Neutral shells and their applications in the design of electromagnetic shields

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Abstract

We present a simple design of electromagnetic shields for both field expelling and field confinement. Motivated by the concept of neutral inclusions in the theory of composites, we introduce two concepts of neutral shells and use neutral shells to construct our designs of electromagnetic shields. We also discuss the relations between electromagnetic shields and cloaking structures and argue that the designed shields are capable of "cloaking" for plane waves in the long wavelength limit.

1. Introduction

Passive electromagnetic shields for field expelling or confinement are necessary for the reliable working of many electronic devices. Examples of shields for field expelling include superconducting quantum interface devices (SQUIDs) for precision measurements of magnetic fields [32]. Examples of shields for field confinement include magnetic resonance imaging (MRI) machines and tokamaks in magnetic confinement fusion [5, 37]. In addition to these examples, electromagnetic shields are commonly used in advanced nanotechnology research facilities, biomedical research laboratories, continuous beam accelerators, and various facilities such as transformer vault and switchgear in electrical power industry [10, 27]. The use of electromagnetic shields reduces the interference between devices and devices or devices and environments.

For frequencies above 100KHz, satisfactory shielding performance can be easily achieved by a conductive shell (Faraday shield), which uses the eddy current to absorb the electromagnetic waves [28]. For a low-frequency electromagnetic wave or static electric or magnetic field, passive shields use, e.g., high-permeability materials, to channel flux lines around the shielded region [4, 18, 19, 2]. There is, however, a limitation in using a passive shield of a high-permeability material. Further, a careful design of the shield can greatly improve the shielding performance without using extra expensive high-permeability materials, as Mager has shown that a shield of double shells of high-low-high permeability materials performs much better than a single shell of the same material, weight and exterior boundary [19]. Since then, various authors [9, 7, 36, 3, 29] have considered multiple cylindrical or spherical shells and achieved a good understanding of such shielding structures.

We present a simple design of electromagnetic shields for both field expelling and field confinement. This design is motivated by the concept of *neutral inclusions* in the theory of composite, based on which we introduce two concepts of *neutral shells*. A second motivation arises from recent waves of theoretical proposals and experimental efforts to realize cloaking by manipulating various materials [31, 16, 24, 25, 26]. Perfect cloaking, by definition, requires that the shield and the objects we desire to hide do not disturb the wave field on the exterior domain for an incident wave of any source. That is, the solution to the Maxwell equations remains exactly the same on the exterior domain as if we set the permittivity $\epsilon(\mathbf{x}) = \epsilon_0$ and permeability $\mu(\mathbf{x}) = \mu_0$ everywhere in space. This requirement is rather stringent but, nevertheless, could be satisfied by using materials including *singular* materials — materials with their physical moduli equal to negative numbers, zero or infinite, see e.g. [31, 16, 24, 35, 25, 26]. Though in theory but not without debates [38, 14, 30], such singular materials may be realized by *metamaterials* [39]. Metamaterials, exhibiting singular properties by resonance, are intrinsically lossy and strongly frequency-dependent. These issues can be addressed by considering normal materials while relaxing the stringent requirements of perfect cloaking. The resulting cloaks, however, must be interpreted with caution.

We propose a relaxed concept of cloaking which requires that (i) the shield and what we desire to hide disturb negligibly the exterior wave field when a plane wave passes them, and (ii) if there is a wave source inside the shield, the wave penetrates negligibly the shield into the exterior domain. In the long wavelength limit, requirement (i) amounts to an electromagnetic shield that excludes electromagnetic fields from the interior domain and does not disturb the exterior fields; requirement (ii) amounts to an electromagnetic shield that confines electromagnetic fields inside the interior domain. We find that some of the ideas in the theory of composites and the design of electromagnetic shields are surprisingly useful for the design of cloaking structures. In particular, we demonstrate that, in the long wavelength limit and for plane waves, the designed electromagnetic shields are capable of "cloaking" in the relaxed sense discussed above.

We remark that the idea behind our designs is different from those of [34, 33], where the distribution of materials in radian direction is optimized in a brute-force manner, though the final layout of materials appear similar, i.e., a sphere of multiple shells. Further, we can generalize our designs to geometries other than coated spheres. The idea is by regarding the requirements on a neutral shell as *overdetermined* conditions. The method presented in [17] can be used to construct neutral shells of various shapes.

The paper is organized as follows. In section 2 we formulate the design problems concerning electromagnetic shields for field expelling and field confinement. In section 3 we introduce the concepts of neutral shells, discuss the methods of constructing neutral shells and present various examples of neutral shells. Using neutral shells as building blocks we then proceed to the solutions of the design problems of electromagnetic shields in section 4. In section 5 we show that the designed electromagnetic shields indeed have the cloaking effects in the long wavelength limit. We conclude in section 6 providing an outlook. In the appendix we derive bounds on the shielding factor of the first kind of neutral shells in terms of the threshold exponents [22].

2. Formulation of the design problems

Let $D \subset \mathbb{R}^n$ be the design region ($n = 2$ or 3), E be the exterior domain, Ω be the region we aim to exclude or confine the fields, and $\mu(\mathbf{x})$ ($\epsilon(\mathbf{x})$) be the permeability (permittivity) of the medium which is equal to μ_0 (ϵ_0) on $\Omega \cup E$ and $\mu_D(\mathbf{x})$ ($\epsilon_D(\mathbf{x})$) on the design region D, see Fig. 1 (a) and (b). We consider the Maxwell equations. At the long wavelength limit, the magnetic and electric fields are decoupled [12] and the magnetic and electric fields can be expressed as $\mathbf{E}(\mathbf{x}, t) = -\nabla\varphi(\mathbf{x}) \exp(i\mathbf{k} \cdot \mathbf{x} - i\omega t)$ and $\mathbf{H}(\mathbf{x}, t) = -\nabla\xi(\mathbf{x}) \exp(i\mathbf{k} \cdot \mathbf{x} - i\omega t)$, where $\varphi(\mathbf{x})$ and $\xi(\mathbf{x})$ are the static electric and magnetic potentials satisfying

$$\mathrm{div}[\epsilon(\mathbf{x})\nabla\varphi] = 0 \qquad \text{on } \mathbb{R}^n \tag{2.1}$$

and

$$\mathrm{div}[\mu(\mathbf{x})\nabla\xi] = 0 \qquad \text{on } \mathbb{R}^n, \tag{2.2}$$

respectively. Since equation (2.1) behaves similarly as (2.2), below we focus on the design of magnetic medium, i.e., $\mu_D(\mathbf{x})$ and assume $\epsilon(\mathbf{x}) = \epsilon_0$ everywhere.

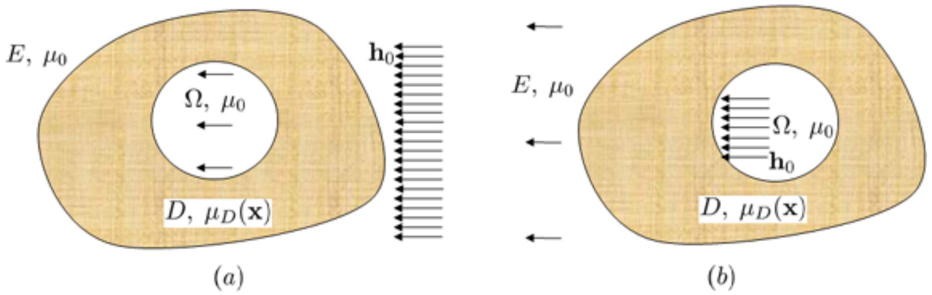

Figure 1. (a) A shield for field expelling; (b) A shield for field confinement.

We consider two types of shields as illustrated in Fig. 1(a) and (b). In the first scenario, a magnetic field is applied externally and we aim to minimize the field inside the shield D, i.e., on Ω; in the second scenario, a magnetic field source is placed inside Ω and we aim to minimize the field outside the shield D, i.e., on E. We shall achieve either goal or both by designing the materials profiles $\mu_D(\mathbf{x})$. Since natural materials have a finite range of permeability, we enforce the constraint

$$\mu_D \in \mathbb{F} := \{\mu : \ 0 < 1/K \le \mu(\mathbf{x})/\mu_0 \le K < +\infty \ \forall \mathbf{x} \in D\}, \tag{2.3}$$

where $K > 1$ is a design constraint.

Mathematically, the design problems are posed as follows. For the first scenario of field expelling, we consider the min-max problem:

$$\min_{\mu_D \in \mathbb{F}} \{\max\{|\nabla\xi(\mathbf{x})| : \ \mathbf{x} \in D\}\}, \tag{2.4}$$

where the magnetostatic potential ξ is determined by the boundary value problem

$$\begin{cases} \mathrm{div}[\mu(\mathbf{x})\nabla\xi] = 0 & \text{on } \mathbb{R}^n, \\ -\nabla\xi(\mathbf{x}) = \mathbf{h}_0 & \text{as } |\mathbf{x}| \to +\infty. \end{cases} \tag{2.5}$$

Here, $\mathbf{h}_0 \in \mathbb{R}^n$ with $|\mathbf{h}_0| = 1$ is interpreted as the polarization of the incident wave or simply the applied magnetic field in the static situation. For the second scenario of field confinement, we consider the min-max problem:

$$\min_{\mu_D \in \mathbb{F}} \{\max\{|\nabla\xi(\mathbf{x})| : \ \mathbf{x} \in E\}\}, \tag{2.6}$$

where the magnetostatic potential ξ is determined by the boundary value problem

$$\begin{cases} \mathrm{div}[\mu(\mathbf{x})\nabla\xi] = 0 & \text{on } \ \mathbb{R}^n \setminus \Omega, \\ -\mathbf{n}\cdot\nabla\xi(\mathbf{x}) = \mathbf{h}_0 \cdot \mathbf{n}, & \text{on } \ \partial\Omega, \\ -\nabla\xi(\mathbf{x}) \to 0, & \text{as } \ |\mathbf{x}| \to +\infty. \end{cases} \tag{2.7}$$

Here $\mathbf{n}$ is the unit normal on the interface $\partial\Omega$, $\mathbf{h}_0 \in \mathbb{R}^n$ with $|\mathbf{h}_0| = 1$ is given. Note that the differences between (2.4) and (2.6) lie on the shielded regions (Ω vs. E) and the boundary value problems that determine the field ((2.5) vs. (2.7)).

3. Neutral Shells

Our solutions to the design problems (2.4) and (2.6) are motivated by the concept of *neutral inclusion* [20]. In the theory of composites, an inclusion inside a homogenized medium is neutral if it does not perturb the effective property of the medium. In the context of a magnetic medium and in terms of the boundary value problem (2.5), the inclusion $D \cup \Omega$ sketched in Fig. 1(a) is a neutral inclusion if the solution to (2.5) satisfies

$$-\nabla\xi = \mathbf{h}_0 \qquad \text{on } \ E \qquad \forall\, \mathbf{h}_0 \in \mathbb{R}^n. \tag{3.1}$$

That is, the exterior field is undisturbed at the presence of the inhomogeneous structure $D \cup \Omega$. From (3.1), it is not hard to see that a composite medium with any number and any size of such neutral inclusions distributed in a matrix of permeability μ_0 has its effective permeability equal to μ_0. The existence of neutral inclusions is well-known, in particular, they include the Hashin's construction of coated spheres [11] and Milton's construction of coated ellipsoids [23].

Below we present two concepts of *neutral shells* which require the followings:

i) the interior medium is the same as the exterior medium, i.e., $\mu(\mathbf{x}) = \mu_0$ if $\mathbf{x} \in \Omega \cup E$,

ii) the solution to the boundary value problem (2.5) satisfies (3.1) and

$$-\nabla\xi = const. \qquad \text{on } \ \Omega. \tag{3.2}$$

We call such a structure D a neutral shell of the first kind, which has been introduced by Milgrom and Shtrikman (1989) [21] for calculating the effective properties of composites. Or,

iii) the solution to the boundary value problem (2.7) satisfies

$$\mathbf{e}_r \cdot \nabla\xi = -\frac{b_0(n-1)(\mathbf{e}_r \cdot \mathbf{h}_0)}{r^n} \qquad \text{on } \ \partial\Omega, \tag{3.3}$$

where $r = |\mathbf{x}|$, $\mathbf{e}_r = |\mathbf{x}|/r$, and $b_0 \in \mathbb{R}$ is a constant. We call such a structure D a neutral shell of the second kind. The physical meaning of (3.3) is that the field on $\partial\Omega$ coincides exactly with a point dipole $\mathbf{h}_0$ at the the origin.

The motivation for the above definitions of neutral shells is that the solutions to (2.5) and (2.7) are exceptionally easy for neutral shells, and neutral shells have the property that a nested neutral shell remains to be a neutral shell, see discussions in section 4.

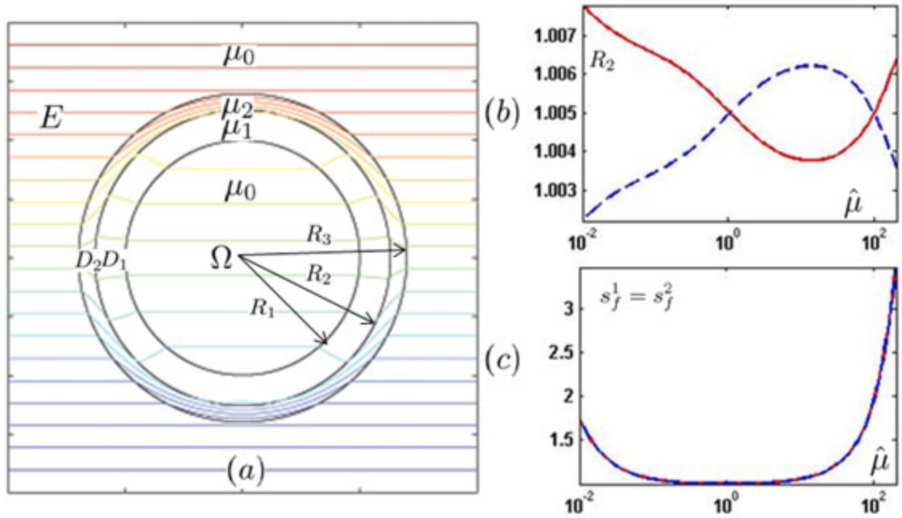

Figure 2. (a) A double-layer neutral shell with radius R_1, R_2, R_3 and permeability μ_1, μ_2. The lines are the contours of the solution ξ to the boundary value problem (2.5). (b) The radius R_2 versus $\hat{\mu}$ such that the structure $D = D_1 \cup D_2$ is a neutral shell: the solid line "—" corresponds to neutral shells of the first kind; the dashed line "– –" corresponds to neutral shells of the second kind. (c) The shielding factors s_f^1 and s_f^2 versus $\hat{\mu}$ of the neutral shells. Note that the two curves $s_f^1 = s_f^1(\hat{\mu})$ and $s_f^2 = s_f^2(\hat{\mu})$ have no noticeable difference.

(a) *Neutral shells of double layers*

We now give various examples of neutral shells. In the simplest situation, we consider domains of spherical symmetry. Dividing our design region D into double spherical shells with radius R_1, R_2 and R_3 as sketched in Fig. 2(a), we denote by χ_V the characteristic function of the domain V, i.e., χ_V is equal to one on V and zero otherwise. Then the permeability on the entire space is given by

$$\mu(\mathbf{x}) = \sum_{i=0}^{3} \mu_i \chi_{D_i}(\mathbf{x}), \tag{3.4}$$

where $\mu_3 = \mu_0$ and

$$D_0 = \Omega, \quad D_i = \{\mathbf{x} : R_i < |\mathbf{x}| < R_{i+1}\} \ \ (i = 1, 2), \quad D_3 = E.$$

Below we show that for appropriate μ_1, μ_2, R_1, R_2 and R_3, the double shell $D = D_1 \cup D_2$ is a neutral shell of the first or the second kind.

By symmetry, we write the solutions to (2.5) or (2.7) as

$$\xi = -\mathbf{h}_0 \cdot \nabla u, \quad \text{i.e.,} \quad \nabla \xi = -(\nabla\nabla u)\mathbf{h}_0, \tag{3.5}$$

where u is given by ($r = |\mathbf{x}|$)

$$u(r) = \begin{cases} \frac{1}{2}a_i r^2 + \frac{b_i}{r} + c_i & \text{if } n = 3 \\ \frac{1}{2}a_i r^2 - b_i \log(r) + c_i & \text{if } n = 2 \end{cases} \quad \text{for } \mathbf{x} \in D_i,\ i = 0, 1, 2, 3, \tag{3.6}$$

and the constants $a_i, b_i \in \mathbb{R}$ are to be determined. We require that $u'(r)$ be continuous for $r > 0$, which implies that for $i = 1, 2, 3$,

$$a_{i-1}R_i - \frac{b_{i-1}}{R_i^{n-1}} = a_i R_i - \frac{b_i}{R_i^{n-1}}. \tag{3.7}$$

By direct calculations we find that

$$\Delta u = \nabla \cdot \nabla u = -\sigma_n b_0 \delta(r) + n \sum_{i=0}^{3} a_i \chi_{D_i}, \tag{3.8}$$

where $\delta(r)$ is the Dirac function, $\sigma_n = 2\pi$ if $n = 2$ and $= 4\pi$ if $n = 3$. Further, we verify that the function $v_{\mathbf{h}_0} = -\mathbf{h}_0 \cdot \nabla u$ satisfies

$$\begin{cases} \Delta v_{\mathbf{h}_0} = 0 & \text{on } D_i \setminus \{0\},\ i = 0, 1, \cdots, 3, \\ \nabla v_{\mathbf{h}_0}(\mathbf{x}) \to -a_3 \mathbf{h}_0 & \text{as } |\mathbf{x}| \to \infty. \end{cases} \tag{3.9}$$

By (3.6) we have

$$\nabla v_{\mathbf{h}_0} = -(\nabla\nabla u)(\mathbf{h}_0) = -[a_i \mathbf{h}_0 - \frac{b_i}{r^n}\mathbf{h}_0 + \frac{n b_i \mathbf{e}_r \cdot \mathbf{h}_0}{r^n}\mathbf{e}_r] \qquad \text{on } D_i,\ i = 0, \cdots, 3,$$

which implies that, for any $\mathbf{x} \in S_i := \{\mathbf{x} : |\mathbf{x}| = R_i\}$ and $i = 1, 2, 3$,

$$\mathbf{e}_r \cdot [\mu_i \nabla v_{\mathbf{h}_0}(\mathbf{x}+) - \mu_{i-1} \nabla v_{\mathbf{h}_0}(\mathbf{x}-)] = \mathbf{e}_r \cdot \mathbf{h}_0 \Big[-\mu_i a_i - \frac{\mu_i (n-1) b_i}{R_i^n} + \mu_{i-1} a_{i-1} + \frac{\mu_{i-1}(n-1) b_{i-1}}{R_i^n} \Big]. \tag{3.10}$$

Here $\mathbf{e}_r = \mathbf{x}/|\mathbf{x}|$ and $\mathbf{x}+$ ($\mathbf{x}-$) denotes the boundary points outside (inside) the sphere S_i. Moreover, we write equation (2.2) in a different form as

$$\begin{cases} \Delta \xi = 0 & \text{on } D_i,\ i = 1, \cdots, 3, \\ \mathbf{e}_r \cdot [\mu_i \nabla \xi(\mathbf{x}+) - \mu_{i-1} \nabla \xi(\mathbf{x}-)] = 0 & \text{on } S_i,\ i = 2, 3. \end{cases} \tag{3.11}$$

Comparing (3.11) with (3.9)-(3.10), we are motivated to require that for any $i = 1, 2, 3$,

$$\mu_i a_i + \frac{\mu_i (n-1) b_i}{R_i^n} = \mu_{i-1} a_{i-1} + \frac{\mu_{i-1}(n-1) b_{i-1}}{R_i^n}, \tag{3.12}$$

which, together with (3.7), can be rewritten as

$$\begin{bmatrix} a_i \\ b_i \end{bmatrix} = \mathbf{M}_i \begin{bmatrix} a_{i-1} \\ b_{i-1} \end{bmatrix}, \qquad \mathbf{M}_i = \frac{1}{n\mu_i} \begin{bmatrix} \mu_{i-1} + (n-1)\mu_i & \frac{(n-1)(\mu_{i-1} - \mu_i)}{R_i^n} \\ (\mu_{i-1} - \mu_i) R_i^n & \mu_i + (n-1)\mu_{i-1} \end{bmatrix}. \tag{3.13}$$

Note that $\det(\mathbf{M}_i) = \mu_{i-1}/\mu_i \neq 0$. It will be useful to define a matrix

$$\mathbf{T} = \mathbf{M}_3\mathbf{M}_2\mathbf{M}_1, \qquad \det(\mathbf{T}) = 1. \tag{3.14}$$

By (3.13) we immediately have

$$\begin{bmatrix} a_3 \\ b_3 \end{bmatrix} = \mathbf{T} \begin{bmatrix} a_0 \\ b_0 \end{bmatrix}, \qquad \mathbf{T} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}. \tag{3.15}$$

We call the matrix $\mathbf{T}$ the *transfer matrix*. The boundary value problems (2.5) and (2.7) can be conveniently solved using this transfer matrix $\mathbf{T}$. To see this, let us first consider (2.5), where the boundary conditions require that $\nabla\xi(r)$ is nonsingular at $r = 0$ and approaches to $-\mathbf{h}_0$ as $r \to +\infty$. These are satisfied by (3.5) if

$$b_0 = 0, \qquad a_3 = 1. \tag{3.16}$$

Meanwhile, the boundary conditions in (2.7) are satisfied by (3.5) if

$$a_0 + \frac{(n-1)b_0}{R_1^n} = 1, \qquad a_3 = 0. \tag{3.17}$$

Therefore, by (3.15) and (3.16) or (3.17), we can solve for all a_i, b_i for $i = 0, \cdots, 3$ and obtain the solution to (2.5) or (2.7), as given by (3.5).

By (3.15) and (3.16), we see that if the matrix element $T_{21} = 0$, then $b_3 = 0$, i.e., the solution to (2.5) satisfies (3.1). The converse is also true. Further, from (3.5), (3.6) and (3.16), we see that the solution to (2.5) automatically satisfies (3.2). Therefore, a double spherical shell is a neutral shell of the first kind if and only if the matrix element $T_{21} = 0$. Moreover, by the divergence theorem and (3.8) we find that $T_{21} = b_3 = 0$ if

$$0 = \int_{\{r=R_3\}} \mathbf{e}_r \cdot \nabla[u(r) - \frac{1}{2}a_3 r^2] = \int_{\{r\le R_3\}} \Delta(u(r) - \frac{1}{2}a_3 r^2) = \sum_{i=0}^{2}(a_i - a_3)|D_i|,$$

where $|D_i|$ denotes the volume of the domain D_i. We define the *shielding factor*

$$s_f^1 = \frac{|\mathbf{h}_0|}{|\nabla\xi(0)|} = \frac{|a_3|}{|a_0|} = |T_{11}|, \tag{3.18}$$

which measures the effectiveness of the shield for field expelling.

Parallel to our discussions about the neutral shell of the first kind, we see that if the matrix element $T_{12} = 0$, then, by (3.15) and (3.17), $a_0 = 0$, i.e., the solution to (2.7) satisfies (3.3). The converse is also true. Therefore, a double spherical shell is a neutral shell of the second kind if and only if the matrix element $T_{12} = 0$. Moreover, by the divergence theorem and (3.8) we find that $T_{12} = a_0 = 0$ if

$$-\sigma_n b_3 = \int_{\{r=R_3\}} \mathbf{e}_r \cdot \nabla u(r) = \int_{\{r\le R_3\}} \Delta u(r) = a_2|D_2| + a_1|D_1| - \sigma_n b_0.$$

We define the *shielding factor*

$$s_f^2 = \frac{|b_0|}{|b_3|} = \frac{1}{|T_{22}|}, \tag{3.19}$$

which measures the effectiveness of the shield for field confinement.

In Fig. 2 we show examples of neutral shells in three dimensions ($n = 3$) and their shielding effects, where we specify

$$\mu_0 = 1, \;\; R_1 = 1, \;\; R_3 = 1.01 \tag{3.20}$$

and assume $\mu_2 = 1/\mu_1 = \hat{\mu}$ so that the transfer matrix $\mathbf{T}$ depends only on $\hat{\mu}$ and R_2: $\mathbf{T} = \mathbf{T}(\hat{\mu}, R_2)$. Note that the thickness of the structure D is only 1% of the radius. For given $\hat{\mu}$, we solve for $R_2(\hat{\mu})$ ($R_2'(\hat{\mu})$) such that $T_{21}(\hat{\mu}, R_2) = 0$ ($T_{12}(\hat{\mu}, R_2) = 0$), and find the corresponding shielding factor $s_f^1 = |T_{11}|$ ($s_f^2 = 1/|T_{22}|$). In Fig. 2(b) we show the curves $R_2 = R_2(\hat{\mu})$ and $R_2 = R_2'(\hat{\mu})$: the solid line "—" of $R_2 = R_2(\hat{\mu})$ corresponds to neutral shells of the first kind; the dashed line "– –" of $R_2 = R_2'(\hat{\mu})$ corresponds to neutral shells of the second kind. In Fig. 2(c) we show the curves of the shielding factors $s_f^1 = s_f^1(\hat{\mu})$ and $s_f^2 = s_f^2(\hat{\mu})$ of the neutral shells. Note that the two shielding factors s_f^1 and s_f^2 have no noticeable difference within the numerical resolution. Further, it is interesting to notice that when $R_2 = 1.0050$ and $\hat{\mu} = 99.0$, the structure D is simultaneously a neutral shell of the first kind and of the second kind. In this case, the transfer matrix defined in (3.15) is a diagonal matrix

$$\mathbf{T} = \begin{bmatrix} s_f^1 & 0 \\ 0 & 1/s_f^2 \end{bmatrix}, \qquad s_f^1 = s_f^2 = 1.95. \tag{3.21}$$

(b) Neutral shells of continuous gradings

We now generalize our constructions of neutral shells to allow continuous gradings. As in the last section, we assume $\Omega = \{\mathbf{x} : |\mathbf{x}| < R_1\}$, $E = \{\mathbf{x} : |\mathbf{x}| > R_3\}$, and $\mu = \mu(r)$ is continuous for $r \in (R_1, R_3)$. Plugging (3.5) into (2.2), we find that

$$(\nabla\nabla u)\nabla\mu + \mu\nabla\Delta u = 0, \quad \text{i.e.,} \quad u''\mu' + \mu(u'' + \frac{n-1}{r}u')' = 0. \tag{3.22}$$

Note that $\mu(r)$ may be discontinuous across $r = R_1$ and $r = R_3$. In this case, equation (2.2) shall be interpreted as $[\![\mu(r)\nabla\xi]\!]\cdot\mathbf{e}_r = 0$. Therefore, across a discontinuous interface of $\mu(r)$ at $r = R$, we have

$$[\![\mu(r)\nabla\nabla u]\!]\mathbf{e}_r = 0, \quad \text{i.e.,} \quad \mu(R-)u''(R-) = \mu(R+)u''(R+), \tag{3.23}$$

where $R = R_1$ or R_3. Further, we require that $u(r)$ is continuously differentiable (C^1) for $r > 0$ and $u'(r)$ satisfies that

$$u'(r) = \begin{cases} a_0 r - \frac{b_0}{r^{n-1}} & \text{if } r \le R_1, \\ a_3 r - \frac{b_3}{r^{n-1}} & \text{if } r > R_3. \end{cases} \tag{3.24}$$

It is interesting to notice that equations (3.22)-(3.24) may be solved from two directions. In one direction, we extend $u(r)$ by interpolation such that $u'(r)$ is at least continuous for all $r > 0$. For example, let us assume

$$u'(r) = \sum_{k=0}^{3} \alpha_k r^k. \tag{3.25}$$

Requiring u' and u'' are continuous at $r = R_1$ and R_3, we obtain

$$\begin{cases} u'(R_1) = a_0 R_1 - \frac{b_0}{R_1^{n-1}}, & u'(R_3) = a_3 R_3 - \frac{b_3}{R_3^{n-1}}, \\ u''(R_1) = a_0 + \frac{(n-1)b_0}{R_1^n}, & u''(R_3) = a_3 + \frac{(n-1)b_3}{R_1^n}. \end{cases} \tag{3.26}$$

Further, equation (3.23) implies

$$\mu(R_1) = \mu(R_3) = \mu_0. \tag{3.27}$$

Plugging (3.25) into (3.22), we obtain a first-order ordinary differential equation (ODE) for $\mu(r)$. Specifying μ_0, R_1, R_3 as in (3.20), we are left with nine unknowns: $a_0, b_0, a_3, b_3, \alpha_0, \cdots, \alpha_3$, and an integration constants associated with the solution of the first-order ODE for μ in (3.22). There are six equations in (3.26)-(3.27). Therefore, presumably we can specify three of the unknowns, e.g., (a_0, b_0) and α_0, and solve for all others. In particular, analogous to (3.15), we can write the relation between (a_0, b_0) and (a_3, b_3) as

$$\begin{bmatrix} a_3 \\ b_3 \end{bmatrix} = \mathbf{T} \begin{bmatrix} a_0 \\ b_0 \end{bmatrix}, \qquad \mathbf{T} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}. \tag{3.28}$$

where the linear dependence of (a_3, b_3) on (a_0, b_0) follows from the ODE (3.22) is linear for $u(r)$. By adjusting the parameter α_0, we could make the matrix element T_{21} (T_{12}) vanish and obtain a neutral shell of the first (second) kind.

In the opposite direction, we specify the functional dependence of μ on r and determine the associated parameters so that the solution to (3.22) and (3.23) can be indeed extended continuously differentiable to satisfy (3.24). In Fig. 2 we show such examples where $\mu(r)$ is a piecewise constant function. Below we assume $\mu(r)$ is linearly graded as

$$\mu(r)/\mu_0 = \tilde{\mu}(r - R_2) + 1 \qquad \text{if } r \in (R_1, R_3), \tag{3.29}$$

where $\tilde{\mu} \in \mathbb{R}$ is the gradient of $\mu(r)$ in the $\mathbf{e}_r$-direction. Then equations (3.22), (3.23) and (3.24) imply

$$\begin{cases} [\tilde{\mu}(r - R_2) + 1](u'' + (n-1)\frac{u'}{r})' + \tilde{\mu} u'' = 0, & \text{if } R_1 < r < R_3, \\ u'(R_1) = a_0 R_1 - \frac{b_0}{R_1^{n-1}}, & \frac{\mu(R_1)}{\mu_0} u''(R_1) = a_0 + \frac{(n-1)b_0}{R_1^n}, \\ u'(R_3) = a_0 R_3 - \frac{b_0}{R_3^{n-1}}, & \frac{\mu(R_3)}{\mu_0} u''(R_3) = a_3 + \frac{(n-1)b_3}{R_3^n}. \end{cases} \tag{3.30}$$

An analytical solution to the above problem is desirable but not obvious; we turn to numerical solutions. Specifying μ_0, R_1, R_3 as in (3.20), we are left with eight unknowns: $a_0, b_0, a_3, b_3, R_2, \tilde{\mu}$, and two integration constants associated with the solution to the second-order ODE for u' in (3.30). Note that there are four boundary conditions in (3.30). Therefore, if four of the unknowns, e.g., (a_0, b_0) and $(\tilde{\mu}, R_2)$ are specified, we can solve for all others. In particular, analogous to (3.15), we can write the relation between (a_0, b_0) and (a_3, b_3) as

$$\begin{bmatrix} a_3 \\ b_3 \end{bmatrix} = \mathbf{T}(\tilde{\mu}, R_2) \begin{bmatrix} a_0 \\ b_0 \end{bmatrix}, \qquad \mathbf{T} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}. \tag{3.31}$$

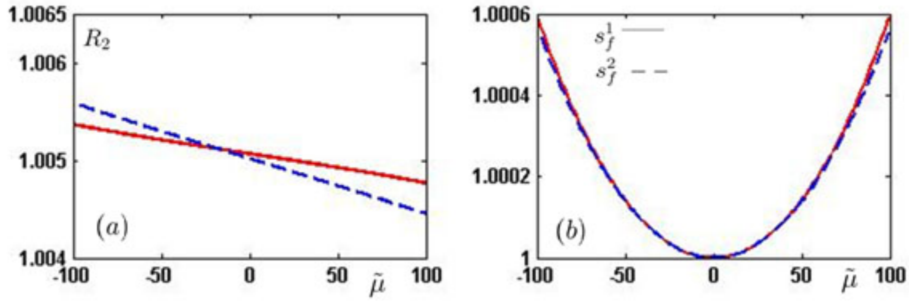

Figure 3. (a) The parameter R_2 versus $\tilde{\mu}$ such that the linearly graded structure $D = \{\mathbf{x} : R_1 < |\mathbf{x}| < R_3\}$ with permeability given by (3.29) is a neutral shell: the solid line "—" of $R_2 = R_2(\tilde{\mu})$ corresponds to neutral shells of the first kind; the dashed line "– –" of $R_2 = R_2'(\tilde{\mu})$ corresponds to neutral shells of the second kind. (b) The shielding factors s_f^1 and s_f^2 versus $\tilde{\mu}$ of the neutral shells. Note that the two curves $s_f^1 = s_f^1(\tilde{\mu})$ and $s_f^2 = s_f^2(\tilde{\mu})$ are slightly different.

where the linear dependence of (a_3, b_3) on (a_0, b_0) follows from the ODE (3.22) is linear for $u(r)$. By adjusting the parameter $(\tilde{\mu}, R_2)$, we could make the matrix element T_{21} (T_{12}) vanish and obtain a neutral shell of the first (second) kind.

In Fig. 3 we show examples of linearly graded neutral shells and their shielding factors, where μ_0, R_1, R_3 are specified by (3.20), and so if $\tilde{\mu}$ and R_2 are given, we can calculate the transfer matrix $\mathbf{T}(\tilde{\mu}, R_2)$ by (3.30). For given $\tilde{\mu}$, we solve for $R_2 = R_2(\tilde{\mu})$ $(R_2 = R_2'(\tilde{\mu}))$ such that $T_{21}(\tilde{\mu}, R_2) = 0$ $(T_{12}(\tilde{\mu}, R_2) = 0)$, and find the corresponding shielding factor $s_f^1 = |T_{11}|$ $(s_f^2 = 1/|T_{22}|)$. Figure 3(a) shows the curves $R_2 = R_2(\tilde{\mu})$ and $R_2 = R_2'(\tilde{\mu})$: the solid line "—" of $R_2 = R_2(\tilde{\mu})$ corresponds to neutral shells of the first kind; the dashed line "– –" of $R_2 = R_2'(\tilde{\mu})$ corresponds to neutral shells of the second kind. Figure 3(b) shows the curves of the shielding factors $s_f^1 = s_f^1(\tilde{\mu})$ and $s_f^2 = s_f^2(\tilde{\mu})$ of the neutral shells. Note that the two shielding factors s_f^1 and s_f^2 are slightly different.

4. Designs of electromagnetic shields

In this section, based on the concepts of neutral shells we construct solutions to the design problems (2.4) and (2.6). First, we verify that the solutions to (2.5) and (2.7) have the transformation property that

$$\mu(\mathbf{x}) \to \mu'(\mathbf{x}) = \mu(\lambda\mathbf{x}) \quad \Rightarrow \quad \xi(\mathbf{x}) \to \xi'(\mathbf{x}) = \xi(\lambda\mathbf{x})/\lambda \qquad \forall\, \lambda > 0.$$

Since $\nabla\xi'_{\mathbf{x}}(\mathbf{x}) = \nabla_{\mathbf{x}}\xi(\lambda\mathbf{x})$, we infer that a uniformly shrunk neutral shell remains to be a neutral shell and the shielding factor remains unchanged. Therefore, if we construct a structure by shrinking and nesting N-neutral shells as illustrated in Fig. 4, the transfer matrix of the overall structure, denoted by $\mathbf{T}_N$, is given by

$$\mathbf{T}_N = \mathbf{T}^N, \tag{4.1}$$

where $\mathbf{T}$ is the transfer matrix of the prototype neutral shell. Since products of lower (upper) triangular matrices remain to be lower (upper) triangular matrix, the overall structure is again a neutral shell of the first (second) kind if the prototype

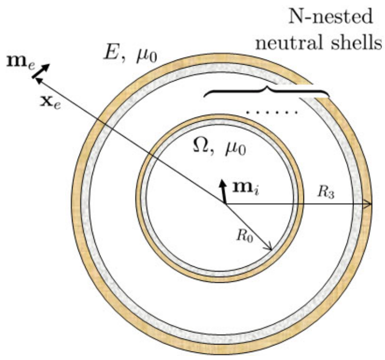

Figure 4. A shield of N-nested neutral shells.

shell is a neutral shell of the first (second) kind. The key observation is that the shielding factor of the overall structure grows *exponentially* whereas the growth of the thickness of the shield decreases *exponentially* as the nesting number increases. For example, let us begin with a double-layer neutral shell specified by (3.4) and illustrated in Fig. 2(a). If $\hat{\mu} = 200$ and $R_2 = 1.0064$, the double-layer shell is a neutral shell of the first kind with shielding factor $s_f^1 = 3.48$, as shown in Fig. 2 (b) and (c). Upon shrinking this prototype neutral shell and nesting N such neutral shells, the permeability on the entire space is given by

$$\mu(\mathbf{x}) = \begin{cases} \mu_0 & \text{if } r > R_3 \\ \hat{\mu} & \text{if } R_2(\frac{R_1}{R_3})^k < r < R_3(\frac{R_1}{R_3})^k, \ k = 0, \cdots, N-1, \\ 1/\hat{\mu} & \text{if } R_1(\frac{R_1}{R_3})^k < r < R_2(\frac{R_1}{R_3})^k, \ k = 0, \cdots, N-1, \\ \mu_0 & \text{if } r < R_0 = R_3(\frac{R_1}{R_3})^N. \end{cases} \tag{4.2}$$

Note that the thickness H and shielding factor S_f^1 of the overall structure is given by

$$H = R_3(1 - (R_1/R_3)^N), \qquad S_f^1 = (s_f^1)^N. \tag{4.3}$$

Thus, if $N = 20$, then $H = 0.18$, the interior radius $R_0 = R_3 - H = 0.83$, and $S_f^1 = 6.79 \times 10^{10}$. For this structure, the solution to the boundary value problem (2.5) has the property that the strength of the field inside the structure is S_f^1 times smaller than the external field. On the other hand, if $R_2 = 1.0036$, the prototype double-layer shell is a neutral shell of the second kind with shielding factor $s_f^2 = s_f^1$, see Fig. 2 (b) and (c). Then for the structure of 20 such nested neutral shells, the solution to the boundary value problem (2.7) has the property that the strength of the field outside the structure is $S_f^2 = 6.79 \times 10^{10}$ times smaller than the interior field.

There are applications that requires simultaneously expelling the external field and confining the interior field. In these applications we can use neutral shells that are both the first and second kind, e.g., the double-layer shell with $\hat{\mu} = 99$ and $R_2 = 1.0050$, see Fig. 2 (b) and (c) and discussions in section 3(a). Then the structure of 20 such nested neutral shells is a neutral shell of both the first and second kind with both shielding factors $S_f = S_f^2 = S_f^1 = 6.32 \times 10^5$, see (3.21).

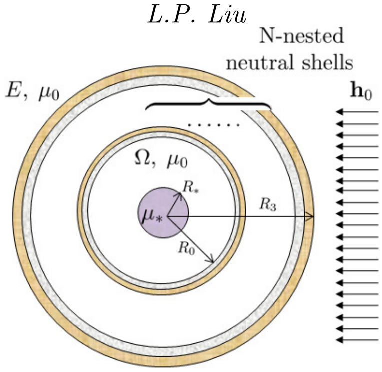

Figure 5. Cloaking effect of a shield of nested neutral shells.

Therefore, for this structure, the solution to the boundary problem (2.5) ((2.7)) has the property that the strength of the field inside (outside) the structure is 6.32×10^5 times smaller than the external (interior) field.

5. Cloaking effect

In this section we verify that the constructed electromagnetic shields have the following cloaking effects at the long wave length limit: (i) when a plane wave passes the shield, the scattered wave field is negligible compared with the scattered wave field at the absence of the shield, and (ii) an magnetic dipole inside the shield gives rise to a negligible radiation field compared with the radiation field at the absence of the shield.

To verify (i), we consider an incident plane wave $\mathbf{h}_0\exp(i(\mathbf{x}\cdot\mathbf{k}-\omega t))$ passes the shield, and for simplicity, assume the permittivity $\epsilon(\mathbf{x})=\epsilon_0$ everywhere but the permeability is given by (4.2) with $N=20$, $\hat{\mu}=99$ and $R_2=1.0050$, i.e., the prototype shell is simultaneously a neutral shell of the first and second kinds with its transfer matrix given by (3.21).

If the wave length (λ) of the incident wave in free space is much larger than the diameter (R_3) of the shield, then besides a harmonic factor $\exp(i(\mathbf{x}\cdot\mathbf{k}-\omega t))$ the scattered wave field on the exterior domain E is given by $\mathbf{h}_{sc}=-\nabla\xi-\mathbf{h}_0$, where ξ is the solution to (2.5). Since the shield is a neutral shell of the first kind, by (3.1) we see that $\mathbf{h}_{sc}=0$ on E, i.e., the shield does not disturb the incident wave on the exterior domain. Moreover, let us hide a spherical particle inside the shield, and for simplicity, assume the particle has permittivity ϵ_0, permeability μ_*, and radius $R_*<R_0$, see Fig. 5. For the particle and the shield, the solution to (2.5) is given by (3.5) with

$$u(r)=\begin{cases}\frac{1}{2}a_*r^2+\frac{b_*}{r} & \text{if } r<R_*,\\ \frac{1}{2}a_0r^2+\frac{b_0}{r} & \text{if } R_*<r<R_0,\\ \frac{1}{2}a_3r^2+\frac{b_3}{r} & \text{if } r>R_3,\end{cases}\tag{5.1}$$

where, by discussions in section a and (4.1), we have

$$\begin{bmatrix}a_3\\ b_3\end{bmatrix}=\mathbf{T}^N\begin{bmatrix}a_0\\ b_0\end{bmatrix}.$$

Further, by (3.13), we have ($n = 3$)

$$\begin{bmatrix} a_0 \\ b_0 \end{bmatrix} = \mathbf{M}_* \begin{bmatrix} a_* \\ b_* \end{bmatrix}, \qquad \mathbf{M}_* = \frac{1}{n\mu_0} \begin{bmatrix} \mu_* + (n-1)\mu_0 & \frac{(n-1)(\mu_*-\mu_0)}{R_*^n} \\ (\mu_* - \mu_0)R_*^n & \mu_0 + (n-1)\mu_* \end{bmatrix}. \tag{5.2}$$

Since the prototype shell is neutral with transfer matrix given by (3.21), and the boundary condition in (2.5) and the nonsingularity at $r = 0$ imply $a_3 = 1$ and $b_* = 0$, we obtain

$$\mathbf{h}_{sc} = b_3 \nabla[\mathbf{h}_0 \cdot \nabla \frac{1}{r}], \qquad b_3 = \frac{(\mu_* - \mu_0)R_*^n}{1.95^{40}(\mu_* + (n-1)\mu_0)}. \tag{5.3}$$

At the absence of the shield, the scattered wave field is given by

$$\mathbf{h}'_{sc} = b'_3 \nabla[\mathbf{h}_0 \cdot \nabla \frac{1}{r}], \qquad b'_3 = \frac{(\mu_* - \mu_0)R_*^n}{(\mu_* + (n-1)\mu_0)}. \tag{5.4}$$

Comparing (5.3) with (5.4), we see that the scattered wave field at the presence of the shield is $1.95^{40} = 3.99 \times 10^{11}$ times smaller than at the scattered wave field at the absence of the shield.

To verify (ii), we assume there is a magnetic dipole $\mathbf{m}^i$ at the origin. At the presence of the shield, the radiation field is determined by

$$\begin{cases} \operatorname{div}[-\mu(\mathbf{x})\nabla\xi^i + \mathbf{m}^i\delta(0)] = 0 & \text{on } \mathbb{R}^3, \\ \xi(\mathbf{x}) \to 0 & \text{as } |\mathbf{x}| \to +\infty. \end{cases} \tag{5.5}$$

Again, assuming the solution to the above problem is given by (3.5) and (5.1) we verify that

$$-\nabla\xi^i(\mathbf{x}) = -\frac{1}{1.95^{20}(4\pi)}\{\nabla[\mathbf{m}^i \cdot \nabla\frac{1}{r}]\} \qquad \forall \mathbf{x} \in E.$$

At the absence of the shield, the radiation field is directly given by $-\frac{1}{(4\pi)}\{\nabla[\mathbf{m}^i \cdot \nabla\frac{1}{r}]\}$, which is $1.95^{20} = 6.32 \times 10^5$ times larger than the wave field at the presence of the shield.

From the energetic viewpoint, the shield of nested neutral shells essentially cuts off the magnetic interactions between bodies inside the shield and outside the shield. To see this, in addition to the magnetic dipole $\mathbf{m}^i$ inside the shield, we assume there is a second magnetic dipoles $\mathbf{m}^e$ at an exterior point $\mathbf{x}_e$, see Fig. 4. At the absence of the shielding structure, the interaction energy between the two dipoles are given by

$$\mathcal{E}'_{int} = \frac{3(\mathbf{m}^i \cdot \hat{\mathbf{x}}_e)(\mathbf{m}^e \cdot \hat{\mathbf{x}}_e) - \mathbf{m}^e \cdot \mathbf{m}^i}{4\pi|\mathbf{x}_e|^3},$$

where $\hat{\mathbf{x}}_e = \mathbf{x}_e/|\mathbf{x}_e|$, see [12]. The interaction energy between the two dipoles at the presence of the shield is given by

$$\mathcal{E}_{int} = -\mathbf{m}^e \cdot \nabla\xi^i(\mathbf{x}_e) = \frac{1}{1.95^{20}}\mathcal{E}'_{int},$$

which is $1.95^{20} = 6.32 \times 10^5$ times smaller than the interaction energy at the absence of the shield.

6. Summary and Discussion

We consider the design of passive electromagnetic shields in the long wavelength limit. By introducing the concepts of neutral shells, we construct our shields simply by shrinking and nesting a number of a prototype neutral shells. The key observation is that the resulting shield remains as a neutral shell and the shielding factor increases exponentially as the number of nesting increases. We also show that the designed shield is capable of cloaking in a relaxed sense discussed in the introduction.

The method of solving the governing boundary value problems (2.5) and (2.7) follows from the observation that the potential of the boundary value problems (2.5) and (2.7) is in fact given by a gradient field, see (3.5). This observation greatly simplifies the procedure of solving the boundary value problems (2.5) and (2.7) and facilitates the definition of transfer matrix, see (3.15). Further, we note that this is a special property of the structure, i.e., multiple spherical shells, but not restricted to structures with spherical symmetries. It can be shown that multiple ellipsoidal shells have a similar property and so neutral shells of confocal ellipsoidal surfaces can be defined as well. More generally, we can construct neutral shells of a variety of geometries by regarding the requirements on neutral shells as *overdetermined* conditions. A generalization of the work [17] yields a method of constructing neutral shells of other shapes.

Two remarks are in order regarding the realizable range of the shielding factors and the relative permeability of materials. Though in Fig. 2(c) and 3(b) it appears that the shielding factors are greater or equal to one, there are situations where the shielding factors are less than one. Further, for given design constraint (2.3), the shielding factors may be bounded from above and below in terms of K. These bounds are closely related with the threshold exponents defined by Milton (1986) [22], see details in the appendix. Further, high-permeability materials, e.g, Mu-metals, are easily available with relative permeability μ/μ_0 up to 2.0×10^4 [13]. Most natural diamagnetic materials such as water and bismuth have relative permeability at the order of 0.99999, but superconductors ideally have relative permeability equal to zero. Therefore, materials with relative permeability between 0.005 and 200 are physically realizable, at least by composite materials with a superconductor phase.

As discussed in this paper, the design of cloaking structures is closely related with the design of electromagnetic shields in the long wavelength limit. It will be interesting to generalize the concepts of neutral shells and the current designs of electromagnetic shields and cloaking structures to waves with finite wavelengths. The interested reader is referred to [6, 1] for works in this direction.

Appendix: Bounds on the shielding factors

Milton (1986) [22] defined the threshold exponents to measure the degree of field concentration in a composite. Below we show that the threshold exponents imply bounds on the shielding factors of neutral shells. To see this, we recall the following definition of the threshold exponents, also see [15, 8]. Let $U \subset \mathbb{R}^n$ be an open bounded domain and $\xi \in W^{1,2}(U)$ be a weak solution to

$$\mathrm{div}[\mu(\mathbf{x})\nabla\xi] = 0 \qquad \text{a.e. on} \quad U,$$

where $\mu(\mathbf{x})$ satisfies

$$0 < 1/K \le \mu(\mathbf{x})/\mu_0 \le K < +\infty \qquad \text{for a.e. } \mathbf{x} \in U, \tag{A.1}$$

It is known that $|\nabla\xi|$ is in $L^p(U)$ for some $p > 2$ and the reciprocal $1/|\nabla\xi|$ is in $L^q(U)$ for some $q \ge 0$, i.e.,

$$\int_U |\nabla\xi|^p < +\infty \qquad \text{and} \qquad \int_U \frac{1}{|\nabla\xi|^q} < +\infty. \tag{A.2}$$

The supremum of such p (q) that the first (second) inequality holds for *any* $\mu(\mathbf{x})$ satisfying (A.1), denoted by p_M (q_M), is called the threshold exponents.

Let $D = \{\mathbf{x} : R_1 < |\mathbf{x}| < R_3\}$ be a spherical neutral shell of the first kind with shielding factor s_f^1, $\mu(\mathbf{x})$ restricted on D satisfies (A.1), and ξ_1 be the solution to (2.4). Consider a shield of N-nested neutral shells of the scaled copies of D, for which the solution to (2.4) is denoted by ξ_N. Let $V_i = \{\mathbf{x} : R_3(\frac{R_1}{R_3})^i < |\mathbf{x}| < R_3\}$ and $V_\infty = \{\mathbf{x} : |\mathbf{x}| < R_3\}$ and, without loss of generality, assume $V_\infty \subset U$. Then from the discussions in section 4, we have that for any $i \ge 1$,

$$\begin{cases} \nabla\xi_{i+1}(\mathbf{x}) = \nabla\xi_i(\mathbf{x}) & \text{if } \mathbf{x} \in V_i, \\ \nabla\xi_{i+1}(\mathbf{x}) = \frac{1}{s_f^1}\nabla\xi_i(\frac{R_3}{R_1}\mathbf{x}) & \text{if } \mathbf{x} \in V_{i+1} \setminus V_i, \\ \nabla\xi_{i+1}(\mathbf{x}) = \frac{1}{s_f^1}\nabla\xi_i(\frac{R_3}{R_1}\mathbf{x}) & \text{if } \mathbf{x} \in V_\infty \setminus V_{i+1}. \end{cases}$$

Thus,

$$\begin{aligned} \int_U |\nabla\xi_N|^p &\ge \int_{V_N} |\nabla\xi_N|^p = (1 + \varrho + \cdots + \varrho^{N-1}) \int_{V_1} |\nabla\xi_1|^p, \\ \int_U \frac{1}{|\nabla\xi_N|^q} &\ge \int_{V_N} \frac{1}{|\nabla\xi_N|^q} = (1 + \varrho' + \cdots + \varrho'^{N-1}) \int_{V_1} \frac{1}{|\nabla\xi_1|^q}, \end{aligned}$$

where $\varrho(p) := (\frac{R_1}{R_3})^n (s_f^1)^{-p}$ and $\varrho'(q) := (\frac{R_1}{R_3})^n (s_f^1)^q$. Sending $N \to +\infty$, by the definition of p_M and q_M we infer that $\varrho(p_M) \le 1$ and $\varrho'(q_M) \le 1$, which implies

$$\left(\frac{R_1}{R_3}\right)^{n/p_M} \le s_f^1 \le \left(\frac{R_1}{R_3}\right)^{-n/q_M}. \tag{A.3}$$

In two dimensions ($n = 2$), $p_M = \frac{2K}{K-1}$ and hence the above inequality implies a lower bound on the shielding factor

$$s_f^1 \ge \left(\frac{R_1}{R_3}\right)^{(K-1)/K}.$$

On the other hand, if a neutral shell of the first kind with s_f^1 could be constructed, by (A.3) we obtain the following nontrivial upper bound for the threshold exponents

$$\begin{cases} p_M \le \frac{n \log \frac{R_1}{R_3}}{\log s_f^1} & \text{if } s_f^1 < 1, \\ q_M \le \frac{n \log \frac{R_3}{R_1}}{\log s_f^1} & \text{if } s_f^1 > 1. \end{cases} \tag{A.4}$$

Acknowledgements This work is supported by the startup funds of the University of Houston. The author thanks Professor Graeme Milton for pointing out the connection between the shield factors and the threshold exponents, which motivates the calculations in the appendix, and the references [6, 21, 15, 8, 1].

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