SOME REMARKS ON ISING-SPIN SYSTEMS

G. GALLAVOTTI and J. L. LEBOWITZ*

Belfer Graduate School of Science, Yeshiva University,
New York, N.Y. 10033, USA

Received 18 June 1973

Synopsis

We show that: (a) the free energy and correlation functions of the two-dimensional Ising-spin system with nearest-neighbour ferromagnetic interactions, remain infinitely differentiable with respect to \( \beta \) and \( h \) as \( \beta \to 0^\pm \) for \( \beta > \beta_c \) (where \( \beta_c \) is the reciprocal of the critical temperature) and, (b) the equilibrium equations for the correlation functions of Ising-spin systems may admit a non-physical solution even in the region, \( \beta < \beta_c \), where they are known to have a unique physical solution.

1. Proof of (a). Consider an Ising-spin system with ferromagnetic pair interactions in a domain \( A \subset \mathbb{Z}^2 \). We shall denote by \( ' + ' \) the boundary condition in which all spins in \( \mathbb{Z}^2 \backslash A \) are \(+1\). Let \( u_2 (x, y; \beta, h, A, +) \) be the pair correlation:

\[
\langle \sigma_x \sigma_y \rangle = \langle \sigma_x \sigma_y \rangle_p(\beta)
\]

for this system, \( x, y \in A \). The argument used in ref. 1 (employing the Griffiths, Hurst and Sherman inequality) then shows that when the magnetic field \( h \) is in the up direction then

\[
u_2 (x, y; \beta, h, A, +) \leq u_2 (x, y; \beta, h, +) \leq u_2 (x, y; \beta, h = 0, +),
\]

where \( u_2 (x, y; \beta, h, +) = \lim_{A \to \infty} u_2 (x, y; \beta, h, A, +) \), the limit being approached monotonically.

We now observe that for the two-dimensional system, \( v = 2 \), with nearest-neighbour attractive interactions, it was shown in ref. 4 that in the infinite-volume limit \( \langle \sigma_x \sigma_y \rangle_p(\beta) = \langle \sigma_x \sigma_y \rangle_p(\beta) \); here \( p \) indicates periodic (or cylindrical) boundary conditions, and the equality holds for all \( \beta \) even when \( h = 0 \). (For \( h \neq 0 \) or \( \beta < \beta_c \), the result was already known before.) Furthermore in ref. 4 it is also shown that \( \lim_{|x-y| \to \infty} \langle \sigma_x \sigma_y \rangle_p = \langle \sigma_x \rangle^2 \). It follows then from the explicit compu-

REFERENCES

2. 487.
3. 469.
4. 40 (1972) 469.
5. 31.

* Supported in part by the A.F.O.S.R. Grant 73-4240.
tation of Wu\(^5\)) that the right side of (1) has an exponential decay\(^1\):

\[ u_2 (x, y; \beta, h = 0, +) \leq \text{const. exp}\ - K| x - y| \]

with \(K > 0\) for \(\beta > \beta_c\). This in turn implies infinite differentiability by the arguments given in ref. 1. (We note here that Martin-Löf obtained the bound (2) directly and communicated it to us prior to our result.)

Actually in ref. 5 the author deals with the case when \(x\) and \(y\) are on the same horizontal or vertical line; the general case follows from a careful examination of the spectrum of the transfer matrix and it is particularly easy to obtain if one content with a weak estimation of the form

\[| \langle \sigma_x \sigma_y \rangle - \langle \sigma_x \rangle^2 | \leq \text{const. exp}\ - \frac{1}{2} K | x - y|, \]

where \(K\) is the horizontal or vertical correlation length.

2. Proof of (b). To prove (b) we consider a one-dimensional system with nearest-neighbour interaction, \(A = [-L, L]\), and 'open' boundary conditions corresponding to no interactions with spins outside \(A\). The Hamiltonian of the system, for \(h = 0\), then is \(H_0(\sigma) = -\sum_{i=1}^{L-1} \sigma_i \sigma_{i+1}\). Let \(H_0(\sigma) - (i\pi/2)_L \times (\sigma_{-L} + \sigma_L)\). We shall denote with a subscript \(L, 0\) or \(L, i\) the average obtained by using \(e^{-\beta H_0}\) or \(e^{-\beta H_1}\) as weights and by a subscript 0 or \(i\) we shall mean the limit as \(L \to \infty\), of the corresponding quantities with subscript \(L, 0\) or \(L, i\).

If \(x_1 < x_2 < \cdots < x_n\) a simple computation leads to the following result:

\[ \langle \sigma_{x_1} \cdots \sigma_{x_{2n+1}} \rangle = \langle \sigma_{x_1} \cdots \sigma_{x_{2n+1}} \rangle_0, \]

\[ \langle \sigma_{x_1} \cdots \sigma_{x_{2n}} \rangle = \prod_{j=1}^{2n-1} \langle \sigma_{x_j} \sigma_{x_{j+1}} \rangle, \]

\[ \langle \sigma_{x_1} \cdots \sigma_{x_2} \rangle = \prod_{j=1}^{2n-1} \langle \sigma_{x_j} \sigma_{x_{j+1}} \rangle_0. \]

Furthermore it is easy to check that:

\[ \langle \sigma_1 \sigma_2 \rangle = \lim_{L \to \infty} \frac{\langle \sigma_1 \sigma_2 \rangle_{L, 0}}{\langle \sigma_L \sigma_L \rangle_{0, 0}} = \frac{1}{\langle \sigma_0 \sigma_0 \rangle_0}. \]

hence \(\langle \sigma_1 \sigma_2 \rangle \geq 1\) and therefore \(\langle \sigma_1 \sigma_2 \rangle\) cannot correspond to a physically acceptable state. It is, however, easy to see from the definition:

\[ \langle \sigma_{x_1} \sigma_{x_2} \cdots \rangle = \lim_{L \to \infty} \frac{\sum_{\sigma} (\sigma_{x_1} \sigma_{x_2} \cdots) \exp -\beta H_1 (\sigma)}{\sum_{\sigma} \exp -\beta H_1 (\sigma)} \]
SOME REMARKS ON ISING-SPIN SYSTEMS

that the \( \langle \sigma_X \rangle \), define a family of local distributions \( f_A(X) \) which verify the equilibrium equations (6) as well as the compatibility and normalization conditions (7) that they would have to satisfy if they came from a probability measure on the space of the spin configurations.

Notice that, since \( \langle \sigma_x \sigma_y \rangle_0 (\beta) = (\text{th} \beta)^{|x-y|} \) the functions \( \langle \sigma_x \sigma_y \rangle_t (\beta) \) are singular around \( \beta = 0 \) which explains why they cannot be obtained by the usual perturbative expansions around \( \beta = 0 \).

It could be directly checked that the Kirkwood–Salsburg equations in zero field i.e. in general, invariant under the transformation \( J_{ij} \rightarrow J_{ij} + \text{i} \epsilon/2 \beta \) (this is because only \( \exp(-4\beta J_{ij}) \) enters into the KS equations) and this remark, applied to our case, could be used to provide a simple direct proof that the correlation functions \( \langle \sigma_X \rangle_t \) are a solution to the KS equations [one merely notices that \( \text{th}(\beta + \frac{1}{2} \text{i}\epsilon) = 1/\text{th} \beta \)].

REFERENCES


* If \( X = (x_1, x_2, \ldots, x_p) \) the functions \( f_A(X) \) are the "probabilities" for finding, inside \( A \), spins up in the points \( x_1 \ldots x_p \) and spins down in the remaining points; i.e.,

\[
 f_A(X) = \left( \frac{\sigma + 1}{2} \right)_{x \in A} \left( \frac{1 - \sigma}{2} \right)_{x \in A^c} .
\]

Also \( \langle \sigma_X \rangle = \left( \frac{1}{P} \sum_{i=1}^{P} \sigma_{x_i} \right) \).