1. Consider the ring of Gaussian integers $\mathbb{Z}[i]$ with function $N: \mathbb{Z}[i] \to \mathbb{Z}$ defined by $N(a + bi) = a + 2b$.

Let $S \subseteq \mathbb{Z}[i]$ be the subset 

$$S = \{a + bi \in \mathbb{Z}[i] : 5|N(a + bi)\}$$

(a) Show that $S$ is a subring of $\mathbb{Z}[i]$.

(b) Show (using the definition) that $S$ is an ideal of $\mathbb{Z}[i]$.

(c) Understand $S$ and its cosets by

i. describing $S$ and its cosets in words, using the function $N$.

ii. plotting $S$ and its cosets in the complex plane. A picture of $\mathbb{Z}[i]$ is provided to get you started.

You may want to use different colors or symbols to distinguish the cosets.

(d) Draw a picture of the principal ideal $(1 + i) \subseteq \mathbb{Z}[i]$ in the complex plane.

(e) Is the ideal $S$ a principal ideal of $\mathbb{Z}[i]$?

(f) Find a homomorphism with $S$ as its kernel, and determine the rings structure of $R/S$. 


2. Let $\mathcal{P}(\mathbb{Z})$ be the power set of $\mathbb{Z}$. Recall that the power set of a set is the set of all its subsets:

$$\mathcal{P}(\mathbb{Z}) := \{ A : A \subset \mathbb{Z} \}$$

We give $\mathcal{P}(\mathbb{Z})$ a ring structure with $\Delta$ (symmetric difference) as ‘addition’ and $\cap$ (intersection) as ‘multiplication.’ Recall the definition of the symmetric difference:

$$A \Delta B = A \cup B \setminus (A \cap B)$$

(a) Convince yourself that $\Delta$ and $\cap$ really do make sense in the roles of ‘addition’ and ‘multiplication’ for this ring by checking the ring axioms.

i. Write down the statement that $\Delta$ and $\cap$ are associative. You need not write full proofs of associativity, but draw Venn diagrams to convince yourself that it holds.

ii. Is $\Delta$ commutative?

iii. What is the $\Delta$ identity (zero element)?

iv. Prove that every element of $\mathcal{P}(\mathbb{Z})$ has a $\Delta$ inverse.

v. Prove that $\cap$ distributes over $\Delta$: $A \cap (B \Delta C) = (A \cap B) \Delta (A \cap C)$.

(b) Is $\mathcal{P}(\mathbb{Z})$ a commutative ring? Does it have a $\cap$ identity? Is it an integral domain? A field?

(c) Does $\mathcal{P}(\mathbb{Z})$ have any idempotents?

(d) Which of the following subsets of $\mathcal{P}(\mathbb{Z})$ are subrings? Which are ideals?

i. $\{ A \subset \mathbb{Z} : 7 \notin A \}$

ii. $\{ B \subset \mathbb{Z} : 17 \notin B \text{ and } 38 \notin B \}$

iii. $\{ C \in \mathbb{Z} : 9 \in C \}$

iv. $\{ n\mathbb{Z} : n \in \mathbb{Z}^{\geq 0} \}$

v. $\{ \emptyset, \{1\}, \{2\}, \{1,2\} \}$

(e) Are there any subrings of $\mathcal{P}(\mathbb{Z})$ that are not ideals?

(f) What is the principal ideal generated by $\{1\} \in \mathcal{P}(\mathbb{Z})$? What is the principal ideal generated by the set of even integers?

(g) Let $S$ be the subset from (d)(ii). Describe the cosets in $\mathcal{P}(\mathbb{Z})/S$, and create addition and multiplication tables for $\mathcal{P}(\mathbb{Z})/S$. What is the multiplicative identity in $\mathcal{P}(\mathbb{Z})/S$?