Linear transformations

Throughout this note, $V$, $W$, and $Z$ are vector spaces over the same field $F$.

Definition. A linear transformation from $V$ to $W$ is a function $T : V \to W$ that preserves linear combinations:

$$T(c_1 \alpha_1 + c_2 \alpha_2) = c_1 T(\alpha_1) + c_2 T(\alpha_2) \quad \text{for all } \alpha_1, \alpha_2 \in V \text{ and } c_1, c_2 \in F.$$ 

We usually perform two separate tests:

$T$ should preserve vector-addition: $T(\alpha + \beta) = T(\alpha) + T(\beta)$ for all $\alpha, \beta \in V$, and $T$ should preserve scalar multiplication: $T(c \cdot \alpha) = c \cdot T(\alpha)$ for all $\alpha \in V$ and $c \in F$.

Theorem 1 (Preservation of subspaces). Let $T : V \to W$ be linear, and let $A$ be a subspace of $V$. Then $T(A) := \{T(x) : x \in A\}$ is a subspace of $W$.

Let $B$ be a subspace of $W$. Then $T^{-1}(B) := \{x \in V : T(x) \in B\}$ is a subspace of $V$.

Definition. The range of $T$ is the set $R(T) = \text{Range}(T) = \text{Im}(T) = \{T(x) : x \in V\}$. The rank of $T$ is the dimension of the range of $T$. The null space (or kernel) of $T$ is the set $N(T) = \text{Null}(T) = \text{Ker}(T) = \{x \in V : T(x) = 0\}$. The nullity of $T$ is the dimension of $N(T)$.

(In view of the theorem above, the range $R(T) = T(V)$ is a subspace of $W$ and the null space $N(T) = T^{-1}(\{0\})$ is a subspace of $V$, so they have dimensions.)

Here are some easy but important facts. For every linear transformation $T : V \to W$,

- $T(0) = 0$ (note the two different 0-vectors: one is in $V$, one in $W$).
- Spanning sets of $V$ (e.g., bases of $V$) are mapped into spanning sets of $R(T)$.
  Hence rank($T$) $\leq$ dim($V$). (Clearly, rank($T$) $\leq$ dim($W$) also holds.)
- In general, for any subspace $U$ of $V$, spanning sets of $U$ are mapped into spanning sets of $T(U)$. Hence dim($T(U)$) $\leq$ dim($U$): “linear transformations cannot increase dimensions!”
- $T$ can be defined arbitrarily on a basis of $V$, and then it is uniquely determined on the whole domain $V$. Consequently, if $T$ and $T'$ are linear transformations from $V$ to $W$, and they agree on a basis of $V$, then they agree everywhere on $V$.
- If $f : V \to W$ and $g : W \to Z$ are linear transformations, their composition $g \circ f : V \to Z$, defined as $(g \circ f)(\alpha) = g(f(\alpha))$ ($\alpha \in V$), is also a linear transformation.

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Theorem 2 (FTLA). Let $T$ be a linear transformation from $V$ to $W$. Then $V$ has a basis that can be written as $A \cup B$, where $A$ and $B$ are disjoint, $A$ is a basis for the null space $N(T)$, and $T(B)$ is a basis for the range $R(T)$. Hence, if $V$ is finite-dimensional, then $\text{rank}(T) + \text{nullity}(T) = \text{dim}(V)$.

Theorem 3 (FTLA – matrix form). Let $M$ be an $m \times n$ matrix over the field $F$. The row space of $M$ is orthogonal to the null space of $M$ (of course), and their dimensions add up to $n$. The row space and the column space have the same dimension (the rank of $M$).
Invertible linear transformations

Throughout this page, \( T \) is a linear transformation from \( V \) to \( W \).

**Definition.** A function \( f : V \to W \) is invertible if there exists a function \( g : W \to V \) such that \( g \circ f = I_V \) and \( f \circ g = I_W \). (\( I_V \) is the identity transformation on \( V \).)

It is easy to see that \( f \) is invertible if and only if it is one-to-one and onto (bijection). It is also easy to see that when such \( g \) exists, it is unique. We usually write \( f^{-1} \) for this unique inverse \( g \). When \( f \) is an invertible linear transformation, then so is \( f^{-1} \) (from \( W \) to \( V \)).

**Definition.** The linear transformation \( T : V \to W \) is non-singular if \( \text{Null}(T) = \{0\} \).

(That is, \( \text{nullity}(T) = 0 \), or \( \alpha \neq 0 \) implies \( T(\alpha) \neq 0 \), or \( T(\alpha) = 0 \) implies \( \alpha = 0 \).)

**Theorem 4.** Let \( T : V \to W \) be a linear transformation.

The Following Are Equivalent

- \( T \) is non-singular.
- \( T \) is one-to-one.
- \( T \) maps linearly independent vectors into linearly independent vectors.
- For every finite-dimensional subspace \( U \) of \( V \), \( \dim(T(U)) = \dim(U) \).

**Theorem 5.** Let \( T : V \to W \) be a linear transformation, and assume that \( V \) has a basis.

The Following Are Equivalent

- \( T \) is one-to-one.
- \( T \) maps every basis into linearly independent vectors.
- \( V \) has a basis which \( T \) maps into linearly independent vectors.

**Theorem 6.** If \( \dim(V) < \infty \), then \( T \) is one-to-one if and only if \( \text{rank}(T) = \dim(V) \).

If \( \dim(W) < \infty \), then \( T \) is onto if and only if \( \text{rank}(T) = \dim(W) \).

**Corollary.** Let \( V \) and \( W \) be finite-dimensional, and assume \( \dim(V) = \dim(W) \).

The Following Are Equivalent

- \( T \) is one-to-one.
- \( T \) is onto.
- \( T \) is invertible.
- \( T \) maps some basis of \( V \) into a basis of \( W \).
- \( T \) maps every basis of \( V \) into a basis of \( W \).

Both assumptions in the corollary are important. Without them, the equivalences may be false even in the case \( V = W \) (\( T \) is a linear operator on \( V \)), as the following example shows:

**Example:** Let \( V = W = F^N = \mathcal{F}(\mathbb{N}, F) \), the space of all \( F \)-sequences. Let \( \text{LS} \) and \( \text{RS} \) be the left-shift and right-shift operators on \( V \):

\[
\text{LS}(x_1, x_2, \ldots) = (x_2, x_3, \ldots), \quad \text{and} \quad \text{RS}(x_1, x_2, x_3, \ldots) = (0, x_1, x_2, \ldots).
\]

Clearly, \( \text{LS} \) is onto but not one-to-one, while \( \text{RS} \) is one-to-one but not onto.