the domain of divergence \( V \) is non-empty. According to Lemma 5.4, any interior chord of \( D \) which bounds \( V \) can terminate only at an endpoint of some segment \( B_i \). Moreover, by the straight line Lemma, the interior of the convex hull of each arc \( C_i \) is contained in \( U \). Consequently, each component of \( V \) must be bounded by a simple closed polygon \( P \) whose vertices are among the endpoints of the segments \( B_i \). By the divergence theorem we have

\[
(6.1) \quad \int_{P} \frac{D}{W_n} \cdot \nu \, ds + \int_{B_i} \frac{D}{W_n} \cdot \nu \, ds = \iint_{P} \frac{1}{W_n} \, dxdy.
\]

Let \( n \to \infty \) by Lemma 5.2 we have \( \gamma - 2\beta \leq \iint_{P} \frac{1}{W_n} \, dxdy < A(P) \), which contradicts the assumed conditions. Therefore \( V \) must be empty and \( \{u_n\} \) converges uniformly on compact subsets of \( D \) to a solution \( u \) in \( D \). By the construction of the sequence \( \{u_n\} \) we have \( u = -\infty \) on \( B_i \). Moreover, by Lemma 5.5 we have \( u = 0 \) on \( C_i \). This proves the theorem. \( \square \)

6.2. \(+\infty\) boundary data. Let \( D \) be a domain satisfies \( \text{diam}(D) < 2 \). Moreover, the boundary of \( D \) contains two sets of open arcs \( \{A_i\} \) and \( \{C_i\} \), satisfying \( \kappa(A_i) = 0 \) and \( \kappa(C_i) \geq 0 \). We suppose that no two of the arcs \( A_i \) have a common endpoint. We are then to find a solution of (3.1) in \( D \) which assumes the value \(+\infty\) on \( A_i \) and assigned continuous data on each of the open arcs \( C_i \).

**Theorem 6.2.** Consider the Dirichlet problem stated above. Then there exists a solution to the Dirichlet problem if \( 2\alpha < \gamma \) for each simple closed polygon \( P \), whose vertices are chosen from among the endpoints \( A_i \). Here \( \alpha \) is the total length of the segments \( A_i \) which are part of the polygon \( P \), and \( \gamma \) is the perimeter of \( P \).

**Proof.** Let \( u_n \) be the solution of (3.1) in \( D \) such that

\[
(6.2) \quad u_n = \begin{cases} 
  n & \text{on } \cup A_i \\
  \min(n, f) & \text{on } \cup C_i
\end{cases}
\]
and let \( u^*_n \) be the solution of minimal surface in \( D \) such that

\[
(6.3) \quad u^*_n = \begin{cases} n & \text{on } \bigcup A_i \\
 \min(n, f) & \text{on } \bigcup C_i, \end{cases}
\]

where \( f \) denotes the assigned data on arcs \( C_i \). By the general maximum principle we have \( \{u^*_n\} \) is increasing and \( u_n \leq u^*_n \). By [Jenkins and Serrin] we know there exists a solution \( u^* \) of the Dirichlet problem for minimal surface equation and \( u^*_n \to u^* \). Since \( u_n \leq u^* \) for any \( n \), we conclude that there exists a subsequence \( \{u_{n_k}\} \to u \in D \), \( u \) is the solution of the Dirichlet problem for equation (3.1), and the convergence is uniform on every compact subset of \( D \). □

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