Subgroup diagrams!

Subgroup diagrams are a way of visualizing the subgroup structure of a group. At the top is the group, at the bottom is the trivial group \( ⟨e⟩ \), and in between are all the subgroups. We draw a line labeled \( n \) from \( G \) down to \( H \) if \( H \) is a subgroup of \( G \) of index \( n \), like so:

\[
\begin{array}{c}
G \\
\downarrow^{n} \\
H
\end{array}
\]

(1) Here are the subgroup diagrams for \( S_3 \) and \( D_4 \). Finish labeling all the edges with the appropriate indices.

(2) Which of the subgroups in the diagrams are normal subgroups of \( S_3 \) or \( D_4 \) respectively? What do you notice about how they sit in the diagrams?

(3) Draw subgroups diagrams for the following Abelian groups: \( \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_6, \mathbb{Z}_8, \mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_4, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \).

(4) Below are some subgroup diagrams with the groups and subgroups missing. What can you say about the groups whose subgroup diagrams they are?

(5) Draw the subgroup diagram for the quaternions \( Q_8 \). Which subgroups are normal in \( Q_8 \)? Hints: it should contain three subgroups of order 4 and one of order 2, and the subgroup \( ⟨i⟩ \) is isomorphic to \( \mathbb{Z}_4 \).
(6) Here’s the subgroup diagram for $A_4$ with all the intermediate groups missing. Fill in the diagram by finding generators for each subgroup. Which ones are normal?