

# Solutions to the Periodic Eshelby Inclusion Problem in Two Dimensions

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*Abstract:* We solve the homogeneous Eshelby inclusion problem on a finite unit cell with periodic boundary conditions. The main result is a representation formula of the strain field which is reminiscent of the familiar Green's representation formula. The formula is valid for any smooth inclusion and divergence-free eigenstress. More, it is shown that a Vigdergauz structure does not have the Eshelby uniformity property for symmetric non-dilatational eigenstress unless it degenerates to a laminate.

*Key Words:* Eshelby inclusion problem, periodic inclusions, complex variable method

## 1. INTRODUCTION

In the theories of composites, fracture mechanics and dislocations, a problem, called the *inhomogeneous* Eshelby inclusion problem, appears often. The governing equation for this problem is

$$\operatorname{div}[\mathbf{L}(\mathbf{x}, \Omega) \nabla \mathbf{v}(\mathbf{x}) + \mathbf{P}^* \chi_{\Omega}] = 0 \quad \text{on } \mathbb{R}^n, \quad (1.1)$$

where  $\chi_{\Omega}$  is the characteristic function of  $\Omega$ ,  $\mathbf{P}^* \in \mathbb{R}^{n \times n}$  is the eigenstress and the elasticity tensor

$$\mathbf{L}(\mathbf{x}, \Omega) = \begin{cases} \mathbf{L}_1 & \text{if } \mathbf{x} \in \Omega, \\ \mathbf{L}_0 & \text{otherwise.} \end{cases} \quad (1.2)$$

Equation (1.1) concerns the equilibrium of an infinite solid of two materials. The elasticity tensor of the inclusion (resp. the matrix) is  $\mathbf{L}_1$  (resp.  $\mathbf{L}_0$ ). The eigenstress  $\mathbf{P}^*$  on the inclusion can be regarded as an applied load. By specifying the tensors  $\mathbf{L}_1$ ,  $\mathbf{L}_0$ , the eigenstress  $\mathbf{P}^*$  and the domain  $\Omega$ , Equation (1.1) is used to model many physical problems. For instance, in the modeling of two-phase composites, Equation (1.1) is the governing equation of the

elastic field in the composite and hence determines the effective moduli of the composites. In the modeling of a crack in an infinite solid, we specify  $\mathbf{L}_1$  to be the zero tensor and  $\Omega$  a surface. In these modelings, usually the goal is to compute the elastic energy and to find how it depends on  $\mathbf{L}_1$ ,  $\mathbf{L}_0$ , the eigenstress  $\mathbf{P}^*$  and the domain  $\Omega$ .

An analytic solution of (1.1) is desirable but rare; an exception is when the domain  $\Omega$  is an ellipsoid. In this case we begin with a simpler problem, called the Eshelby *homogeneous* inclusion problem, for  $\mathbf{v} : \mathbb{R}^n \rightarrow \mathbb{R}^n$

$$\operatorname{div}[\mathbf{L}_0 \nabla \mathbf{v} + \mathbf{P}^0 \chi_\Omega] = 0 \quad \text{on } \mathbb{R}^n, \quad (1.3)$$

where  $\mathbf{P}^0 \in \mathbb{R}^{n \times n}$  is the equivalent eigenstress to be determined. For an ellipsoidal inclusion  $\Omega$ , as first noticed by Eshelby [1], the induced strain  $\nabla \mathbf{v}$  is uniform on  $\Omega$  and can be written as

$$\nabla \mathbf{v} = \mathbf{R} \mathbf{P}^0 \quad \text{on } \Omega, \quad \forall \mathbf{P}^0 \in \mathbb{R}^{n \times n}, \quad (1.4)$$

where the fourth-order symmetric tensor  $\mathbf{R} : \mathbb{R}^{n \times n} \rightarrow \mathbb{R}^{n \times n}$  is the *Eshelby tensor*. This remarkable property of ellipsoids, i.e., the induced strain  $\nabla \mathbf{v}$  is uniform on the ellipsoid for any applied eigenstress  $\mathbf{P}^0 \in \mathbb{R}^{n \times n}$ , is now referred to as the *Eshelby uniformity property*. Eshelby realized that under mild hypotheses the homogeneous problem (1.3) can be used to solve the inhomogeneous problem (1.1) provided that the induced field  $\nabla \mathbf{v}$  for the homogeneous problem is constant on  $\Omega$ . Roughly speaking, the uniformity of  $\nabla \mathbf{v}$  allows us to match on  $\partial\Omega$  the interfacial jump conditions of (1.1) with those of (1.3). At the same time, on the exterior of  $\Omega$  Equation (1.1) coincides with (1.3) whereas on the interior of  $\Omega$  Equation (1.1) is automatically satisfied since  $\nabla \mathbf{v}$  is constant on  $\Omega$ . After some algebraic calculations, we find that the solution of (1.3) also solves (1.1) if

$$\mathbf{P}^* + \mathbf{L}_1 \mathbf{R} \mathbf{P}^0 = \mathbf{P}^0 + \mathbf{L}_0 \mathbf{R} \mathbf{P}^0. \quad (1.5)$$

This solution technique is called the *equivalent inclusion method*, see [1, 2]. More, important quantities and relations can be obtained by solving an array of algebraic equations, though the original problem requires solving a system of partial differential equations. For example, by the divergence theorem the total elastic energy can be written as

$$\begin{aligned} \frac{1}{2} \int_{\mathbb{R}^n} \nabla \mathbf{v} \cdot \mathbf{L}(\mathbf{x}, \Omega) \nabla \mathbf{v} &= -\frac{1}{2} \int_{\mathbb{R}^n} \mathbf{P}^* \chi_\Omega \cdot \nabla \mathbf{v} \\ &= -\frac{1}{2} \mathbf{P}^* \cdot \int_{\Omega} \nabla \mathbf{v} = -\frac{1}{2} |\Omega| \mathbf{P}^* \cdot \mathbf{R} \mathbf{P}^0, \end{aligned} \quad (1.6)$$

which is explicitly known for any given ellipsoid  $\Omega$  and applied eigenstress  $\mathbf{P}^*$ . The applications of Eshelby's solution, in particular (1.5)–(1.6), include but are not limited to determining the shape and orientation of precipitates in alloys, the stress intensity factor around a crack, the effective properties of composite materials, and microstructural evolution in inhomogeneous solids with defects. For a survey of applications of the Eshelby solution, the reader is referred to [2].

Two or more ellipsoids do not enjoy the Eshelby uniformity property; we cannot use the equivalent inclusion method to solve the inhomogeneous problem when the domain  $\Omega$  consists of more than one ellipsoid. Therefore, in the applications of the Eshelby's solution, only one ellipsoid can be present and the results apply to an isolated inclusion or situations where inclusions are far apart. In other words, if interactions between inclusions become important, e.g. two cracks near to each other and composites in the non-dilute limit, it is insufficient to model them by a single inclusion and the Eshelby's solution.

Both the usefulness and the limitation of ellipsoidal inclusions motivate the following question: *are there any non-ellipsoidal shapes that have the Eshelby uniformity property?* Many authors have speculated on the possibility that other domains may have the Eshelby uniformity property [3], and successful attempts, in the author's opinion, include the two dimensional periodic structures constructed by Vigdergauz [4] and the E-inclusions constructed by Liu et al. [5]. To describe these examples and their properties, we introduce the counterparts of (1.1) and (1.3) in a periodic setting which we call *periodic Eshelby inclusion problems*. Below we set up these problems.

Let  $\{\mathbf{e}_1, \dots, \mathbf{e}_n\} \subset \mathbb{R}^n$  be linearly independent vectors,  $\mathcal{L} = \{\sum_{i=1}^n \nu_i \mathbf{e}_i : \nu_1, \dots, \nu_n \in \mathbb{Z}\}$  a Bravais lattice,  $Y = \{\sum_{i=1}^n y_i \mathbf{e}_i : 0 < y_1, \dots, y_n < 1\} \subset \mathbb{R}^n$  an open unit cell, and  $\Omega \subset Y$  a smooth (possibly multiply-connected) domain. Then the periodic *inhomogeneous* (resp. *homogeneous*) Eshelby inclusion problem for  $\mathbf{v} : Y \rightarrow \mathbb{R}^n$  is

$$\begin{cases} \operatorname{div}[\mathbf{L}(\mathbf{x}, \Omega) \nabla \mathbf{v} + \mathbf{P}^I \chi_\Omega] = 0 & \text{on } Y, \\ \text{periodic boundary conditions} & \text{on } \partial Y, \end{cases} \quad (1.7)$$

resp.

$$\begin{cases} \operatorname{div}[\mathbf{L}_0 \nabla \mathbf{v} + \mathbf{P}^H \chi_\Omega] = 0 & \text{on } Y, \\ \text{periodic boundary conditions} & \text{on } \partial Y, \end{cases} \quad (1.8)$$

where  $\mathbf{L}(\mathbf{x}, \Omega)$  is as in (1.2) and  $\mathbf{P}^I, \mathbf{P}^H : \Omega \rightarrow \mathbb{R}^{n \times n}$  are the applied (possibly nonuniform) eigenstress.

Our motivation is to solve the inhomogeneous problem (1.7), if applicable, using the solution of the homogeneous problem (1.8) and the equivalent inclusion method. To proceed, it is useful to introduce the following harmonic and biharmonic potentials  $h, b : Y \rightarrow \mathbb{R}$  determined by

$$\begin{cases} \Delta h = \theta - \chi_\Omega & \text{on } Y, \\ \text{periodic boundary conditions} & \text{on } \partial Y, \end{cases} \quad (1.9)$$

and

$$\begin{cases} \Delta^2 b = \theta - \chi_\Omega & \text{on } Y, \\ \text{periodic boundary conditions} & \text{on } \partial Y, \end{cases} \quad (1.10)$$

respectively, where  $\theta = |\Omega|/|Y|$  is the volume fraction of the inclusion  $\Omega$ . If  $Y = \mathbb{R}^n$ , we replace the periodic boundary conditions in (1.7)–(1.10) by appropriate boundary conditions at the infinity. If  $Y$  is bounded, we require the averages of  $\mathbf{v}$ ,  $b$  and  $h$  over the unit cell  $Y$  vanish. For simplicity, from now on we assume  $\mathbf{L}_0$  is an isotropic elasticity tensor with Lamé constants  $\mu$ ,  $\lambda$ . In the index form,

$$(\mathbf{L}_0)_{piqj} = \mu\delta_{pq}\delta_{ij} + \mu\delta_{pj}\delta_{iq} + \lambda\delta_{pi}\delta_{qj}.$$

The elasticity problem (1.8) is closely related with the potential problems (1.9)–(1.10). Following Eshelby [1, 6], we show that if the eigenstress  $\mathbf{P}^H = \mathbf{P}^0 \in \mathbb{R}^{n \times n}$  is uniform on  $\Omega$ , then the solution of (1.8) satisfies (see Section 2)

$$(\nabla \mathbf{v})_{pi} = \frac{1}{\mu} h_{,ki}(\mathbf{P}^0)_{pk} - \frac{(\mu + \lambda)}{\mu(2\mu + \lambda)} b_{,piqk}(\mathbf{P}^0)_{qk} \quad \text{on } Y. \quad (1.11)$$

Note that Equation (1.11) holds for any domains  $\Omega$  and any unit cells  $Y$ , including the case  $Y = \mathbb{R}^n$ .

If  $Y = \mathbb{R}^n$  and  $\Omega$  is an ellipsoidal inclusion, Eshelby [1, 6] showed that

$$\nabla \nabla h \quad \text{and} \quad \nabla \nabla \nabla \nabla b \quad \text{are uniform on } \Omega, \quad (1.12)$$

from which Equation (1.4) follows.

The unit cell  $Y$  is bounded in a periodic setting. Vigdergauz [4] first showed the existence of “equal-strength” periodic holes in two dimensions, which is subsequently called *Vigdergauz structures*. Liu et al. [5] observed that a Vigdergauz structure can be defined as a simply-connected bounded inclusion  $\Omega \subset Y$  such that the solution of (1.9) satisfies

$$\nabla \nabla h = -(1 - \theta)\mathbf{Q} \quad \text{on } \Omega, \quad (1.13)$$

where

$$\mathbf{Q} \in \mathbb{Q} := \{\mathbf{M} \in \mathbb{R}_{sym}^{n \times n} : \mathbf{M} \text{ is positive semi-definite with } \text{Tr}(\mathbf{M}) = 1\}. \quad (1.14)$$

From this characterization, the author et al (2008) generalized the concept of Vigdergauz structures which we called *periodic E-inclusions*. In two dimensions we use the term “Vigdergauz structure” interchangeably with “simply-connected periodic E-inclusion”. Further, we proved the existence of periodic E-inclusions, and in particular simply-connected periodic E-inclusions in any finite dimensional spaces. Formally we recall the following theorem.

**Theorem 1.** *Let  $Y$  be an open bounded unit cell associated with a lattice  $\mathcal{L}$ . For any matrix  $\mathbf{Q} \in \mathbb{Q}$  and volume fraction  $\theta \in (0, 1)$ , there exists a simply-connected open inclusion  $\Omega \subset Y$  such that the solution of (1.9) satisfies (1.13). We call such inclusion  $\Omega$  a simply-connected periodic E-inclusion with lattice  $\mathcal{L}$ , matrix  $\mathbf{Q}$  and volume fraction  $\theta$ .*

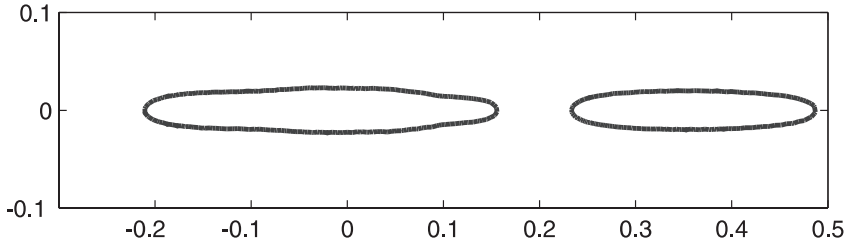


Figure 1. A two-component E-inclusion with a large aspect ratio.

Periodic E-inclusions are a natural generalization of ellipsoids in present context and ellipsoids can be regarded as the limits of simply-connected periodic E-inclusions as  $Y \rightarrow \mathbb{R}^n$  and  $\theta \rightarrow 0$ . We remark that a periodic E-inclusion can be multiply-connected in a unit cell and there are nonperiodic E-inclusions. Physically, a periodic E-inclusion can be characterized as a periodic array of bodies with the property that any uniform magnetization of the bodies induces a uniform magnetic field on these bodies (Liu et al 2007). Geometrically, the shape of a periodic E-inclusion is prescribed by the lattice  $\mathcal{L}$ , the matrix  $\mathbf{Q}$ , the volume fractions  $\theta$ , the number of components of the E-inclusion and their mutual distances and directions. In particular, the matrix  $\mathbf{Q}$  determines the aspect ratio of the E-inclusion, as the demagnetization matrix of an ellipsoid determines the aspect ratio of an ellipsoid. For example, a periodic E-inclusion degenerates to a simple laminate if all but one of the eigenvalues of the matrix  $\mathbf{Q}$  vanish. Figure 1 shows a two dimensional two-component E-inclusion  $\Omega$  with a large aspect ratio such that the solution of (1.9) satisfies (1.13) for  $\mathbf{Q} = \text{diag}[0.05, 0.95]$  and  $\theta = 0$  (i.e.,  $Y = \mathbb{R}^2$ ). The reader is invited to explore more examples of E-inclusions in [5, 7].

From Equations (1.11) and (1.13), we immediately see that a periodic E-inclusion  $\Omega$  enjoys the following partial Eshelby uniformity property. That is, for the periodic homogeneous problem (1.8), if the eigenstress

$$\mathbf{P}^H = \mathbf{I} := \text{the identity matrix} \quad \text{on } \Omega \quad (1.15)$$

is uniform and dilatational, then the elastic strain is also uniform on  $\Omega$  and can be written as

$$\nabla \mathbf{v} = \frac{1}{2\mu + \lambda} \nabla \nabla h = -\frac{1 - \theta}{2\mu + \lambda} \mathbf{Q} \quad \text{on } \Omega. \quad (1.16)$$

By adapting the equivalent inclusion method to the periodic setting, we can show that  $\mathbf{v} = \frac{1}{2\mu + \lambda} \nabla h$  solves the periodic inhomogeneous problem (1.7) if the eigenstress

$$\mathbf{P}^I = \mathbf{I} - \frac{1 - \theta}{2\mu + \lambda} (\mathbf{L}_0 - \mathbf{L}_1) \mathbf{Q} \quad \text{on } \Omega. \quad (1.17)$$

Further, the elastic energy averaged over the unit cell  $Y$  can be written as

$$\frac{1}{2|Y|} \int_Y \nabla \mathbf{v} \cdot \mathbf{L}(\mathbf{x}, \Omega) \nabla \mathbf{v} = \frac{\theta(1-\theta)}{2(2\mu + \lambda)} \left[ 1 - \frac{1-\theta}{2\mu + \lambda} \mathbf{Q} \cdot (\mathbf{L}_0 - \mathbf{L}_1) \mathbf{Q} \right], \quad (1.18)$$

see Liu et al. [5] for details. In analogy with the Eshelby's solution and the equivalent inclusion method for ellipsoidal inclusions, we conclude that periodic E-inclusions are indeed the proper generalizations of ellipsoids. As for ellipsoids, we can use the above solutions (see (1.16)–(1.18)) to study various physical problems, e.g., the interactions of periodic E-inclusions with a far field. The advantage of periodic E-inclusions is that their volume fractions can be any number between zero and one and the interactions between individual inclusions are accounted precisely. For applications of periodic E-inclusions in composite materials, the reader is referred to [5].

The analogy between ellipsoids and periodic E-inclusions is, however, not perfect yet in the sense that it is not known if the induced strain of a non-dilatational eigenstress is uniform on  $\Omega$  for the homogeneous problem (1.8).

We will give an answer to the above question in two dimensions. The complex representation of the solution, originally introduced by Kolosov [8], has been found useful in elasticity, see [9–11]. Our solution strategy for (1.10) and (1.8) is to use complex functions to progressively construct the solutions of the related harmonic and biharmonic equations. It turns out that this strategy can give us solutions in terms of Cauchy-type integrals as long as the eigenstress  $\mathbf{P}^H : \Omega \rightarrow \mathbb{R}^{2 \times 2}$  is divergence-free on  $\Omega$  (see (1.8))

$$\operatorname{div} \mathbf{P}^H = 0 \quad \text{on } \Omega. \quad (1.19)$$

Physically the above divergence-free condition implies that there is no applied body but surface load. When specialized to situations where the inclusion  $\Omega$  is a Vigdergauz structure and the eigenstress  $\mathbf{P}^H = \mathbf{P}^0 \in \mathbb{R}_{sym}^{2 \times 2}$  is uniform, we find that the induced strain  $\nabla \mathbf{v}$  is not uniform unless  $\mathbf{P}^0$  is dilatational or  $\Omega$  is a laminate. This means that the equivalent inclusion method is no longer applicable to solve the inhomogeneous problem (1.7) if the eigenstress  $\mathbf{P}^H$  does not satisfy (1.17) within a multiplicative constant.

This somewhat “disappointing” fact of Vigdergauz structures can be understood intuitively as follows. Comparing (1.13) with (1.12), we see that to have the Eshelby uniformity property a Vigdergauz structure  $\Omega$  necessarily satisfies that the fourth gradient of the solution of (1.10), i.e.  $\nabla \nabla \nabla \nabla b$ , is uniform on the periodic E-inclusion. This happens to be true for an ellipsoid but not so for a Vigdergauz structure. After all,  $\nabla \nabla \nabla \nabla b$  being uniform on  $\Omega$  is a much stronger condition than  $\nabla \nabla h$  being uniform on  $\Omega$ .

Associated with this “disappointing” fact, we raise the following question. For the homogeneous problem (1.8) and a given non-dilatational uniform eigenstress  $\mathbf{P}^H = \mathbf{P}^0 \in \mathbb{R}_{sym}^{2 \times 2}$ , are there any non-laminate shapes  $\Omega$  such that the induced strain  $\nabla \mathbf{v}$  is uniform on  $\Omega$ ? We conjecture the answer is affirmative. In this paper we do not pursue this issue.

Besides giving an answer to the Eshelby uniformity property of Vigdergauz structures, the solutions in this paper can be used independently in other ways. In applications to magnetism, we give a simple formula of the induced magnetic field under the hypothesis that there is no internal pole inside the magnetized body, see Theorem 2, (3.23). In applications to elasticity, Equations (4.1)–(4.3) express the induced strain field of (1.8) in terms of Cauchy-type integrals. Such explicit formulas are useful for the effective solutions of the elasticity problems that arise from the modelings of crack, dislocations and phase transfor-

mations [2, 12]. From a mathematical viewpoint, the formulas (e.g. (3.33) and (3.86)) are reminiscent of the Green's representation formulas. We remark that a naive summation of the Green's functions over a lattice does not converge. That is why we need to consider the Weierstrass functions and elaborate on various special functions.

The paper is organized as follows. In Section 2 we use the Fourier method to establish the relations between the periodic Eshelby inclusion problem (1.8) and two simpler problems concerning harmonic and biharmonic operators (see (2.1) and (2.2)). From Section 3 we restrict our discussions to two dimensions. After reviewing some relevant facts from complex analysis in Section 3.1, we construct the solutions of (2.1) and (2.2) in Sections 3.2 and 3.3, respectively. In particular we establish a necessary and sufficient condition for a simply-connected inclusion to be a Vigdergauz structure in Section 3.2.1. Interior solutions of (2.1) and (2.2) are exceptionally simple for a Vigdergauz structure, which enable us to show that the induced strain is not uniform if the uniform eigenstress is symmetric but not dilatational and the inclusion is not a laminate. Finally we summarize our results in Section 4.

## 2. REPRESENTATION OF THE SOLUTIONS BY FOURIER SERIES

If  $Y = \mathbb{R}^n$  and the eigenstress is uniform on  $\Omega$ , as shown by Eshelby [1] solutions of (1.8) are related with those of (1.9) and (1.10) by (1.11). Eshelby's arguments are based on the known Green's functions of (1.8), (1.9) and (1.10). In a periodic setting, their relation can be conveniently demonstrated by the Fourier method. More, relation of this sort can be generalized to situations where Equation (1.8) has a nonuniform eigenstress  $\mathbf{P}^H$ .

To make progress, we consider the *magnetostatic problem* for  $u_1 : Y \rightarrow \mathbb{R}$

$$\begin{cases} \operatorname{div}(\nabla u_1 + \mathbf{m}\chi_\Omega) = 0 & \text{on } Y, \\ \text{periodic boundary conditions} & \text{on } \partial Y, \end{cases} \quad (2.1)$$

and an associated Poisson problem for  $u_2 : Y \rightarrow \mathbb{R}$

$$\begin{cases} \Delta u_2 = u_1 & \text{on } Y, \\ \text{periodic boundary conditions} & \text{on } \partial Y, \end{cases} \quad (2.2)$$

where  $\mathbf{m} : \Omega \rightarrow \mathbb{R}^n$  denotes the magnetization on  $\Omega$ . Note that if  $Y$  is bounded, we require the solutions of (2.1) and (2.2) have zero averages on  $Y$

$$\oint_Y u_1 \mathbf{d}\mathbf{x} = 0 \quad \text{and} \quad \oint_Y u_2 \mathbf{d}\mathbf{x} = 0, \quad (2.3)$$

where  $\oint_V \cdot \mathbf{d}\mathbf{x} = \frac{1}{|V|} \int_V \cdot \mathbf{d}\mathbf{x}$  denotes the average of the integrand over the set  $V$ . This requirement eliminates the arbitrary additive constants associated with the solutions of (2.1) and (2.2). If the magnetization  $\mathbf{m} = \mathbf{m}^0 \in \mathbb{R}^n$  is uniform on  $\Omega$ , to emphasize the linear

dependence of  $u_1, u_2$  on  $\mathbf{m}^0$ , we sometimes write the solutions of (2.1) and (2.2) as  $u_1(\mathbf{x}, \mathbf{m}^0)$  and  $u_2(\mathbf{x}, \mathbf{m}^0)$ , respectively.

Liu et al. [7] showed that solutions of (1.9) and (2.1) are related by

$$(\nabla \nabla h(\mathbf{x}))\mathbf{m}^0 = \nabla u_1(\mathbf{x}, \mathbf{m}^0) \quad \forall \mathbf{m}^0 \in \mathbb{R}^n, \mathbf{x} \in Y. \quad (2.4)$$

Below we will show that the solutions of (1.10) and (2.2) satisfy

$$(\nabla \nabla \nabla \nabla b(\mathbf{x}))\mathbf{m}^0 = \nabla \nabla \nabla u_2(\mathbf{x}, \mathbf{m}^0) \quad \forall \mathbf{m}^0 \in \mathbb{R}^n, \mathbf{x} \in Y. \quad (2.5)$$

More generally, we can establish this type of relations between the solutions of (1.8) and those of (2.1) and (2.2) for nonuniform eigenstress  $\mathbf{P}^H$ . To see this, let

$$\hat{\mathbf{P}}_{\mathbf{k}} = \int_Y \mathbf{P}^H(\mathbf{x}) \chi_\Omega \exp(-i\mathbf{k} \cdot \mathbf{x}) d\mathbf{x} \quad (2.6)$$

be the Fourier coefficients of the eigenstress. The Fourier inversion theorem implies that

$$\mathbf{P}^H(\mathbf{x}) = \theta \mathbf{P}^0 + \sum_{\mathbf{k} \in \mathcal{K} \setminus \{0\}} \hat{\mathbf{P}}_{\mathbf{k}} \exp(i\mathbf{k} \cdot \mathbf{x}) := \begin{bmatrix} \mathbf{m}^{(1)}(\mathbf{x}) \\ \mathbf{m}^{(2)}(\mathbf{x}) \\ \vdots \\ \mathbf{m}^{(n)}(\mathbf{x}) \end{bmatrix}, \quad (2.7)$$

where  $\mathbf{P}^0 = \int_\Omega \mathbf{P}^H(\mathbf{x}) d\mathbf{x}$ ,  $\theta = |\Omega|/|Y|$  is the volume fraction of the inclusion  $\Omega$ ,  $\mathcal{K}$  is the collection of all reciprocal vectors  $\mathbf{k}$  that satisfy  $\mathbf{k} \cdot \mathbf{r} = 2\nu\pi$  for some  $\mathbf{r} \in \mathcal{L}$  and some integer  $\nu$ , and  $\mathbf{m}^{(p)} : \Omega \rightarrow \mathbb{R}^n$  ( $p = 1, \dots, n$ ) are the row vectors of  $\mathbf{P}^H(\mathbf{x})$ . From (1.8), direct calculations reveal that (see [2])

$$\nabla \mathbf{v}(\mathbf{x}) = - \sum_{\mathbf{k} \in \mathcal{K} \setminus \{0\}} \left[ \frac{1}{\mu |\mathbf{k}|^2} (\hat{\mathbf{P}}_{\mathbf{k}} \mathbf{k}) \otimes \mathbf{k} - \frac{(\mu + \lambda) \mathbf{k} \cdot (\hat{\mathbf{P}}_{\mathbf{k}} \mathbf{k})}{\mu(2\mu + \lambda) |\mathbf{k}|^4} \mathbf{k} \otimes \mathbf{k} \right] \exp(i\mathbf{k} \cdot \mathbf{x}). \quad (2.8)$$

Let  $u_1^{(p)}$  be the solution of (2.1) with the magnetization given by  $\mathbf{m}^{(p)}$ , and  $u_2^{(p)}$  be the solution of (2.2) with  $u_1$  replaced by  $u_1^{(p)}$  ( $p = 1, \dots, n$ ). Then the gradients of  $u_1^{(p)}$  and the third-order gradients of  $u_2^{(p)}$  can be represented as

$$\nabla u_1^{(p)}(\mathbf{x}) = - \sum_{\mathbf{k} \in \mathcal{K} \setminus \{0\}} \frac{1}{|\mathbf{k}|^2} \mathbf{k} (\mathbf{k} \cdot \hat{\mathbf{m}}_{\mathbf{k}}^{(p)}) \exp(i\mathbf{k} \cdot \mathbf{x}) \quad (2.9)$$

and

$$\nabla \nabla \nabla u_2^{(p)}(\mathbf{x}) = - \sum_{\mathbf{k} \in \mathcal{K} \setminus \{0\}} \frac{\mathbf{k} \cdot \hat{\mathbf{m}}_{\mathbf{k}}^{(p)}}{|\mathbf{k}|^4} \mathbf{k} \otimes \mathbf{k} \otimes \mathbf{k} \exp(i\mathbf{k} \cdot \mathbf{x}), \quad (2.10)$$



respectively, where  $\hat{\mathbf{m}}_{\mathbf{k}}^{(p)} = \int_Y \mathbf{m}^{(p)}(\mathbf{x}) \exp(-i\mathbf{k} \cdot \mathbf{x}) d\mathbf{x}$  are the Fourier coefficients of  $\mathbf{m}^{(p)}$ . Note that from the definition (2.7),  $\hat{\mathbf{m}}_{\mathbf{k}}^{(p)}$  is the  $p$ th-row vector of the matrix  $\hat{\mathbf{P}}_{\mathbf{k}}$ . Comparing (2.8) with (2.9) and (2.10), we see that for every  $p, i = 1, \dots, n$ ,

$$[\nabla \mathbf{v}(\mathbf{x})]_{pi} = \frac{1}{\mu} [\nabla u_1^{(p)}(\mathbf{x})]_i - \frac{(\mu + \lambda)}{\mu(2\mu + \lambda)} \sum_{q=1}^n [\nabla \nabla \nabla u_2^{(q)}(\mathbf{x})]_{qpi}. \quad (2.11)$$

Note that the above equation is valid for any eigenstress  $\mathbf{P}^H : \Omega \rightarrow \mathbb{R}^{n \times n}$  and any inclusion  $\Omega$ .

When the eigenstress  $\mathbf{P}^H = \mathbf{P}^0$  is uniform on  $\Omega$ , we have

$$\hat{\mathbf{P}}_{\mathbf{k}} = \mathbf{P}^0 \hat{\chi}_{\Omega}(\mathbf{k}), \quad \hat{\chi}_{\Omega}(\mathbf{k}) = \int_Y \chi_{\Omega} \exp(-i\mathbf{k} \cdot \mathbf{x}) d\mathbf{x}, \quad (2.12)$$

and therefore,

$$[\hat{\mathbf{m}}_{\mathbf{k}}^{(p)}]_i = (\mathbf{P}^0)_{pi} \hat{\chi}_{\Omega}(\mathbf{k}) \quad \forall p, i = 1, \dots, n. \quad (2.13)$$

Further, the solutions of (1.9) and (1.10) can be represented as

$$\nabla \nabla h(\mathbf{x}) = - \sum_{\mathbf{k} \in \mathcal{K} \setminus \{0\}} \frac{\mathbf{k} \otimes \mathbf{k}}{|\mathbf{k}|^2} \hat{\chi}_{\Omega}(\mathbf{k}) \exp(i\mathbf{k} \cdot \mathbf{x}) \quad \forall \mathbf{x} \in Y \quad (2.14)$$

and

$$\nabla \nabla \nabla \nabla b(\mathbf{x}) = - \sum_{\mathbf{k} \in \mathcal{K} \setminus \{0\}} \frac{\mathbf{k} \otimes \mathbf{k} \otimes \mathbf{k} \otimes \mathbf{k}}{|\mathbf{k}|^4} \hat{\chi}_{\Omega}(\mathbf{k}) \exp(i\mathbf{k} \cdot \mathbf{x}) \quad \forall \mathbf{x} \in Y, \quad (2.15)$$

respectively. Substituting (2.13) into (2.9) and (2.10) and comparing the results with (2.14) and (2.15), we obtain (2.4) and (2.5). By (2.11), we obtain (1.11), which generalizes Eshelby's result ([1], Equation (2.8)) to a periodic setting.

From Equation (2.11), as far as the strain  $\nabla \mathbf{v}$  is concerned, it suffices to solve the magnetostatic problem (2.1) and the Poisson problem (2.2). Our focus is henceforth on (2.1) and (2.2). From now on we will restrict ourselves to two-dimensional space ( $n = 2$ ).

### 3. REPRESENTATION OF THE SOLUTIONS BY ANALYTIC FUNCTIONS

#### 3.1. Cauchy Integrals, Plemelj Formula and Elliptic Functions

We will construct solutions of (2.1) and (2.2) in terms of analytic functions. The main ingredients of our construction are the *Cauchy integral* and the *Plemelj formula*. Below we recall a few relevant facts from complex analysis. To follow the notations the reader may find Figure 2 useful.

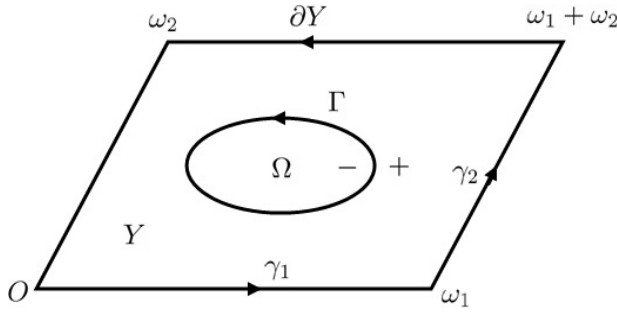


Figure 2. A unit cell with a simply-connected inclusion.

In two dimensions, a complex number is denoted by  $z = x_1 + ix_2 \in \mathbb{C}$ ;  $x_1, x_2 \in \mathbb{R}$ . We identify the complex plane  $\mathbb{C}$  with  $\mathbb{R}^2$  in this obvious manner. The conjugate of  $z$  is  $\bar{z} = x_1 - ix_2$ . These definitions imply the following transformation:

$$\begin{bmatrix} z \\ \bar{z} \end{bmatrix} = \mathbf{T} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \mathbf{T} = \begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix}, \quad (3.1)$$

which is clearly linear and nonsingular.

Let  $\Omega \subset \mathbb{C}$  be a simply-connected open domain and  $u, v : \Omega \rightarrow \mathbb{R}$  be real-valued differentiable functions. The complex function  $\zeta(z) = u(x_1, x_2) + iv(x_1, x_2)$  is analytic on  $\Omega$  if and only if

$$\frac{\partial u}{\partial x_1} = \frac{\partial v}{\partial x_2}, \quad \frac{\partial u}{\partial x_2} = -\frac{\partial v}{\partial x_1} \quad \text{on } \Omega, \quad (3.2)$$

see [13] Vol. I, p. 110). The above equations are the *Cauchy–Riemann equations*. The chain rule implies

$$\frac{\partial}{\partial z} = \frac{1}{2} \left( \frac{\partial}{\partial x_1} - i \frac{\partial}{\partial x_2} \right) \quad \text{and} \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left( \frac{\partial}{\partial x_1} + i \frac{\partial}{\partial x_2} \right). \quad (3.3)$$

Thus, the harmonic operator can be identified as

$$\Delta = \nabla^2 = 4 \frac{\partial^2}{\partial z \partial \bar{z}}. \quad (3.4)$$

Let  $\Gamma = \partial\Omega$  be the simple counterclockwise contour around the simply-connected open inclusion  $\Omega$ , see Figure 2. Assume that  $\Gamma$  is smooth and admits the parametrization

$$\begin{cases} x_1 = x_1(s) \\ x_2 = x_2(s) \end{cases} \quad 0 \leq s < L, \quad (3.5)$$

where  $\sqrt{x_1'^2(s) + x_2'^2(s)} = 1$  and  $L$  is the length of  $\Gamma$ . For a smooth function  $f : \Gamma \rightarrow \mathbb{C}$ , the derivative of  $f$  along  $\Gamma$  is defined as (see [9], Equation (69.6))

$$f^{(1)}(t) := \lim_{t_1 \rightarrow t, t_1 \in \Gamma} \frac{f(t_1) - f(t)}{t_1 - t} = e^{-i\gamma(s)} \frac{df(t(s))}{ds} \quad \forall t \in \Gamma, \quad (3.6)$$

where  $\gamma(s)$  is the angle between the positive tangent to  $\Gamma$  and the positive real axis, and hence

$$\cos \gamma(s) = \frac{dx_1(s)}{ds}, \quad \sin \gamma(s) = \frac{dx_2(s)}{ds}. \quad (3.7)$$

Note that this concept of differentiation should not be confused with the operator  $\frac{\partial}{\partial z}$  defined in (3.3). For instance,  $\frac{\partial \bar{z}}{\partial z} = 0$  but by (3.6),

$$\bar{t}^{(1)} = e^{-i\gamma} (x_1'(s) - ix_2'(s)) = e^{-i2\gamma} \quad \forall t \in \Gamma. \quad (3.8)$$

Progressively, higher order derivatives of  $f$  along  $\Gamma$  are defined as

$$f^{(n)}(t) = \lim_{t_1 \rightarrow t, t_1 \in \Gamma} \frac{f^{(n-1)}(t_1) - f^{(n-1)}(t)}{t_1 - t}.$$

A smooth function  $f : \Gamma \rightarrow \mathbb{C}$  can be alternately regarded as a smooth periodic function  $s \mapsto f(x_1(s) + ix_2(s))$  on  $\mathbb{R}$  if we extend the domain of the parametrization (3.5) by  $x_i(s + \nu L) = x_i(s)$  ( $i = 1, 2$ ) for  $s \in [0, L)$  and  $\nu \in \mathbb{Z}$ . Note that this mapping is one-to-one and we do not distinguish them in notation. Further, let  $g(z)$  be an analytic function on a neighborhood of  $\Gamma$  and  $G_n(z)$  be its *single-valued*  $n$ th-order antiderivative such that  $d^n G_n(z)/dz^n = g(z)$ . Integrating by parts, we have

$$\int_{\Gamma} f(t)g(t)dt = (-1)^n \int_{\Gamma} f^{(n)}(t)G_n(t)dt. \quad (3.9)$$

This formula will be repeatedly used in the subsequent discussions.

The Cauchy integral

$$F(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(t)}{t - z} dt \quad (3.10)$$

is probably the most important tool in complex analysis. For a smooth  $f : \Gamma \rightarrow \mathbb{C}$ , it is clear that  $F(z)$  is analytic in each component of  $\mathbb{C} \setminus \Gamma$ . When  $z$  coincides with a point  $t_0 \in \Gamma$ , the integral (3.10) has no meaning in ordinary (Riemann) sense. However, the Cauchy principle value could be well defined as

$$\frac{1}{2\pi i} \text{PV} \int_{\Gamma} \frac{f(t)}{t - t_0} dt = \lim_{t_1, t_2 \rightarrow t_0} \frac{1}{2\pi i} \int_{\Gamma - \gamma(t_1, t_2)} \frac{f(t)}{t - t_0} dt \quad \forall t_0 \in \Gamma, \quad (3.11)$$

where the arc  $\gamma(t_1, t_2) \subset \Gamma$  contains  $t_0$  and the limit is taken by assuming  $|t_1 - t_0| = |t_2 - t_0|$ . Further, the outer boundary value (resp. the inner boundary value), denoted by

$$F(t^+) = \lim_{z \rightarrow t, z \in \mathbb{C} \setminus \bar{\Omega}} F(z) \quad \left( \text{resp. } F(t^-) = \lim_{z \rightarrow t, z \in \Omega} F(z) \right), \quad (3.12)$$

is well-defined on  $\Gamma$  if the above limit exists for every  $t \in \Gamma$ . It is summarized below the conditions for the existence of the Cauchy principle value and its relations with the boundary values  $F(t^+)$  and  $F(t^-)$ , see the textbooks of Muskhelishvili [9] (p. 263) and Markushevich [13] (Vol. I, p. 299) for proofs.

1. Existence of the Cauchy principle value. If  $f : \Gamma \rightarrow \mathbb{C}$  satisfies

$$|f(t_2) - f(t_1)| \leq C |t_2 - t_1|^\mu \quad \forall t_1, t_2 \in \Gamma \quad (3.13)$$

for some constant  $C > 0$ ,  $\mu \in (0, 1]$ , then the Cauchy principle value (3.11) exists for every  $t \in \Gamma$ , and so the outer (resp. inner) boundary value  $F(t^+)$  (resp.  $F(t^-)$ ) exists on  $\Gamma$ .

2. Plemelj formulas. If  $f : \Gamma \rightarrow \mathbb{C}$  is smooth on  $\Gamma$ , then for every  $t \in \Gamma$ ,

$$\begin{aligned} \lim_{z \rightarrow t, z \in \mathbb{C} \setminus \Omega} \frac{d^n F(z)}{dz^n} + \lim_{z \rightarrow t, z \in \Omega} \frac{d^n F(z)}{dz^n} &= \frac{1}{\pi i} \text{PV} \int_{\Gamma} \frac{f^{(n)}(t_1)}{t_1 - t} dt_1, \\ \left[ \left[ \frac{d^n F(z)}{dz^n} \right] \right]_{z=t} &= \lim_{z \rightarrow t, z \in \mathbb{C} \setminus \Omega} \frac{d^n F(z)}{dz^n} - \lim_{z \rightarrow t, z \in \Omega} \frac{d^n F(z)}{dz^n} = -f^{(n)}(t), \end{aligned} \quad (3.14)$$

where  $\llbracket \cdot \rrbracket$  denotes the jump across the interface  $\Gamma$  from the outside to the inside of the inclusion, see Figure 2.

In complex analysis, a meromorphic function is an elliptic function if it is doubly periodic. Let  $\{\omega_1, \omega_2\} \in \mathbb{C}$  with  $\text{Im}[\omega_1/\omega_2] \neq 0$  be the periods and  $\mathcal{L} = \{v_1\omega_1 + v_2\omega_2 : v_1, v_2 \in \mathbb{Z}\}$  be the lattice. The simplest elliptic function is the Weierstrass  $\wp$ -function (see [14], Chapter 7 and [13], Vol. III, Chapter 4)

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \in \mathcal{L} \setminus \{0\}} \left( \frac{1}{(z - \omega)^2} - \frac{1}{\omega^2} \right). \quad (3.15)$$

The Weierstrass  $\zeta$ -function is defined as

$$\zeta(z) = \frac{1}{z} + \sum_{\omega \in \mathcal{L} \setminus \{0\}} \left( \frac{1}{z + \omega} - \frac{1}{\omega} + \frac{z}{\omega^2} \right). \quad (3.16)$$

Note that  $\zeta(z)$  satisfies  $d\zeta(z)/dz = -\wp(z)$  and

$$\zeta(z + \omega_1) = \zeta(z) + \eta_1, \quad \zeta(z + \omega_2) = \zeta(z) + \eta_2, \quad \forall z \notin \mathcal{L}, \quad (3.17)$$

where  $\eta_1$  and  $\eta_2$  are constants satisfying the *Legendre's relation*:

$$\eta_1\omega_2 - \eta_2\omega_1 = 2\pi i. \quad (3.18)$$

Further, it is convenient to normalize the periods such that the area of  $Y = \{x_1\omega_1 + x_2\omega_2 : 0 < x_1, x_2 < 1\}$  is one. That is,

$$2i = 2i|Y| = \int_{\partial Y} \bar{t} dt = -\omega_1\bar{\omega}_2 + \omega_2\bar{\omega}_1. \quad (3.19)$$

Before proceeding to details, we introduce some more notation. We denote by  $\bar{\Omega}$  the closure of the open inclusion  $\Omega$ . For sets such as  $\Gamma$ ,  $\Omega$ ,  $\bar{\Omega}$  and  $Y$ , we use the subscript  $\omega$  to denote the translated sets and the subscript  $per$  to denote the periodic extension of them. For instance,  $\Gamma_\omega := \{z + \omega : z \in \Gamma\}$  and  $\Gamma_{per} := \{z + \omega : z \in \Gamma, \omega \in \mathcal{L}\}$ . The meanings of  $\Omega_\omega$ ,  $\Omega_{per}$ ,  $\bar{\Omega}_{per}$ ,  $Y_\omega$ , etc., are likewise.

### 3.2. Solutions to Problem (2.1)

In this section we give solutions to (2.1) in terms of Cauchy-type integrals. We require the magnetization  $\mathbf{m} : \Omega \rightarrow \mathbb{R}^2$  is smooth and satisfies

$$\operatorname{div} \mathbf{m} = 0 \quad \text{on } \Omega. \quad (3.20)$$

Note that the above condition is satisfied for every row vector in  $\mathbf{P}^H$  (i.e.,  $\mathbf{m}^{(p)}$  in (2.7)) if the eigenstress  $\mathbf{P}^H$  satisfies (1.19). Equation (3.20) has two consequences and both of them are critical for subsequent constructions. The first one is that the solution  $u_1$  of (2.1) is harmonic on  $\Omega$ ; the second is that the function  $f : \Gamma \rightarrow \mathbb{R}$

$$\begin{aligned} f(t(s)) &= f(x_1(s) + ix_2(s)) \\ &= - \int_0^s \mathbf{m}(x_1(s_1), x_2(s_1)) \cdot \mathbf{n}(x_1(s_1), x_2(s_1)) ds_1 \quad \forall s \in [0, L] \end{aligned} \quad (3.21)$$

is smooth on  $\Gamma$ . To see this, by the divergence theorem we have

$$f(t(L)) = - \int_{\Omega} \operatorname{div} \mathbf{m} \, dx = 0.$$

Since  $f(t(0)) = 0$ , we see that  $f$  is continuous on  $\Gamma$  and smooth on  $\Gamma \setminus \{t(s) : s = 0\}$ . Direct differentiations show that any order derivatives of  $f$  are also continuous at  $s = 0$  and hence  $f$  is smooth on  $\Gamma$ .

The following theorem will be useful.

**Theorem 2.** Let  $f : \Gamma \rightarrow \mathbb{R}$  be a smooth real-valued function, and

$$\xi_{10}(z) = \frac{1}{2\pi i} \int_{\Gamma} i f(t) \zeta(t-z) dt, \quad (3.22)$$

where  $\zeta(z)$  is the Weierstrass  $\zeta$ -function, see (3.16). Then

$$\tilde{u}_1(z, \bar{z}) = \begin{cases} \operatorname{Re}[\xi_{10}(z)] = \frac{1}{2}[\xi_{10}(z) + \overline{\xi_{10}(z)}] & \text{if } z \in \mathbb{C} \setminus \Gamma_{per}, \\ \lim_{z_1 \rightarrow z, z_1 \in \mathbb{C} \setminus \Gamma_{per}} \operatorname{Re}[\xi_{10}(z_1)] & \text{if } z \in \Gamma_{per} \end{cases} \quad (3.23)$$

is continuous on  $\mathbb{C}$  and satisfies

$$\begin{cases} \Delta \tilde{u}_1 = 0 & \text{on } \mathbb{C} \setminus \Gamma_{per}, \\ \llbracket \nabla \tilde{u}_1 \rrbracket = -\frac{d f(t(s))}{ds} \mathbf{n} & \text{on } \Gamma, \\ \nabla \tilde{u}_1(z + \omega, \bar{z} + \bar{\omega}) = \nabla \tilde{u}_1(z, \bar{z}) \quad \forall \omega \in \mathcal{L} \quad \text{and } z \in \mathbb{C}, \end{cases} \quad (3.24)$$

where  $\mathbf{n} = [\sin \gamma, -\cos \gamma]$  is the outward normal on  $\Gamma$ .

**Proof.** Clearly  $\xi_{10}(z)$  is analytic on each component of  $\mathbb{C} \setminus \Gamma_{per}$  and hence (see (3.2))

$$\Delta \tilde{u}_1 = 0 \text{ on } \mathbb{C} \setminus \Gamma_{per}.$$

Note that  $\frac{1}{t-z}$  is the only term in  $\zeta(t-z)$  that can give rise to discontinuity of  $\xi_{10}(z)$ . From Equations (3.6)–(3.7) and the Plemelj formulas (3.14), we have that for every  $t \in \Gamma$ ,

$$\begin{aligned} \llbracket \xi_{10}(t) \rrbracket &= -i f(t) \quad \text{and} \\ \left[ \left[ \frac{d\xi_{10}(z)}{dz} \right] \right]_{z=t} &= -i f^{(1)}(t) = (-i \cos \gamma - \sin \gamma) \frac{d f(t(s))}{ds}, \end{aligned} \quad (3.25)$$

where in the second of the above equation we have used

$$\frac{d\xi_{10}(z)}{dz} = \frac{1}{2\pi i} \int_{\Gamma} i f(t) \frac{d}{dz} \zeta(t-z) dt = \frac{1}{2\pi i} \int_{\Gamma} i f^{(1)}(t) \zeta(t-z) dt.$$

Further, from the properties of  $\wp(z)$  and  $\zeta(z)$  (see (3.15) and (3.17)), for any  $z \in \mathbb{C} \setminus \Gamma_{per}$  we have

$$\frac{d\xi_{10}(z+\omega)}{dz} = \frac{1}{2\pi} \int_{\Gamma} f(t) \wp(t-z-\omega) dt = \frac{d\xi_{10}(z)}{dz} \quad \forall \omega \in \mathcal{L}. \quad (3.26)$$

Note that (see (3.2)–(3.3))

$$\frac{\partial \tilde{u}_1(z, \bar{z})}{\partial x_1} - i \frac{\partial \tilde{u}_1(z, \bar{z})}{\partial x_2} = \frac{d\check{\zeta}_{10}(z)}{dz}. \quad (3.27)$$

From (3.26) we arrive at the third equation in (3.24). The first in (3.25) implies that  $\tilde{u}$  is continuous on  $\mathbb{C}$ , whereas the second in (3.25) and (3.27) imply the second in (3.24). The proof is completed.  $\square$

Using Theorem 2, we can write the solutions of (2.1) in terms of Cauchy-type integrals. To this end, we reformulate (2.1) as

$$\begin{cases} \Delta u_1 = 0 & \text{on } Y \setminus \Gamma, \\ \llbracket \nabla u_1 \rrbracket = (\mathbf{m} \cdot \mathbf{n}) \mathbf{n} & \text{on } \Gamma, \\ \text{periodic boundary conditions} & \text{on } \partial Y. \end{cases} \quad (3.28)$$

Define  $f : \Gamma \rightarrow \mathbb{R}$  by (3.21). By Theorem 2, we see that  $\tilde{u}_1$  defined by (3.23) satisfies all the equations but the periodic boundary conditions in (3.28). In another word, we have

$$\Delta(u_1 - \tilde{u}_1) = 0 \quad \text{on } \mathbb{C},$$

where  $u_1$  is the solution of (2.1) or (3.28). From the definition of  $\tilde{u}_1$ , (3.26) and (3.27), we see that  $|\tilde{u}_1(z, \bar{z})| < C_1|z| + C_0$  for constants  $C_1, C_0 > 0$ , whereas  $u_1 - \tilde{u}_1$  being real harmonic on  $\mathbb{C}$  can be expressed as the real part of a power series of  $z$  ([14], Chapter 5). Therefore, there exist constants  $z_{11}, z_{10} \in \mathbb{C}$  such that

$$u_1(z, \bar{z}) = \operatorname{Re}[\check{\zeta}_{10}(z) + z_{11}z + z_{10}] = \frac{1}{2}[\check{\zeta}_{10}(z) + z_{11}z + z_{10} + \overline{\check{\zeta}_{10}(z) + z_{11}z + z_{10}}]. \quad (3.29)$$

To find the constant  $z_{11}$ , we notice that, by (3.17),

$$\begin{aligned} \check{\zeta}_{10}(z + \omega_1) &= \frac{1}{2\pi} \int_{\Gamma} f(t) \check{\zeta}(t - z - \omega_1) dt = \check{\zeta}_{10}(z) - \eta_1 M, \\ \check{\zeta}_{10}(z + \omega_2) &= \frac{1}{2\pi} \int_{\Gamma} f(t) \check{\zeta}(t - z - \omega_2) dt = \check{\zeta}_{10}(z) - \eta_2 M, \end{aligned} \quad (3.30)$$

where

$$M = \frac{1}{2\pi} \int_{\Gamma} f(t) dt. \quad (3.31)$$

From the periodicity of  $u_1$ , we have  $u_1(z + \omega, \bar{z} + \bar{\omega}) = u_1(z, \bar{z})$  for all  $\omega \in \mathcal{L}$ . In particular, choosing  $\omega = \omega_1, \omega_2$ , we obtain

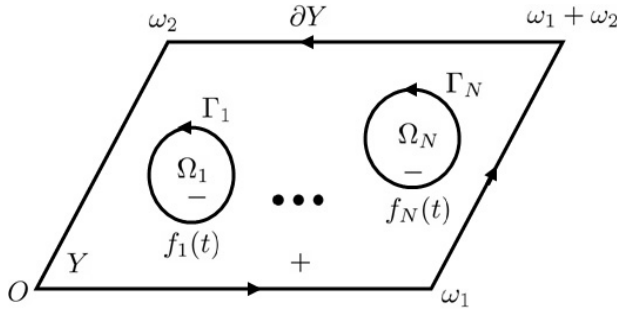


Figure 3. A unit cell with  $N$  mutually-disjoint simply-connected inclusions.

$$\begin{cases} \operatorname{Re}[-\eta_1 M + z_{11}\omega_1] = 0 \\ \operatorname{Re}[-\eta_2 M + z_{11}\omega_2] = 0 \end{cases} \implies z_{11} = \frac{i}{2} M(\eta_1\bar{\omega}_2 - \eta_2\bar{\omega}_1) + \pi \bar{M}, \quad (3.32)$$

where Equations (3.18)–(3.19) have been used. For many applications, it is the gradient of the potential  $u_1$  that is important. From (3.27), we have

$$\frac{\partial u_1(z, \bar{z})}{\partial x_1} - i \frac{\partial u_1(z, \bar{z})}{\partial x_2} = \frac{d\zeta_{10}(z)}{dz} + z_{11} = \frac{1}{2\pi} \int_{\Gamma} f(t)\wp(t-z)dt + z_{11}. \quad (3.33)$$

We remark that Equation (3.33) is reminiscent of the familiar Green's function. In practice, to compute the integral (3.33) we can use the formula

$$\wp(z) = \left(\frac{\pi}{\omega_1}\right)^2 \left[ -\frac{1}{3} + \sum_{n \in \mathbb{Z}} \operatorname{csc}^2\left(\frac{z - n\omega_2}{\omega_1}\pi\right) - \sum_{n \in \mathbb{Z} \setminus \{0\}} \operatorname{csc}^2\left(\frac{n\omega_2}{\omega_1}\pi\right) \right],$$

which converges more rapidly than the original series (3.15), see [15], p. 434.

**Remark 3.** Let  $\Omega \subset Y$  be the union of mutually-disjoint smooth simply-connected inclusions  $\Omega_i$  ( $i = 1, \dots, N$ ), see Figure 3. Assume (3.20). Then by Theorem 2 and the principle of superposition, we see that the solution of (2.1) can be written as

$$u_1(z, \bar{z}) = \operatorname{Re} \left[ \frac{1}{2\pi} \sum_{i=1}^N \int_{\Gamma_i} f_i(t)\zeta(t-z)dt + z_{11}z + z_{10} \right], \quad (3.34)$$

where  $f_i : \Gamma_i \rightarrow \mathbb{R}$  is defined as in (3.21) for every  $i = 1, \dots, N$ ,  $z_{11}$  is the same as in (3.32) but with

$$M = \sum_{i=1}^N \frac{1}{2\pi} \int_{\Gamma_i} f_i(t)dt,$$

and  $z_{10}$  is again a constant such that the average of  $u_1$  vanishes over  $Y$ .



**Remark 4.** Let  $\Omega$ ,  $\mathbf{m}$ ,  $f_i$  and  $\Gamma_i$  be as in Remark 3. If  $Y = \mathbb{C}$ , we should replace the periodic boundary condition in (2.1) by the boundary condition  $u_1(z) \rightarrow 0$  as  $|z| \rightarrow +\infty$ . Then, the solution of

$$\begin{cases} \Delta u_1 = 0 & \text{on } \mathbb{C} \setminus \cup_{i=1}^N \Gamma_i, \\ \llbracket \nabla u_1 \rrbracket = (\mathbf{m} \cdot \mathbf{n}) \mathbf{n} & \text{on } \cup_{i=1}^N \Gamma_i, \\ u_1(z) \rightarrow 0 & \text{as } |z| \rightarrow +\infty \end{cases} \quad (3.35)$$

is given by

$$u_1(z, \bar{z}) = \operatorname{Re} \left[ \frac{1}{2\pi} \sum_{i=1}^N \int_{\Gamma_i} \frac{f_i(t)}{t-z} dt \right].$$

### 3.2.1. Specification to a Vigdergauz structure

Now let us consider the particular situation of (2.1) where the magnetization  $\mathbf{m} = \mathbf{m}^0 = (m_1^0, m_2^0)$  is uniform on  $\Omega$ . By Equation (3.21) we have

$$f(t) = - \int_0^s (m_1^0 \sin \gamma(s_1) - m_2^0 \cos \gamma(s_1)) ds_1 = \frac{-i}{2} (\bar{t} \bar{z}_m - t z_m) + a_0 \quad \forall t \in \Gamma, \quad (3.36)$$

where  $a_0 \in \mathbb{R}$  is a real constant and

$$z_m = m_1^0 - i m_2^0. \quad (3.37)$$

Note that

$$\int_{\Gamma} t dt = 0 \quad \text{and} \quad \int_{\Gamma} \bar{t} dt = 2i|\Omega| = 2i\theta, \quad (3.38)$$

where  $|\Omega|$  denotes the area of  $\Omega$  and  $\theta$  can be interpreted as the area fraction of  $\Omega$  since  $|Y| = 1$ , see (3.19). By (3.31) and (3.32) we have

$$M = \theta \frac{\bar{z}_m}{2\pi} \quad \text{and} \quad z_{11} = \theta \left[ \frac{i \bar{z}_m}{4\pi} (\eta_1 \bar{\omega}_2 - \eta_2 \bar{\omega}_1) + \frac{z_m}{2} \right]. \quad (3.39)$$

Moreover, if the inclusion  $\Omega$  is a Vigdergauz structure with matrix  $\mathbf{Q}$  and volume fraction  $\theta$ , from (1.13) and (2.4) we have

$$\nabla u_1 = -(1-\theta) \mathbf{Q} \mathbf{m}^0 \quad \text{on } \Omega. \quad (3.40)$$

Let

$$\begin{bmatrix} h_1 \\ h_2 \end{bmatrix} = (1 - \theta) \begin{bmatrix} (\mathbf{Q})_{11} & (\mathbf{Q})_{12} \\ (\mathbf{Q})_{12} & (\mathbf{Q})_{22} \end{bmatrix} \begin{bmatrix} m_1^0 \\ m_2^0 \end{bmatrix}, \quad (3.41)$$

$$z_h = h_1 - ih_2 \quad \text{and} \quad (1 - \theta)z_q \bar{z}_m = 2z_h - (1 - \theta)z_m. \quad (3.42)$$

Noticing  $\text{Tr}(\mathbf{Q}) = 1$ , we have

$$z_q = (\mathbf{Q})_{11} - (\mathbf{Q})_{22} - 2i(\mathbf{Q})_{12}. \quad (3.43)$$

By (3.33), (3.40) and (3.42) we have

$$\frac{d\xi_{10}(z)}{dz} + z_{11} = \frac{1}{2\pi} \int_{\Gamma} f(t)\wp(t - z)dt + z_{11} = -z_h \quad \forall z \in \Omega_{per}. \quad (3.44)$$

Integrating on every simply-connected component of  $\Omega_{per}$ , we conclude that there exist constants  $z_{\omega}^0 \in \mathbb{C}$ , depending on  $\omega$  but not  $z$ , such that

$$\xi_{10}(z) = \begin{cases} (-z_h - z_{11})z + z_{\omega}^0 & \text{if } z \in \Omega_{\omega}, \omega \in \mathcal{L}, \\ \frac{1}{2\pi} \int_{\Gamma} f(t)\zeta(t - z)dt & \text{if } z \in \mathbb{C} \setminus \bar{\Omega}_{per}. \end{cases} \quad (3.45)$$

Further, from (3.14), (3.36) and (3.45), we have that for every  $t \in \Gamma$ ,

$$\llbracket \xi_{10}(t) \rrbracket = -i f(t) \implies -\frac{1}{2}(\bar{t}\bar{z}_m - tz_m) = \xi_{10}(t^+) + (z_h + z_{11})t - z_0^0. \quad (3.46)$$

The latter of the above equation can be rewritten as  $\bar{t} = \xi_{00}(t)$  ( $\forall t \in \Gamma$ ) and

$$\begin{aligned} \xi_{00}(z) &:= -\frac{2}{\bar{z}_m} \left[ \xi_{10}(z) + \left( z_h + z_{11} - \frac{z_m}{2} \right) z - z_0^0 \right] \quad \forall z \in \mathbb{C} \setminus \bar{\Omega}_{per} \\ &= -\frac{2}{\bar{z}_m} \left[ \frac{1}{2\pi} \int_{\Gamma} \frac{-i}{2} (\bar{t}\bar{z}_m - tz_m) \zeta(t - z) dt + \left( z_h + z_{11} - \frac{z_m}{2} \right) z - z_0^0 \right] \\ &= -\frac{1}{2\pi i} \int_{\Gamma} \bar{t} \zeta(t - z) dt - \frac{1}{\bar{z}_m} \left[ (2z_h z + \theta z \frac{i\bar{z}_m}{2\pi} (\eta_1 \bar{\omega}_2 - \eta_2 \bar{\omega}_1) - (1 - \theta) z_m z - 2z_0^0) \right] \\ &= -\frac{1}{2\pi i} \int_{\Gamma} \bar{t} \zeta(t - z) dt - z \left[ \frac{i\theta}{2\pi} (\eta_1 \bar{\omega}_2 - \eta_2 \bar{\omega}_1) + (1 - \theta) z_q \right] + \frac{2z_0^0}{\bar{z}_m}, \quad (3.47) \end{aligned}$$

where we have plugged in (3.36) to arrive at the second equality, in the third equality we have plugged in (3.39) and used  $\int_{\Gamma} t \zeta(t - z) dt = 0$  for  $z \in \mathbb{C} \setminus \bar{\Omega}_{per}$ , and in the final equality we have used the second in (3.42). Direct calculations reveal that  $\xi_{00}$  satisfies

$$\begin{cases} \xi_{00}(z + \omega_1) = \xi_{00}(z) + \rho_1 \\ \xi_{00}(z + \omega_2) = \xi_{00}(z) + \rho_2 \end{cases} \quad \forall z \in \mathbb{C} \setminus \overline{\Omega}_{per}, \quad (3.48)$$

where, by (3.47) and the formula  $\int_{\Gamma} \bar{t} dt = 2i|\Omega| = 2i\theta$ ,

$$\begin{cases} \rho_1 = \theta \left[ \frac{\eta_1}{\pi} + \frac{\omega_1}{2\pi i} (\eta_1 \bar{\omega}_2 - \eta_2 \bar{\omega}_1) \right] - (1 - \theta) z_q \omega_1, \\ \rho_2 = \theta \left[ \frac{\eta_2}{\pi} + \frac{\omega_2}{2\pi i} (\eta_1 \bar{\omega}_2 - \eta_2 \bar{\omega}_1) \right] - (1 - \theta) z_q \omega_2. \end{cases} \quad (3.49)$$

Note that the above equation uniquely determines the constants  $\rho_1, \rho_2$  for given matrix  $\mathbf{Q}$  and volume fraction  $\theta$ , and vice versa. Multiply (3.49)<sub>1</sub> by  $\omega_2$  (resp.  $\bar{\omega}_2$ ) and then subtract (3.49)<sub>2</sub> multiplied by  $\omega_1$  (resp.  $\bar{\omega}_1$ ). From (3.18) and (3.19) we obtain the identities

$$\begin{cases} \rho_1 \omega_2 - \rho_2 \omega_1 = \frac{\theta}{\pi} (\eta_1 \omega_2 - \eta_2 \omega_1) = 2i\theta, \\ \rho_1 \bar{\omega}_2 - \rho_2 \bar{\omega}_1 = 2i(1 - \theta) z_q. \end{cases} \quad (3.50)$$

The above calculations motivate the following theorem.

**Theorem 5.** *Let  $\{\omega_1, \omega_2\}$  with  $\text{Im}[\omega_1/\omega_2] \neq 0$  be the periods,  $\mathcal{L} = \{v_1\omega_1 + v_2\omega_2 : v_1, v_2 \in \mathbb{Z}\}$  be the lattice,  $Y$  be the unit cell  $\{x_1\omega_1 + x_2\omega_2 : 0 < x_1, x_2 < 1\}$ , and  $\Omega \subset Y$  be a simply-connected open inclusion with smooth boundary. Then the inclusion  $\Omega$  is a periodic E-inclusion with matrix  $\mathbf{Q}$  and volume fraction  $\theta$  if, and only if there exists a function  $\xi_{00} : \mathbb{C} \setminus \Omega_{per} \rightarrow \mathbb{C}$  such that*

$$\bar{t} = \xi_{00}(t) \quad \forall t \in \partial\Omega, \quad (3.51)$$

where  $\xi_{00}(z)$  is analytic on  $\mathbb{C} \setminus \overline{\Omega}_{per}$ , continuous up to the boundary  $\partial\Omega_{per}$ , and satisfies (3.48) for the constants  $\rho_1, \rho_2$  in (3.49).

**Proof.** From the above discussions, we see Equations (3.48)–(3.51) are necessary for  $\Omega$  to be a periodic E-inclusion. Below we show  $\Omega$  is a periodic E-inclusion if Equations (3.48)–(3.51) hold.

From Theorem 2, (3.33) and (3.36), the gradient of the solution of (2.1) can be written as

$$\begin{aligned} \frac{\partial u_1(z, \bar{z})}{\partial x_1} - i \frac{\partial u_1(z, \bar{z})}{\partial x_2} &= \frac{1}{2\pi} \int_{\Gamma} \left[ \frac{-i}{2} (\bar{t} z_m - t z_m) + z_0 \right] \wp(t - z) dt + z_{11} \\ &\forall z \in \mathbb{C} \setminus \Gamma_{per}. \end{aligned} \quad (3.52)$$

Since  $\frac{1}{2\pi i} \int_{\Gamma} t \wp(t - z) dt = 1$  for any  $z \in \Omega$  and  $\int_{\Gamma} z_0 \wp(t - z) dt = 0$  for any  $z \notin \Gamma_{per}$ , by (3.51) it suffices to show the integral

$$\int_{\Gamma} \xi_{00}(t) \wp(t-z) dt = C_{00} \quad \forall z \in \Omega, \quad (3.53)$$

where  $C_{00}$  is a constant independent of  $z$ . Since the function  $t \mapsto \xi_{00}(t) \wp(t-z)$  is analytic on  $\mathbb{C} \setminus \overline{\Omega}_{per}$ , by the Cauchy integral theorem ([14], Chapter 4) we have

$$\int_{\Gamma} \xi_{00}(t) \wp(t-z) dt = \int_{\partial Y} \xi_{00}(t) \wp(t-z) dt \quad \text{on } \forall z \in \Omega, \quad (3.54)$$

where  $\partial Y$  denote the counterclockwise contour around the boundary of  $Y$ . This is the step we use the condition  $z \in \Omega$ . Let  $\gamma_i$  ( $i = 1, 2, 3, 4$ ) be the straight line subdivision of the contour  $\partial Y$ , see Figure 2. Then, for every  $z \in \Omega$  we have

$$\begin{aligned} C_{00} &= \sum_{i=1}^4 \int_{\gamma_i} \xi_{00}(t) \wp(t-z) dt \\ &= \int_{\gamma_1} [\xi_{00}(t) \wp(t-z) - \xi_{00}(t+\omega_2) \wp(t+\omega_2-z)] dt \\ &\quad + \int_{\gamma_2} [\xi_{00}(t) \wp(t-z) - \xi_{00}(t-\omega_1) \wp(t+\omega_1-z)] dt \\ &= -\rho_2 \int_{\gamma_1} \wp(t-z) dt + \rho_1 \int_{\gamma_2} \wp(t-z) dt \\ &= -\rho_2 \zeta(t-z) \Big|_0^{\omega_1} + \rho_1 \zeta(t-z) \Big|_{\omega_1}^{\omega_1+\omega_2} = -\rho_2 \eta_1 + \rho_1 \eta_2, \end{aligned} \quad (3.55)$$

which is indeed independent of  $z$ . Substituting (3.49) into (3.55), and then into (3.52), we find the right-hand side of (3.52) is equal to  $z_h$  (see (3.42)), and henceforth complete the proof of the theorem.  $\square$

### 3.3. Solutions to Problem (2.2)

In this section we construct the solution of (2.2) with  $u_1$  given by (3.29). As in Section 3.2, we will write the second gradient of  $u_2$  in terms of Cauchy-type integrals and their derivatives. To fix the idea, we notice that

$$\Delta^2 u_2 = 0 \quad \text{on } \mathbb{C} \setminus \Gamma_{per},$$

which motivates us to consider  $\hat{u}_2 : \mathbb{C} \rightarrow \mathbb{R}$

$$\hat{u}_2(z, \bar{z}) = \frac{1}{2} \left[ \bar{z} \zeta_{21}(z) + \zeta_{20}(z) + \overline{\bar{z} \zeta_{21}(z) + \zeta_{20}(z)} \right], \quad (3.56)$$

where  $\zeta_{21}(z)$ ,  $\zeta_{20}(z)$ , analytic on every component of  $\mathbb{C} \setminus \Gamma_{per}$ , are to be determined.

Direct calculations of the second-order derivatives of (3.56) reveal that on  $\mathbb{C} \setminus \Gamma_{per}$ ,

$$\hat{G}_\alpha(z, \bar{z}) = \begin{cases} \frac{\partial^2 \hat{u}_2}{\partial z^2} = \frac{1}{2} \left[ \bar{z} \frac{d^2 \zeta_{21}(z)}{dz^2} + \frac{d^2 \zeta_{20}(z)}{dz^2} \right] & \text{if } \alpha = (20), \\ \frac{\partial^2 \hat{u}_2}{\partial z \partial \bar{z}} = \frac{1}{4} \Delta \hat{u}_2 = \frac{1}{2} \left[ \frac{d \zeta_{21}(z)}{dz} + \overline{\left( \frac{d \zeta_{21}(z)}{dz} \right)} \right] & \text{if } \alpha = (11), \\ \frac{\partial^2 \hat{u}_2}{\partial \bar{z}^2} = \overline{\left( \frac{\partial^2 \hat{u}_2}{\partial z^2} \right)} & \text{if } \alpha = (02), \end{cases} \quad (3.57)$$

where  $\alpha = (\alpha_1 \alpha_2)$  are multi-indices for derivatives. In short, we can write the above the equation as  $\hat{G}_\alpha(z, \bar{z}) = \frac{\partial^2 \hat{u}_2}{\partial z^{\alpha_1} \partial \bar{z}^{\alpha_2}}$  for every  $\alpha_1 + \alpha_2 = 2$ . Let

$$4 \frac{d \zeta_{21}(z)}{dz} = \zeta_{10}(z) + z_{11}z + z_{10} \quad \forall z \in \mathbb{C} \setminus \Gamma_{per}, \quad (3.58)$$

and hence (see (3.22) and (3.29))

$$\frac{d^2 \zeta_{21}(z)}{dz^2} = \frac{1}{8\pi} \int_{\Gamma} f(t) \wp(t-z) dz + \frac{z_{11}}{4} \quad \forall z \in \mathbb{C} \setminus \Gamma_{per}. \quad (3.59)$$

By (3.29) and (3.57)<sub>2</sub>, we have

$$\Delta \hat{u}_2 = u_1 \quad \text{on } \mathbb{C}. \quad (3.60)$$

Since  $u_1$  is continuous on  $\mathbb{C}$ , if  $\hat{u}_2$  were a solution of (2.2), it would necessarily have continuous second gradient, i.e., the right-hand sides of (3.57) should all be continuous on the complex plane  $\mathbb{C}$ . Therefore, for every  $t \in \Gamma_\omega$  and every  $\omega \in \mathcal{L}$ , the quantity

$$2 \left[ \left. \frac{\partial^2 \hat{u}_2}{\partial z^2} \right] \Big|_{z=t} = \left[ \left. \bar{z} \frac{d^2 \zeta_{21}(z)}{dz^2} + \frac{d^2 \zeta_{20}(z)}{dz^2} \right] \Big|_{z=t} = -\frac{i}{4} t f^{(1)}(t-\omega) + \left[ \left. \frac{d^2 \zeta_{20}(z)}{dz^2} \right] \Big|_{z=t} \quad (3.61)$$

should vanish, where the Plemelj formulas (3.14) have been used.

We now express  $d^2 \zeta_{20}(z)/dz^2$  in terms of Cauchy-type integrals. Let

$$K_\omega(z) = \begin{cases} \frac{1}{z} & \text{if } \omega = 0 \\ \frac{1}{z+\omega} - \frac{1}{\omega} + \frac{z}{\omega^2} & \text{if } \omega \neq 0 \end{cases}$$

$$\hat{K}_\omega(z) = \begin{cases} \frac{1}{z} & \text{if } \omega = 0 \\ \frac{1}{z+\omega} - \frac{1}{\omega} + \frac{z}{\omega^2} - \frac{z^2}{\omega^3} & \text{if } \omega \neq 0 \end{cases} \quad (3.62)$$

and

$$\begin{aligned}\psi_{20}^{(\omega)}(z) &= \frac{1}{2\pi} \int_{\Gamma_\omega} \left[ \frac{\overline{t-\omega}}{4} f^{(1)}(t-\omega) \right] K_\omega(t-\omega-z) dt, \\ \phi_{20}^{(\omega)}(z) &= \frac{1}{2\pi} \int_{\Gamma_\omega} \frac{\overline{\omega}}{4} f^{(1)}(t-\omega) \hat{K}_\omega(t-\omega-z) dt.\end{aligned}\quad (3.63)$$

Since the only term in  $K_\omega(t-\omega-z)$  and  $\hat{K}_\omega(t-\omega-z)$  that can give rise to discontinuity is  $\frac{1}{t-z}$ , we see that both  $\psi_{20}^{(\omega)}(z)$  and  $\phi_{20}^{(\omega)}(z)$  are analytic on  $\mathbb{C} \setminus \Gamma_\omega$  and, by the Plemelj formulas (3.14), satisfy

$$\begin{aligned}\left[ \psi_{20}^{(\omega)}(t) \right] &= -\frac{i}{4} \frac{1}{t-\omega} f^{(1)}(t-\omega) \quad \text{and} \\ \left[ \phi_{20}^{(\omega)}(t) \right] &= -\frac{i}{4} \overline{\omega} f^{(1)}(t-\omega) \quad \forall t \in \Gamma_\omega.\end{aligned}\quad (3.64)$$

Note that  $K_\omega(z)$  ( $\omega \in \mathcal{L}$ ) are the summands in the infinite series (3.16) and that

$$K_\omega(z) = \frac{z^2}{(z+\omega)\omega^2}, \quad \hat{K}_\omega(z) = \frac{-z^3}{(z+\omega)\omega^3} \quad \text{if } \omega \neq 0. \quad (3.65)$$

Define

$$\sigma(z) = \sum_{\omega \in \mathcal{L}} \overline{\omega} \hat{K}_\omega(z) \quad \text{and} \quad \sigma'(z) = \frac{d\sigma(z)}{dz} = \sum_{\omega \in \mathcal{L}} \overline{\omega} \frac{d}{dz} \hat{K}_\omega(z). \quad (3.66)$$

From (3.65), we see that the series in (3.66) converge absolutely in  $\omega$  and uniformly for  $z$  in any compact subset of  $\mathbb{C} \setminus \mathcal{L}$ . Let

$$\phi_{20}(z) = \sum_{\omega \in \mathcal{L}} \phi_{20}^{(\omega)}(z) = \frac{1}{8\pi} \int_{\Gamma} f^{(1)}(t) \sigma(t-z) dt = -\frac{1}{8\pi} \int_{\Gamma} f(t) \sigma'(t-z) dt \quad (3.67)$$

and

$$\begin{aligned}\psi_{20}(z) &= \sum_{\omega \in \mathcal{L}} \frac{1}{2\pi} \int_{\Gamma_\omega} \left[ \frac{\overline{t-\omega}}{4} f^{(1)}(t-\omega) \right] K_\omega(t-\omega-z) dt \\ &= \frac{1}{8\pi} \int_{\Gamma} \overline{t} f^{(1)}(t) \zeta(t-z) dt.\end{aligned}\quad (3.68)$$

From (3.65) we see that both summations in (3.67)–(3.68) converge absolutely on any compact subset of  $\mathbb{C} \setminus \Gamma_{per}$ , and so are analytic on every component of  $\mathbb{C} \setminus \Gamma_{per}$ . From the definitions (3.67), (3.68) and (3.64), we have that for every  $\omega \in \mathcal{L}$  and every  $t \in \Gamma_\omega$ ,

$$\llbracket \psi_{20}(t) \rrbracket = -\frac{i}{4} \overline{t - \omega} f^{(1)}(t - \omega) \quad \text{and} \quad \llbracket \phi_{20}(t) \rrbracket = -\frac{i}{4} \overline{\omega} f^{(1)}(t - \omega). \quad (3.69)$$

Therefore, the following function:

$$\frac{d^2 \zeta_{20}(z)}{dz^2} := -\psi_{20}(z) - \phi_{20}(z) \quad (3.70)$$

is analytic on every component of  $\mathbb{C} \setminus \Gamma_{per}$  and satisfies

$$\left[ \left[ \frac{d^2 \zeta_{20}(z)}{dz^2} \right] \right]_{z=t} = \frac{i}{4} \overline{t} f^{(1)}(t - \omega) \quad \forall t \in \Gamma_{\omega},$$

which implies the quantity (3.61) indeed vanishes for every  $t \in \Gamma_{\omega}$  and every  $\omega \in \mathcal{L}$ .

We now calculate the translational property of  $\hat{G}_{\alpha}(z, \bar{z})$  under  $z \rightarrow z + \omega_i$  for  $i = 1, 2$ . From (3.68), by the same calculation analogous to (3.30) we obtain

$$\psi_{20}(z + \omega_i) = \frac{1}{8\pi} \int_{\Gamma} \overline{t} f^{(1)}(t) \zeta(t - z - \omega_i) dt = \psi_{20}(z) + \eta_i \overline{M}/4 \quad (i = 1, 2), \quad (3.71)$$

where we have used

$$\frac{1}{8\pi} \int_{\Gamma} \overline{t} f^{(1)}(t) dt = -\frac{1}{8\pi} \int_{\Gamma} f(t) d\bar{t} = -\frac{1}{4} \overline{\left( \frac{1}{2\pi} \int_{\Gamma} f(t) dt \right)} = -\frac{\overline{M}}{4}.$$

Also, from (3.67) and (A.1) we have for  $i = 1, 2$ ,

$$\begin{aligned} \phi_{20}(z + \omega_i) &= -\frac{1}{8\pi} \int_{\Gamma} f(t) \sigma'(t - z - \omega_i) dt = \phi_{20}(z) \\ &+ \frac{\overline{\omega}_i}{8\pi} \int_{\Gamma} f(t) \wp(t - z) dt + S_i M/4, \end{aligned} \quad (3.72)$$

where  $S_i$  ( $i = 1, 2$ ) are constants defined in (A.2). Direct verification of the above equations is rather cumbersome and we postpone it to the Appendix. From (3.56), (3.59), (3.70), (3.71) and (3.72), direct calculations reveal that for  $i = 1, 2$ ,

$$\begin{aligned} 2 \frac{\partial^2 \hat{u}(z + \omega_i, \bar{z} + \overline{\omega}_i)}{\partial z^2} &= (\bar{z} + \overline{\omega}_i) \frac{d^2 \zeta_{21}(z + \omega_i)}{dz^2} + \frac{d^2 \zeta_{20}(z + \omega_i)}{dz^2} \\ &= \bar{z} \frac{d^2 \zeta_{21}(z)}{dz^2} + \overline{\omega}_i \frac{d^2 \zeta_{21}(z)}{dz^2} + \frac{d^2 \zeta_{20}(z)}{dz^2} - \eta_i \overline{M}/4 \\ &- \frac{\overline{\omega}_i}{8\pi} \int_{\Gamma} f(t) \wp(t - z) dt - S_i M/4 \\ &= 2 \frac{\partial^2 \hat{u}(z, \bar{z})}{\partial z^2} + \overline{\omega}_i z_{11}/4 - \eta_i \overline{M}/4 - S_i M/4, \end{aligned} \quad (3.73)$$

and henceforth

$$\frac{\partial^3 \hat{u}(z + \omega_i, \bar{z} + \bar{\omega}_i)}{\partial z^3} = \frac{\partial^3 \hat{u}(z, \bar{z})}{\partial z^3}.$$

From (3.57), we conclude

$$\frac{\partial^3 \hat{u}(z + \omega_i, \bar{z} + \bar{\omega}_i)}{\partial z^{\alpha_1} \partial \bar{z}^{\alpha_2}} = \frac{\partial^3 \hat{u}(z, \bar{z})}{\partial z^{\alpha_1} \partial \bar{z}^{\alpha_2}} \quad \forall z \in \mathbb{C} \setminus \Gamma_{per} \quad \text{and} \quad \alpha_1 + \alpha_2 = 3. \quad (3.74)$$

At this point, it seems appealing to express the solution of (2.2) in terms of  $\hat{u}_2$  defined in (3.56). However, with  $d\xi_{21}/dz$ ,  $d^2\xi_{20}/dz^2$  defined as in (3.58) and (3.70),  $\xi_{21}$ ,  $\xi_{20}$  and so  $\hat{u}_2$  might be multiple-valued functions. To remedy this issue, we define  $\tilde{u}_2 : \mathbb{C} \rightarrow \mathbb{R}$  through the following steps.

First, we notice that  $\hat{G}_\alpha(z, \bar{z}) = \frac{\partial^2 \hat{u}(z, \bar{z})}{\partial z^{\alpha_1} \partial \bar{z}^{\alpha_2}}$  is continuous on  $\mathbb{C}$  for all  $\alpha_1 + \alpha_2 = 2$ . Since the transformation between  $(x_1, x_2)$  and  $(z, \bar{z})$  is linear and nonsingular, by (3.1) and (3.3) we can write every  $\tilde{G}_\alpha(z, \bar{z}) := \frac{\partial^2 \tilde{u}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2}}$  as a linear combination of  $\hat{G}_\alpha(z, \bar{z})$ . Direct calculations yield

$$\begin{bmatrix} \tilde{G}_{(20)} & \tilde{G}_{(11)} \\ \tilde{G}_{(11)} & \tilde{G}_{(02)} \end{bmatrix} = \begin{bmatrix} \hat{G}_{(20)} + 2\hat{G}_{(11)} + \hat{G}_{(02)} & i(\hat{G}_{(20)} - \hat{G}_{(02)}) \\ i(\hat{G}_{(20)} - \hat{G}_{(02)}) & -\hat{G}_{(20)} + 2\hat{G}_{(11)} - \hat{G}_{(02)} \end{bmatrix}, \quad (3.75)$$

which are clearly real-valued and continuous on  $\mathbb{C}$ . Further, we verify the following identities:

$$\begin{cases} \frac{\partial \tilde{G}_{(20)}}{\partial x_2} = \frac{\partial \tilde{G}_{(11)}}{\partial x_1} \\ \frac{\partial \tilde{G}_{(11)}}{\partial x_2} = \frac{\partial \tilde{G}_{(02)}}{\partial x_1} \end{cases} \quad \text{on } \mathbb{C} \setminus \Gamma_{per}. \quad (3.76)$$

Second, let  $z_0 = x_1^0 + ix_2^0 \in \mathbb{C}$  be an arbitrary but fixed point. We define

$$\begin{cases} g_1(z, \bar{z}) = \int_{\gamma(z_0, z)} \tilde{G}_{(20)} dx_1 + \tilde{G}_{(11)} dx_2, \\ g_2(z, \bar{z}) = \int_{\gamma(z_0, z)} \tilde{G}_{(11)} dx_1 + \tilde{G}_{(02)} dx_2, \end{cases} \quad (3.77)$$

where  $\gamma(z_0, z)$  denotes a rectifiable integration path from  $z_0$  to  $z = x_1 + ix_2$ . From Equation (3.76), Green's theorem and the continuity of  $\tilde{G}_\alpha$ , we infer that the integrals in (3.77) depend only on the end points  $z_0$  and  $z$ , and hence  $g_1, g_2 : \mathbb{C} \rightarrow \mathbb{R}$  are well-defined on  $\mathbb{C}$ . More, since

$$\frac{\partial g_1(z, \bar{z})}{\partial x_2} = \tilde{G}_{11}(z, \bar{z}) = \frac{\partial g_2(z, \bar{z})}{\partial x_1},$$



Green's theorem implies that the integral

$$\tilde{u}_2(z, \bar{z}) = \int_{\gamma(z_0, z)} g_1 dx_1 + g_2 dx_2 \quad (3.78)$$

depends only on the end points, and henceforth  $\tilde{u}_2 : \mathbb{C} \rightarrow \mathbb{R}$  is well-defined. In short, the above elaborate definition of  $\tilde{u}_2$  is equivalent to the second-order tensor field (3.75) is the second gradient of a scalar field.

We now study the properties of  $\tilde{u}_2$ . It is standard to verify that  $\tilde{u}$  defined by (3.77)–(3.78) is continuously differentiable up to the second order (more precisely,  $\tilde{u}_2$  belongs to  $C_{loc}^{2,1}(\mathbb{C})$ ) and satisfies that

$$\frac{\partial^2 \tilde{u}(z, \bar{z})}{\partial z^{\alpha_1} \partial \bar{z}^{\alpha_2}} = \hat{G}_\alpha(z, \bar{z}) = \frac{\partial^2 \hat{u}(z, \bar{z})}{\partial z^{\alpha_1} \partial \bar{z}^{\alpha_2}} \quad \text{on } \mathbb{C} \quad \forall \alpha_1 + \alpha_2 = 2. \quad (3.79)$$

From (3.74), we infer that

$$\frac{\partial^3 \tilde{u}(z + \omega, \bar{z} + \bar{\omega})}{\partial z^{\alpha_1} \partial \bar{z}^{\alpha_2}} = \frac{\partial^3 \tilde{u}(z, \bar{z})}{\partial z^{\alpha_1} \partial \bar{z}^{\alpha_2}} \quad \forall z \in \mathbb{C} \setminus \Gamma_{per}, \quad \omega \in \mathcal{L} \quad \text{and} \quad \alpha_1 + \alpha_2 = 3. \quad (3.80)$$

From the explicit representations (see (3.59), (3.68) and (3.67)) and the property of Cauchy integrals (see (3.13)), we know all the third-order derivatives of  $\tilde{u}$ , i.e.,  $\frac{\partial^3 \tilde{u}(z + \omega, \bar{z} + \bar{\omega})}{\partial z^{\alpha_1} \partial \bar{z}^{\alpha_2}}$  ( $\alpha_1 + \alpha_2 = 3$ ), are uniformly bounded on  $Y \setminus \Gamma$ . Therefore, there exists constants  $C_1, C_2 > 0$  such that

$$|\tilde{u}_2(z, \bar{z})| \leq C_1 |z|^3 + C_2 \quad \text{on } \mathbb{C}. \quad (3.81)$$

Further, from (2.2), (3.60) and (3.79), we verify that

$$\Delta \tilde{u}_2 = \Delta u_2 = u_1 \quad \text{on } \mathbb{C},$$

where  $u_2$  is the solution of (2.2) and  $u_1$  is the solution of (2.1) given by (3.29). Therefore,  $u_2 - \tilde{u}_2$  is real and harmonic on  $\mathbb{C}$ , and hence can be expressed as the real part of a power series of  $z$ . From (3.81), we conclude that

$$\begin{aligned} & u_2(z, \bar{z}) - \tilde{u}_2(z, \bar{z}) \\ &= \frac{1}{2} \left[ z_{23} z^3 + z_{22} z^2 + z_{21} z + z_{20} + \overline{z_{23} z^3 + z_{22} z^2 + z_{21} z + z_{20}} \right] \quad \text{on } \mathbb{C}, \end{aligned} \quad (3.82)$$

where  $z_{2k}$  ( $k = 0, 1, 2, 3$ ) are complex constants such that  $u_2$ ,  $\nabla u_2$ , and  $\nabla \nabla u_2$  are periodic functions. In particular,  $z_{23}$  is determined by

$$\frac{\partial^2 u_2(z + \omega_i, \bar{z} + \bar{\omega}_i)}{\partial z^2} = \frac{\partial^2 u_2(z, \bar{z})}{\partial z^2} \quad (i = 1, 2). \quad (3.83)$$

Let

$$\eta'_i = \frac{1}{8}(\bar{\omega}_i z_{11} - \eta_i \bar{M} - S_i M) \quad (i = 1, 2).$$

From (3.73), (3.82) and (3.83), we have

$$\begin{cases} \operatorname{Re}[3z_{23}\omega_1 + \eta'_1] = 0 \\ \operatorname{Re}[3z_{23}\omega_2 + \eta'_2] = 0 \end{cases} \implies z_{23} = \frac{i}{6}[(\eta'_2 + \bar{\eta}'_2)\bar{\omega}_1 - (\eta'_1 + \bar{\eta}'_1)\bar{\omega}_2]. \quad (3.84)$$

In analogy with Theorem 2, we summarize below.

**Theorem 6.** Let  $f : \Gamma \rightarrow \mathbb{R}$  be a smooth real-valued function,  $\zeta_{21}$  and  $\zeta_{20}$  be defined as in (3.59) and (3.70),  $\phi_{20}$  and  $\psi_{20}$  be defined as in (3.67) and (3.68),  $\hat{G}_\alpha$  and  $\tilde{G}_\alpha$  be defined as in (3.57) and (3.75), and  $\tilde{u}_2 : \mathbb{C} \rightarrow \mathbb{R}$  be defined as in (3.77) and (3.78). Then  $\tilde{u}_2$  is twice continuously differentiable on  $\mathbb{C}$ , smooth on  $\mathbb{C} \setminus \Gamma_{per}$ , and satisfies

$$\begin{cases} \Delta \Delta \tilde{u}_2 = 0 & \text{on } \mathbb{C} \setminus \Gamma_{per}, \\ \llbracket \nabla \Delta \tilde{u}_2 \rrbracket = -\frac{d f(t(s))}{ds} \mathbf{n} & \text{on } \Gamma, \\ \nabla \nabla \nabla \tilde{u}_2(z + \omega, \bar{z} + \bar{\omega}) = \nabla \nabla \nabla \tilde{u}_2(z, \bar{z}) \quad \forall \omega \in \mathcal{L} \quad \text{and} \quad z \in \mathbb{C} \setminus \Gamma_{per}, \end{cases} \quad (3.85)$$

where  $\mathbf{n} = [\sin \gamma, -\cos \gamma]$  is the outward normal on  $\Gamma$ . Further, if  $u_2 : \mathbb{C} \rightarrow \mathbb{R}$  is the solution of (3.85) with zero average on  $Y$ , i.e. the solution of (2.2) with  $u_1$  being the solution of (2.1), then  $u_2 - \tilde{u}_2$  can be different at most by a harmonic cubic polynomial (see (3.82)), and henceforth the third-order derivatives of  $u_2$  can be expressed as

$$\begin{cases} \frac{\partial^3 u_2}{\partial z^3} = 3z_{23} + \frac{\bar{z}}{16\pi} \int_\Gamma f(t) \frac{d\wp(t-z)}{dz} dt - \frac{1}{2} \frac{d}{dz} [\psi_{20}(z) + \phi_{20}(z)], \\ \frac{\partial^3 u_2}{\partial z^2 \partial \bar{z}} = \frac{1}{4} \frac{\partial u_1}{\partial z} = \frac{1}{16\pi} \int_\Gamma f(t) \wp(t-z) dt + \frac{z_{11}}{8}, \\ \frac{\partial^3 u_2}{\partial z \partial \bar{z}^2} = \overline{\left( \frac{\partial^3 u_2}{\partial z^2 \partial \bar{z}} \right)}, \quad \frac{\partial^3 u_2}{\partial \bar{z}^3} = \overline{\left( \frac{\partial^3 u_2}{\partial z^3} \right)}, \end{cases} \quad (3.86)$$

where  $f : \Gamma \rightarrow \mathbb{R}$  is given by (3.21),  $z_{11}$ ,  $z_{23}$  are constants given by (3.32) and (3.84), and from (3.67)–(3.68),

$$\frac{d}{dz} [\psi_{20}(z) + \phi_{20}(z)] = \frac{1}{8\pi} \left[ \int_\Gamma \bar{t} f^{(1)}(t) \wp(t-z) dt - \int_\Gamma f^{(1)}(t) \sigma'(t-z) dt \right]. \quad (3.87)$$

We note that analogous to Remarks 3 and 4, Theorem 6 can be generalized to situations where  $\Omega$  is a collection of multiple simply-connected inclusions or  $Y = \mathbb{C}$ . The reader is invited to write down the formulas corresponding to (3.86) for these situations.

### 3.3.1. Specification to a Vigdergauz structure

We now consider the case  $u_1$  is induced by a uniform magnetization on  $\Omega$  (see (2.1)) and  $\Omega$  is a Vigdergauz structure with matrix  $\mathbf{Q}$  and volume fraction  $\theta$ , see Section 3.2.1. In this case, the function  $f$  given by (3.36) and  $\zeta_{10}(z)$  satisfies (3.44). In particular, we notice

$$\frac{1}{2\pi} \int_{\Gamma} f(t) \wp(t-z) dt = -z_h - z_{11} \quad \text{and} \quad \int_{\Gamma} f(t) \frac{d\wp(t-z)}{dz} dt = 0 \quad \forall z \in \Omega.$$

Therefore, to evaluate the third gradient of  $u_2$  on  $\Omega$ , we only need to evaluate (3.87), see (3.86).

By Theorem 5, (3.6), (3.36) and (3.51), we have

$$f^{(1)}(t) = \frac{-i}{2} [\bar{z}_m \zeta'_{00}(t) - z_m]. \quad \forall t \in \Gamma, \quad (3.88)$$

where  $\zeta'_{00}(z) = \frac{d}{dz} \zeta_{00}(z)$ . Note that the functions  $t \mapsto \zeta_{00}(t)$ ,  $t \mapsto \wp(t-z)$  and  $t \mapsto \sigma(t-z)$  are analytic on  $\mathbb{C} \setminus \overline{\Omega}_{per}$ . Substituting (3.51) and (3.88) into (3.87), by the Cauchy integral theorem we obtain that for every  $z \in \Omega$ ,

$$\begin{aligned} \frac{d}{dz} [\psi_{20}(z) + \phi_{20}(z)] &= \frac{-z_m}{16\pi i} \int_{\partial Y} [\zeta_{00}(t) \wp(t-z) - \sigma'(t-z)] dt \\ &+ \frac{\bar{z}_m}{16\pi i} \int_{\partial Y} \zeta'_{00}(t) [\zeta_{00}(t) \wp(t-z) - \sigma'(t-z)] dt \\ &=: I_1(z) + I_2(z). \end{aligned} \quad (3.89)$$

The first term in the right-hand side of (3.89) is clearly a constant since (see (3.53))

$$\begin{aligned} I_1(z) &= \frac{-z_m}{16\pi i} \int_{\partial Y} [\zeta_{00}(t) \wp(t-z) - \sigma'(t-z)] dt \\ &= \frac{-z_m}{16\pi i} \int_{\partial Y} \zeta_{00}(t) \wp(t-z) dt = \frac{-z_m C_{00}}{16\pi i}. \end{aligned}$$

By (3.48) and (A.1), the second term in the right-hand side of (3.89) can be written as (see Figure 2)

$$\begin{aligned} I_2(z) &= \frac{\bar{z}_m}{16\pi i} \int_{\partial Y} \zeta'_{00}(t) [\zeta_{00}(t) \wp(t-z) - \sigma'(t-z)] dt \\ &= \frac{\bar{z}_m}{16\pi i} \left[ \int_{\gamma_1} (-\rho_2 + \bar{\omega}_2) \zeta'_{00}(t) \wp(t-z) dt \right. \\ &\quad \left. + \int_{\gamma_2} (\rho_1 - \bar{\omega}_1) \zeta'_{00}(t) \wp(t-z) dt + S_2 \rho_1 - S_1 \rho_2 \right]. \end{aligned} \quad (3.90)$$

In general,  $I_2(z)$  cannot be a constant. Otherwise,  $I_2(z)$  would be equal to this constant on the entire complex plane  $\mathbb{C}$ , and by the Plemelj formulas (3.14) we would have

$$\begin{cases} 0 = \llbracket I_2(t) \rrbracket = \frac{-\bar{z}_m}{8}(-\rho_2 + \bar{\omega}_2) \frac{d^2 \zeta_{00}(t)}{dt^2} & \forall t \in \gamma_1, \\ 0 = \llbracket I_2(t) \rrbracket = \frac{-\bar{z}_m}{8}(-\rho_1 + \bar{\omega}_1) \frac{d^2 \zeta_{00}(t)}{dt^2} & \forall t \in \gamma_2. \end{cases} \quad (3.91)$$

For nonzero  $z_m$ , the above equation implies either (1)

$$\rho_1 = \bar{\omega}_1 \quad \text{and} \quad \rho_2 = \bar{\omega}_2, \quad (3.92)$$

or (2)  $\frac{d^2 \zeta_{00}(t)}{dt^2} = 0$  for every  $t \in \gamma_1$  or every  $t \in \gamma_2$ . From the analyticity of  $\zeta_{00}(z)$  on  $\mathbb{C} \setminus \bar{\Omega}_{per}$ , the latter situation is equivalent to (see [14], Chapter 4)

$$\frac{d^2}{dz^2} \zeta_{00}(z) = 0 \quad \forall z \in \mathbb{C} \setminus \Omega_{per}. \quad (3.93)$$

Equation (3.92) is clearly impossible if  $\theta \neq 1$  since it implies (see (3.19))

$$\rho_1 \omega_2 - \rho_2 \omega_1 = \bar{\omega}_1 \omega_2 - \bar{\omega}_2 \omega_1 = 2i,$$

which contradicts (3.50)<sub>1</sub>. On the other hand, if Equation (3.93) holds, we see that  $\zeta_{00}(z)$  is a linear function of  $z$ . By (3.51), the boundary of  $\Omega$  must consist of parallel lines, i.e.,  $\Omega$  is a laminate. In this case, from (1.9) and (1.13) we infer that the matrix  $\mathbf{Q}$  must have a zero eigenvalue. The above discussions show that the third gradient of  $u_2$  is in general not uniform on  $\Omega$  even if the magnetization is uniform on  $\Omega$  and the inclusion is a Vigdergauz structure.

In regard of problem (1.8), we have the following theorem.

**Theorem 7.** *Consider the periodic Eshelby inclusion problem (1.8). Assume that the inclusion  $\Omega$  is a Vigdergauz structure with a positive definite matrix  $\mathbf{Q} \in \mathbb{Q}$  and volume fraction  $\theta \in (0, 1)$  and the eigenstress  $\mathbf{P}^H = \mathbf{P}^0 \in \mathbb{R}_{sym}^{2 \times 2}$  is uniform on  $\Omega$ . If  $\mathbf{P}^0 \neq a \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$  for some  $a \in \mathbb{R}$ , then the induced strain  $\nabla \mathbf{v}$  is not uniform on  $\Omega$ .*

**Proof.** Let  $\{\mathbf{e}_1^0, \mathbf{e}_2^0\} \subset \mathbb{R}^2$  be the basis of our rectangular coordinate system and  $b$  the solution of (1.10). From Equation (2.5) and the fact that both  $b$  and  $u_2$  are bounded on  $\mathbb{C}$  and have zero average over  $Y$ , we see that

$$\frac{\partial b(z, \bar{z})}{\partial z} = \frac{1}{2} \left( \frac{\partial b(z, \bar{z})}{\partial x_1} - i \frac{\partial b(z, \bar{z})}{\partial x_2} \right) = \frac{1}{2} (u_2(z, \bar{z}, \mathbf{e}_1^0) - i u_2(z, \bar{z}, \mathbf{e}_2^0)),$$

and hence for every  $z \in \Omega$ ,

$$\begin{aligned}
 \frac{\partial^4 b(z, \bar{z})}{\partial z^4} &= \frac{1}{2} \left( \frac{\partial^3 u_2(z, \bar{z}, \mathbf{e}_1^0)}{\partial z^3} - i \frac{\partial^3 u_2(z, \bar{z}, \mathbf{e}_2^0)}{\partial z^3} \right) \\
 &= C_1 - \frac{1}{4} \frac{d}{dz} [\psi_{20}(z, \mathbf{e}_1^0) + \phi_{20}(z, \mathbf{e}_1^0) - i\psi_{20}(z, \mathbf{e}_2^0) - i\phi_{20}(z, \mathbf{e}_2^0)] \\
 &= C_2 + \frac{1}{32\pi i} \left[ \int_{\gamma_1} (-\rho_2 + \bar{\omega}_2) \zeta'_{00}(t) \wp(t-z) dt \right. \\
 &\quad \left. + \int_{\gamma_2} (\rho_1 - \bar{\omega}_1) \zeta'_{00}(t) \wp(t-z) dt \right], \tag{3.94}
 \end{aligned}$$

where  $C_1, C_2$  are constants. In particular, we have used (3.86)<sub>1</sub> in the the second equality, and the final equality follows from

$$\begin{aligned}
 \frac{d}{dz} [\psi_{20}(z, \mathbf{e}_1^0) + \phi_{20}(z, \mathbf{e}_1^0)] &= \frac{1}{8\pi} \left[ \int_{\Gamma} \frac{-i}{2} (\bar{t}^{(1)} - 1) [\bar{t} \wp(t-z) - \sigma'(t-z)] dt \right], \\
 \frac{d}{dz} [\psi_{20}(z, \mathbf{e}_2^0) + \phi_{20}(z, \mathbf{e}_2^0)] &= \frac{1}{8\pi} \left[ \int_{\Gamma} \frac{-i}{2} (i\bar{t}^{(1)} + i) [\bar{t} \wp(t-z) - \sigma'(t-z)] dt \right],
 \end{aligned}$$

and similar calculations as from (3.88) to (3.90). Note that the above equations are obtained by plugging (3.88) into (3.87) since  $z_m = 1$  if  $\mathbf{m}^0 = \mathbf{e}_1^0$  and  $z_m = -i$  if  $\mathbf{m}^0 = \mathbf{e}_2^0$ , see (3.37).

Since the transformation between  $(x_1, x_2)$  and  $(z, \bar{z})$  is linear and nonsingular, by (3.1) we have

$$b_{,piqj} = \frac{\partial^4 b}{\partial x_p \partial x_i \partial x_q \partial x_j} = (\mathbf{T})_{p\bar{p}} (\mathbf{T})_{i\bar{i}} (\mathbf{T})_{q\bar{q}} (\mathbf{T})_{j\bar{j}} \tilde{b}_{,\bar{p}\bar{i}\bar{q}\bar{j}}, \tag{3.95}$$

where

$$\tilde{b}_{,\bar{i}} = \begin{cases} \frac{\partial b}{\partial z} & \text{if } \bar{i} = 1, \\ \frac{\partial b}{\partial \bar{z}} & \text{if } \bar{i} = 2 \end{cases} \quad \text{and} \quad b_{,i} = \begin{cases} \frac{\partial b}{\partial x_1} & \text{if } i = 1, \\ \frac{\partial b}{\partial x_2} & \text{if } i = 2. \end{cases}$$

In particular, we have

$$b_{,11qj}(\mathbf{P}^0)_{qj} = c_{40} \frac{\partial^4 b(z, \bar{z})}{\partial z^4} + c_{04} \frac{\partial^4 b(z, \bar{z})}{\partial \bar{z}^4} + L \left( \frac{\partial^4 b}{\partial z^3 \bar{z}}, \frac{\partial^4 b}{\partial z^2 \bar{z}^2}, \frac{\partial^4 b}{\partial z \bar{z}^3} \right), \tag{3.96}$$

where  $L(\dots)$  denotes a linear combination of its arguments. We remark that  $L(\dots)$  is uniform on  $\Omega$  for a Vigdergauz structure since  $4 \frac{\partial^2 b}{\partial z \partial \bar{z}} = \Delta b = h$ , see (1.13). Also, by (3.1) and (3.95) the coefficients  $c_{40}$  and  $c_{04}$  can be written as

$$\begin{aligned}
c_{40} &= (\mathbf{T})_{11}(\mathbf{T})_{11}(\mathbf{T})_{q1}(\mathbf{T})_{j1}(\mathbf{P}^0)_{qj} = (\mathbf{T})_{q1}(\mathbf{T})_{j1}(\mathbf{P}^0)_{qj}, \\
c_{04} &= (\mathbf{T})_{22}(\mathbf{T})_{22}(\mathbf{T})_{q2}(\mathbf{T})_{j2}(\mathbf{P}^0)_{qj} = -(\mathbf{T})_{q2}(\mathbf{T})_{j2}(\mathbf{P}^0)_{qj}.
\end{aligned}$$

If the quantity in (3.96) is uniform on  $\Omega$ , we infer that

$$c_{40} \frac{\partial^4 b(z, \bar{z})}{\partial z^4} + c_{04} \frac{\partial^4 b(z, \bar{z})}{\partial \bar{z}^4} = (c_{40} + c_{04})u + i(c_{40} - c_{04})v \quad \text{is uniform on } \Omega, \quad (3.97)$$

where  $u$  (resp.  $v$ ) is the real (resp. imaginary) part of  $\frac{\partial^4 b(z, \bar{z})}{\partial z^4}$ . Also, from (3.94) we see that  $\frac{\partial^4 b(z, \bar{z})}{\partial z^4}$  is in fact analytic on  $\Omega$ , and hence  $u, v$  satisfy the Cauchy–Riemann Equation (3.2) on  $\Omega$ . If  $\mathbf{P}^0$  is symmetric, direct calculations show that  $c_{40}$  and  $c_{04}$  cannot simultaneously vanish unless  $\mathbf{P}^0$  is dilatational, i.e. a constant times the identity matrix. Therefore, if  $\mathbf{P}^0$  is symmetric but not dilatational, Equations (3.97) and (3.2) imply  $\frac{\partial^4 b(z, \bar{z})}{\partial z^4}$  would be uniform on  $\Omega$ , which, by (3.94) and similar calculations from (3.89) to (3.91), implies either (3.92) or (3.93) would hold. From the discussions following (3.93), we see that both (3.92) and (3.93) contradict the hypotheses of the theorem. This completes our proof of the theorem.  $\square$

#### 4. SUMMARY AND DISCUSSION

We have expressed the solution to the periodic Eshelby inclusion problem (1.8) in terms of Cauchy-type integrals. To be explicit, for given smooth divergence-free eigenstress  $\mathbf{P}^H : \Omega \rightarrow \mathbb{R}^{2 \times 2}$  in (1.19) we define  $f_p : \Gamma \rightarrow \mathbb{R}$  for  $p = 1, 2$  as (see (3.21))

$$\begin{bmatrix} f_1(s) \\ f_2(s) \end{bmatrix} = - \int_0^s \mathbf{P}^H \mathbf{n} ds_1.$$

Accordingly, the constants  $z_{11}^p$  and  $z_{23}^p$  can be calculated by (3.32) and (3.84), respectively. Then from (2.11), (3.33) and (3.86) the induced strain for the periodic homogeneous Eshelby inclusion problem (1.8) can be written as

$$[\nabla \mathbf{v}(\mathbf{x})]_{pi} = \frac{1}{\mu} \left[ \nabla u_1^{(p)}(\mathbf{x}) \right]_i - \frac{(\mu + \lambda)}{\mu(2\mu + \lambda)} \sum_{q=1}^n \left[ \nabla \nabla \nabla u_2^{(q)}(\mathbf{x}) \right]_{qpi}, \quad (4.1)$$

where for  $p = 1, 2$ ,

$$\frac{\partial u_1^{(p)}(z, \bar{z})}{\partial x_1} - i \frac{\partial u_1^{(p)}(z, \bar{z})}{\partial x_2} = \frac{1}{2\pi} \int_{\Gamma} f_p(t) \wp(t - z) dt + z_{11}^p \quad (4.2)$$

and

$$\left\{ \begin{array}{l} \frac{\partial^3 u_2^{(p)}}{\partial z^3} = 3z_{23}^p + \frac{\bar{z}}{16\pi} \int_{\Gamma} f_p(t) \frac{d\wp(t-z)}{dz} dt \\ \quad - \frac{1}{16\pi} \frac{d}{dz} \left[ \int_{\Gamma} \bar{t} f_p^{(1)}(t) \wp(t-z) dt - \int_{\Gamma} f_p^{(1)}(t) \sigma'(t-z) dt \right], \\ \frac{\partial^3 u_2^{(p)}}{\partial z^2 \partial \bar{z}} = \frac{1}{4} \frac{\partial u_1^{(p)}}{\partial z} = \frac{1}{16\pi} \int_{\Gamma} f_p(t) \wp(t-z) dt + \frac{1}{8} z_{11}^p, \\ \frac{\partial^3 u_2^{(p)}}{\partial z \partial \bar{z}^2} = \overline{\left( \frac{\partial^3 u_2^{(p)}}{\partial z^2 \partial \bar{z}} \right)}, \quad \frac{\partial^3 u_2^{(p)}}{\partial \bar{z}^3} = \overline{\left( \frac{\partial^3 u_2^{(p)}}{\partial z^3} \right)}. \end{array} \right. \quad (4.3)$$

The explicit representation formulas (4.1)–(4.3) on one hand imply a necessary and sufficient condition (see Theorem 5) for an inclusion to be a Vigdergauz structure, and on the other hand, enable us to show that the strain induced by a non-dilatational symmetric eigenstress cannot be uniform on a Vigdergauz structure (see Theorem 7).

We remark that much of the analysis in this paper are valid as long as the Cauchy principle values exist for almost all points on the boundary  $\Gamma$ . Therefore, the requirements of the contour  $\Gamma$  being smooth and the magnetization being smooth on  $\Omega$  can be relaxed considerably. For instance, the formulas (4.2) and (4.3) are valid if  $\Gamma$  is a piecewise smooth contour.

The lack of the Eshelby uniformity property of Vigdergauz structures prevents us from using the equivalent inclusion method [1, 2] to solve a generic *inhomogeneous* Eshelby inclusion problem (1.7) in a periodic setting. Nevertheless, under suitable conditions, we can solve an important class of inhomogeneous inclusion problems for which the equivalent eigenstress is dilatational for the homogeneous problem (1.8), see (1.15)–(1.18). This enables us to calculate the effective modulus of periodic composites with Vigdergauz structures and to show they are optimal microstructures in certain sense, see Grabovsky and Kohn (1995) and Liu et al (2008). In the meantime, it is interesting to extend the approach in this paper to solve inhomogeneous inclusion problems in both periodic and nonperiodic settings. The Plemelj formulas and the calculations in this paper will be useful.

## APPENDIX. VERIFICATION OF EQUATION (3.72)

Equation (3.72) follows from the following property of  $\sigma'(z)$  (see (3.66)):

$$\sigma'(z + \omega_i) = \sigma'(z) + \bar{\omega}_i \wp(z) + S_i \quad i = 1, 2, \quad (A.1)$$

where the  $z$ -independent constants

$$S_i = -\frac{\bar{\omega}_i}{\omega_i^2} + \sum_{\omega \in \mathcal{L} \setminus \{\omega_i, 0\}} \frac{\omega_i^3 \bar{\omega} - 3|\omega|^2 \omega_i^2}{\omega^2 (\omega - \omega_i)^3}. \quad (A.2)$$

To show this, by (3.62) we have

$$\begin{aligned}
\sigma'(z + \omega_i) &= \sum_{\omega \in \mathcal{L} \setminus \{0\}} \bar{\omega} \left[ -\frac{1}{(z + \omega_i + \omega)^2} + \frac{1}{\omega^2} - \frac{2(z + \omega_i)}{\omega^3} \right] \\
&= \sum_{\omega' \in \mathcal{L} \setminus \{\omega_i\}} (\bar{\omega}' - \bar{\omega}_i) \left[ -\frac{1}{(z + \omega')^2} + \frac{1}{(\omega' - \omega_i)^2} - \frac{2(z + \omega_i)}{(\omega' - \omega_i)^3} \right], \quad (\text{A.3})
\end{aligned}$$

where  $\omega' = \omega + \omega_i$ . Also,

$$\sigma'(z) = \sum_{\omega' \in \mathcal{L} \setminus \{0\}} \bar{\omega}' \left[ -\frac{1}{(z + \omega')^2} + \frac{1}{\omega'^2} - \frac{2z}{\omega'^3} \right]. \quad (\text{A.4})$$

Since both series in (A.4) and (A.3) converge absolutely, subtracting (A.4) from (A.3) term by term for  $\omega' \in \mathcal{L} \setminus \{0, \omega_i\}$ , we are left with

$$\begin{aligned}
\sigma'(z + \omega_i) - \sigma'(z) &= \bar{\omega}_i \left[ \frac{1}{z^2} - \frac{1}{\omega_i^2} + \frac{2(z + \omega_i)}{(-\omega_i)^3} \right] \\
&+ \sum_{\omega' \in \mathcal{L} \setminus \{\omega_i, 0\}} \bar{\omega}_i \left[ \frac{1}{(z + \omega')^2} - \frac{1}{(\omega' - \omega_i)^2} + \frac{2(z + \omega_i)}{(\omega' - \omega_i)^3} \right] \\
&+ \sum_{\omega' \in \mathcal{L} \setminus \{\omega_i, 0\}} \bar{\omega}' \left[ \frac{1}{(\omega' - \omega_i)^2} - \frac{2(z + \omega_i)}{(\omega' - \omega_i)^3} - \frac{1}{\omega'^2} + \frac{2z}{\omega'^3} \right] \\
&+ \bar{\omega}_i \left[ \frac{1}{(z + \omega_i)^2} - \frac{1}{\omega_i^2} + \frac{2z}{\omega_i^3} \right] \\
&= \bar{\omega}_i \varphi(z) + L_i(z), \quad (\text{A.5})
\end{aligned}$$

where

$$\begin{aligned}
L_i(z) &= \bar{\omega}_i \left[ -\frac{1}{\omega_i^2} + \frac{2(z + \omega_i)}{(-\omega_i)^3} \right] + \sum_{\omega' \in \mathcal{L} \setminus \{\omega_i, 0\}} \bar{\omega}_i \left[ \frac{1}{\omega'^2} - \frac{1}{(\omega' - \omega_i)^2} + \frac{2(z + \omega_i)}{(\omega' - \omega_i)^3} \right] \\
&+ \sum_{\omega' \in \mathcal{L} \setminus \{\omega_i, 0\}} \bar{\omega}' \left[ \frac{1}{(\omega' - \omega_i)^2} - \frac{2(z + \omega_i)}{(\omega' - \omega_i)^3} - \frac{1}{\omega'^2} + \frac{2z}{\omega'^3} \right] + \bar{\omega}_i \frac{2z}{\omega_i^3}. \quad (\text{A.6})
\end{aligned}$$

Clearly,  $L_i(z)$  is a linear function of  $z$ . The coefficient associated with  $z$  is

$$\begin{aligned}
\frac{dL_i(z)}{dz} &= -\frac{2\bar{\omega}_i}{\omega_i^3} + \sum_{\omega \in \mathcal{L} \setminus \{\omega_i, 0\}} \frac{2\bar{\omega}_i}{(\omega - \omega_i)^3} + \sum_{\omega \in \mathcal{L} \setminus \{\omega_i, 0\}} \bar{\omega} \left[ -\frac{2}{(\omega - \omega_i)^3} + \frac{2}{\omega^3} \right] + \frac{2\bar{\omega}_i}{\omega_i^3} \\
&= \frac{2\bar{\omega}_i}{\omega_i^3} + \sum_{\omega \in \mathcal{L} \setminus \{\omega_i, 0\}} \bar{\omega} \left[ -\frac{2}{(\omega - \omega_i)^3} + \frac{2}{\omega^3} \right], \quad (\text{A.7})
\end{aligned}$$



where we have noticed

$$\frac{2\bar{\omega}_i}{\omega_i^3} + \sum_{\omega' \in \mathcal{L} \setminus \{\omega_i, 0\}} \frac{-2\bar{\omega}_i}{(\omega' - \omega_i)^3} = 0 \quad \text{or} \quad \frac{2\bar{\omega}_i}{\omega_i^3} + \sum_{\omega' \in \mathcal{L} \setminus \{\omega_i, 0\}} \frac{2\bar{\omega}_i}{(\omega')^3} = 0. \quad (\text{A.8})$$

Let  $\omega' = \omega_i - \omega$ . The right-hand side of (A.7) can be written as

$$\begin{aligned} & \frac{2\bar{\omega}_i}{\omega_i^3} + \sum_{\omega' \in \mathcal{L} \setminus \{\omega_i, 0\}} \overline{\omega_i - \omega'} \left[ -\frac{2}{(-\omega')^3} + \frac{2}{(\omega_i - \omega')^3} \right] \\ &= \frac{2\bar{\omega}_i}{\omega_i^3} + \sum_{\omega' \in \mathcal{L} \setminus \{\omega_i, 0\}} \left[ \frac{2\bar{\omega}_i}{(\omega')^3} - \frac{2\bar{\omega}_i}{(\omega' - \omega_i)^3} \right] - \sum_{\omega' \in \mathcal{L} \setminus \{\omega_i, 0\}} \bar{\omega}' \left[ \frac{2}{(\omega')^3} - \frac{2}{(\omega' - \omega_i)^3} \right] \\ &= -\frac{2\bar{\omega}_i}{\omega_i^3} - \sum_{\omega' \in \mathcal{L} \setminus \{\omega_i, 0\}} \bar{\omega}' \left[ \frac{2}{(\omega')^3} - \frac{2}{(\omega' - \omega_i)^3} \right] = -\frac{dL_i(z)}{dz}, \end{aligned} \quad (\text{A.9})$$

From (A.7) and (A.9) we have

$$\frac{dL_i(z)}{dz} = -\frac{dL_i(z)}{dz} \implies \frac{dL_i(z)}{dz} = 0. \quad (\text{A.10})$$

Next we calculate the constant term in  $L_i(z)$ . From (A.6), (A.8) and (A.10) we have

$$\begin{aligned} L_i(z) &= \bar{\omega}_i \left[ -\frac{1}{\omega_i^2} + \frac{2\omega_i}{(-\omega_i)^3} \right] + \sum_{\omega \in \mathcal{L} \setminus \{\omega_i, 0\}} \bar{\omega}_i \left[ \frac{1}{\omega^2} - \frac{1}{(\omega - \omega_i)^2} + \frac{2\omega_i}{(\omega - \omega_i)^3} \right] \\ &+ \sum_{\omega \in \mathcal{L} \setminus \{\omega_i, 0\}} \bar{\omega} \left[ \frac{1}{(\omega - \omega_i)^2} - \frac{1}{\omega^2} - \frac{2\omega_i}{(\omega - \omega_i)^3} \right] \\ &= -\frac{\bar{\omega}_i}{\omega_i^2} + \sum_{\omega \in \mathcal{L} \setminus \{\omega_i, 0\}} \frac{\omega_i^3 \bar{\omega} - 3|\omega|^2 \omega_i^2}{\omega^2 (\omega - \omega_i)^3} = S_i. \end{aligned} \quad (\text{A.11})$$

where we have used

$$\begin{aligned} T_i &:= \sum_{\omega \in \mathcal{L} \setminus \{\omega_i, 0\}} \bar{\omega}_i \left[ \frac{1}{\omega^2} - \frac{1}{(\omega - \omega_i)^2} + \frac{2\omega_i}{(\omega - \omega_i)^3} \right] \\ &= \sum_{\omega' \in \mathcal{L} \setminus \{\omega_i, 0\}} \bar{\omega}_i \left[ \frac{1}{(\omega' - \omega_i)^2} - \frac{1}{\omega'^2} + \frac{-2\omega_i}{\omega'^3} \right] \\ &= -T_i + \sum_{\omega \in \mathcal{L} \setminus \{\omega_i, 0\}} \bar{\omega}_i \left[ \frac{2\omega_i}{(\omega - \omega_i)^3} + \frac{-2\omega_i}{\omega^3} \right] = -T_i + \frac{4|\bar{\omega}_i|^2}{\omega_i^3}. \end{aligned}$$

Equation (A.1) follows from (A.5) and (A.11).

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