Ergodic properties of simple model system with collisions\textsuperscript{*}

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We investigate the ergodic properties of the discrete time evolution of a particle in a two-dimensional torus with velocity in the unit square. The dynamics consists of free motion for a unit time interval followed by a baker's transformation of the velocity.

1. INTRODUCTION

We are interested in the ergodic properties of dilute gas systems. These may be thought of as Hamiltonian dynamical systems in which the particles move freely except during binary "collisions". In a collision the velocities of the colliding particles undergo a transformation with "good" mixing properties (cf. Sinai's study of the billiard problem\textsuperscript{3}). To gain an understanding of such systems we have studied the following simple discrete time model: The system consists of a single particle with coordinate $\mathbf{x} = (x, y)$ in a two-dimensional torus with sides of length $(L_x, L_y)$, and "velocity" $v = (v_x, v_y)$, in the unit square $v_x \in [0, 1], v_y \in [0, 1]$. The phase space $T$ is thus a direct product of the torus and the unit square. The transformation $T$ which takes the system from a dynamical state $(x, v)$ at time $j$ to a new dynamical state $(x', v')$ at time $j+1$ may be pictured as resulting from the particle moving freely during the unit time interval between $j$ and $j+1$ and then undergoing a "collision" in which its velocity changes according to the baker's transformation, i.e.,

$$T(v, x) = (x + v, Bv),$$

with

$$B(v, x) = \begin{cases} (2v_x, 1), & 0 \leq v_x < \frac{1}{2} \\ (2v_x - 1, 0), & \frac{1}{2} \leq v_x \leq 1. \end{cases}$$

The normalized Lebesgue measure $d\mu = dx dy dv$, $L_x L_y = dx dv L_x L_y$ in $T$ is left invariant by $T$. We call $U_T$ the unitary transformation induced by $T$ on $L^2(d\mu)$. Our interest lies then in the ergodic properties of $T$ and in the spectrum of $U_T$.

We note first that the transformation $T$ on the velocities is, when taken by itself as a transformation of the unit square with measure $d\mu$, well known to be isomorphic to a Bernoulli shift. It therefore has very good mixing properties. The isomorphism is obtained by setting

$$v_x = \sum_{j=-\infty}^{\infty} 2^{-j} u_{x, j}, \quad v_y = \sum_{j=-\infty}^{\infty} 2^{-j} u_{y, j},$$

with the $u_{x, j}$ independent random variables taking the values 0 and 1 each with probability $\frac{1}{2}$. We then have

$$(Bv)_x = \sum_{j=-\infty}^{\infty} 2^{-j} u_{x, j+1} + 2v_x - u_1,$$

$$(Bv)_y = \sum_{j=-\infty}^{\infty} 2^{-j} u_{y, j} + \frac{1}{2} v_y + \frac{1}{2} u_1,$$

2. ERGODIC PROPERTIES

The ergodic properties of our system which combines $B$ with free motion turn out to depend on whether $L_x^2$ and $L_y^4$ satisfy the independence condition (i),

$$n_x L_x^{-1} + n_y L_y^{-1} \notin Z$$

for $n_x$ and $n_y$ integers unless $n_x = n_y = 0$. (1)

Theorem 1: When (i) holds, the spectrum of $U_T$ on the complement of the one-dimensional subspace generated by the constants, is absolutely continuous with respect to Lebesgue measure and has infinite multiplicity.

It follows from Theorem 1 that when (i) holds the dynamical system $(T, \mu)$ is at least mixing. We do not know at present whether it is also a Bernoulli shift or at least a $K$ system.

Theorem 2: When (i) does not hold the system $(T, \mu)$ is not ergodic.

The proof of Theorem 1 has two parts: a general characterization of unitary operators with Lebesgue spectrum and a set of estimates.

Lemma: Let $U$ be a unitary operator on a Hilbert space $H$, with spectral representation $U = \int_{\hat{X}} \hat{U} d\mu(\hat{U})$. Assume that there exists a total set of vectors $\{\phi_i\}$ such that $\sum_{i=1}^{\infty} |\langle \phi_i | \phi \rangle|^2 < \infty$ for all $\phi$. (A set of vectors is said to be total if the finite linear span of this set of vectors is dense.) Then the spectral measure $P(\hat{U})$ is absolutely continuous with respect to Lebesgue measure, i.e., if $E$ is a Borel set of Lebesgue measure 0, then $P(E) = 0$.

Proof: We have

$$|\langle \hat{U} \phi | \phi \rangle| = |\int_{\hat{X}} \langle \phi | \hat{U} \phi \rangle d\mu(\hat{U})|,$$

so the numerical measure $P(\hat{U})| \phi \rangle = \int_{\hat{X}} d\mu(\hat{U})| \phi \rangle$, so the numerical measure $P(\hat{U})| \phi \rangle$ is absolutely continuous with respect to Lebesgue measure. If $E$ is a Borel set of Lebesgue measure 0,

$$\|P(E)| \phi \rangle\|^2 = |P(E)| \phi \rangle| \phi \rangle = 0,$$

$P(E)| \phi \rangle = 0$ for all $\phi$. 

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But the vectors \( \{\varphi_j\} \) form a total set, so \( P(E) = 0 \) as desired.

Now the estimates: Let \( \chi(1) = 1, \chi(0) = -1 \). For each finite subset \( X \) of \( Z \), we define

\[
X_k(u) = \prod_{j \in X} \chi(u_j).
\]

The \( X_k \) form an orthonormal basis for \( L^2(d\mu) \). Similarly, the functions \( \exp(k \cdot \eta \eta) \cdot \varphi(x) = \varphi(x - k \cdot \eta) \), form an orthonormal basis for \( L^2(d\eta) \). Thus, the functions \( \varphi_{x,k}^\eta = \exp(k \cdot \eta) \cdot \varphi(x) \) form an orthonormal basis for \( L^2(d\mu) \). We will prove that

\[
\sum_{n=1}^\infty \langle U_\eta \varphi_{x,1,\eta}, \varphi_{x,2,\eta} \rangle < \infty \quad \text{unless} \quad k_1 = k_2 = 0,
\]

\[
X_1 = X_2 = 0.
\]

By straightforward computation,

\[
U_\eta \varphi_{x,1,\eta}(y) = \varphi_{x,1,\eta}(y) \exp(i k \cdot \eta) \times \exp(i B \mu(y) \cdot k_1 B \mu(y) = \sum_{i=1}^\infty u_j, 2^{-i}, \]

so we assume \( k_1 = k_2 = 0 \). Also,

\[
U_\eta \varphi_{x,0,\eta}(y) = 0 \quad \text{unless} \quad X_2 = X_1 + n,
\]

so the result is trivially true for \( h = 0 \). We therefore assume \( k = 0 \).

Now

\[
(L_x L_x)^{-1} \int_{-\infty}^{\infty} \langle U_\eta \varphi_{x,1,\eta}, \varphi_{x,2,\eta} \rangle \, dx = \int_{-\infty}^{\infty} \langle U_\eta \varphi_{x,1,\eta}, \varphi_{x,2,\eta} \rangle \, dx \exp(i k \cdot \eta) = \sum_{i=1}^\infty u_j \cdot 2^{-i},
\]

and \( \alpha_\eta \) defined as follows:

\[
B \mu \eta \sum_{i=0}^{n-1} u_j \cdot 2^{-i} = \sum_{i=1}^\infty \sum_{i=1}^\infty \sum_{i=1}^\infty u_j \cdot 2^{-i}.
\]

Hence, this equation defines \( \alpha_\eta \).

\[
B \mu \eta \sum_{i=0}^{n-1} v_j \cdot 2^{-i} = \sum_{i=0}^\infty \sum_{i=1}^\infty u_j \cdot 2^{-i}.
\]

Now let \( L_2 = \max(X_2), L_1 = \inf(X_1) \wedge 0 \).

Then

\[
\sum_{i=0}^{n-1} \langle U_\eta \varphi_{x,1,\eta}, \varphi_{x,2,\eta} \rangle = \sum_{i=1}^\infty \sum_{i=1}^\infty \exp(i k \cdot \eta) = \sum_{i=1}^\infty \sum_{i=1}^\infty 2^{-i}.
\]

By independence, the integral of the product on the right is the product of the integrals, and the unspecified function of the \( u_j \)'s, \( i \in (l_0, n + l_1) \) is no greater than one in absolute value, so

\[
\int d\mu \cdot (\sum_{i=0}^{n-1} B \mu \eta) = \int d\mu \cdot (\sum_{i=0}^{n-1} B \mu \eta) = \sum_{i=0}^{n-1} B \mu \eta = \sum_{i=0}^{n-1} B \mu \eta.
\]

For \( i \)'s within the limits of the product, we have

\[
\alpha_\eta = \sum_{i=1}^\infty \beta \mu \eta = \sum_{i=1}^\infty \beta \mu \eta.
\]

Thus, for most of the terms in the product, \( \alpha_\eta \approx \beta_\eta \approx 1 \), and the number of terms is \( n - \text{const for large } n \). In particular, if we put

\[
\gamma = \frac{1}{\beta} \exp(i k \cdot \eta) + 1 \quad \text{for all sufficiently large } n,
\]

we have

\[
\sum_{i=0}^{n-1} \langle U_\eta \varphi_{x,1,\eta}, \varphi_{x,2,\eta} \rangle < \gamma^\mu \quad \text{as desired.}
\]

The fact that the multiplicity is infinite is trivial. We have \( L^2(d\mu) \subset L^2(d\eta \mu) \), and we already know that the spectrum of \( U_\eta \) restricted to \( L^2(d\mu) \) has infinite multiplicity.

To obtain a proof of Theorem 2, we note that ergodicity is equivalent to

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} \langle U_\eta \varphi, \varphi \rangle = \langle \langle U_\eta \varphi, \varphi \rangle \rangle = \langle d\mu \varphi \rangle = \langle d\mu \varphi \rangle.
\]

For \( \varphi \) or \( \psi \) orthogonal to the constants we must then have Cesaro convergence to zero when the system is ergodic. We prove that the system is nonergodic by finding \( \varphi \) or \( \psi \) orthogonal to the constants such that the above integral converges (strictly) to a nonzero number.

Let \( n_x, n_y \) be such that \( n_x / L_x + n_y / L_y \in Z \) and \( n_y \) and \( n_x \) are not both \( 0 \), and let \( k_x = 2m_x / L_x, k_y = 2m_y / L_y \). We set \( \varphi = \psi = \varphi_{0,k}, \psi_{0,k} \) and compute as before the relevant integrals:

\[
I_x = \int d\mu \cdot (\sum_{i=0}^{n-1} B \mu \eta) = \int d\mu \cdot (\sum_{i=0}^{n-1} B \mu \eta) = \int d\mu \cdot (\sum_{i=0}^{n-1} B \mu \eta) = \int d\mu \cdot (\sum_{i=0}^{n-1} B \mu \eta).
\]

Here

\[
\alpha_\eta = \sum_{i=1}^\infty \beta \mu \eta = \sum_{i=1}^\infty \beta \mu \eta = \sum_{i=1}^\infty \beta \mu \eta.
\]

For \( i > 0 \) and vanishes for \( i = 0 \), and

\[
\beta_\eta = \sum_{i=1}^\infty \beta \mu \eta = \sum_{i=1}^\infty \beta \mu \eta = \sum_{i=1}^\infty \beta \mu \eta.
\]

for \( i > n \) and vanishes for \( i \geq n \).

We thus have found that
\[
I_n = \prod_{i=-\infty}^0 \left[ 1 + \exp(i(2^i - 2^{-i})k_x) \right] \\
\times \prod_{i=1}^{n-1} \left[ 1 + \exp(i((1 - 2^{-i})k_x + (1 - 2^{-i})k_y)) \right] \\
\times \prod_{i=n}^\infty \left[ 1 + \exp(i(k_x(2^{-i} - 2^{-(n+i)} - 2^{-i})) \right] \\
= F_n^1(k) F_n^2(k) F_n^3(k)
\]
with
\[
F_n^1(k) = F_n^2(k) = \prod_{m=1}^{\infty} \left[ 1 + \exp(i k_x (2^{-m} - 2^{-(m+n)}) \right], \\
F_n^3(k) = F_n^4(k) = F_n^5(k).
\]

Since \( k_x + k_y \in 2\pi Z \), we have
\[
F_n^2(k) = \prod_{i=1}^{n-1} \left[ 1 + \exp(i((1 - 2^{-i})k_x + (1 - 2^{-i})k_y)) \right].
\]

We now assert that (for \( k_x + k_y \in \pi Z \))
\[
\lim_{n \to \infty} F_n^i(k) = \alpha^i = 0, \quad i = 1, 2, 3.
\]

This is verified by observing that the log \( F_n^i(k) \) coverage to a finite limit, thus completing the proof.

(If \( k_x \) and \( k_y \) are such that some of the terms at the beginning of the series which one obtains from the log \( F_n^i(k) \) are singular, one easily removes the difficulty by an appropriate change in the functions \( \phi \) and \( \psi \) introduced at the beginning of the proof of Theorem 2. We also note that for the case where \( L_x/L_y \) is rational we can find explicitly a nonconstant function \( f \) which is left invariant by \( U_x \). From the fact that \( U_y(u_x + 2v_y) = 2u_x + v_y \), it follows that \( f(x - y - v_x - 2v_y) \) is invariant if \( f \) is doubly periodic with periods \( L_x \) and \( L_y \), so that we can construct an infinite family of orthonormal invariant functions \( f_n : f_n = \exp((2\pi i n/L)(x - y - v_x - 2v_y)) \) with \( L_x/r = L_y/s = L, r \) and \( s \) integers.)

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