Math 351  
Workshop 3  
Spring 2019

1. Field of Remainders. Consider \( \mathbb{R}[x] \), and let \( \mathbb{R}_1[x] = \{ a + bx : a, b \in \mathbb{R} \} \). Define the map \( \varphi : \mathbb{R}[x] \to \mathbb{R}_1[x] \) by letting \( \varphi(f(x)) \) be the remainder, \( r(x) \), when \( f(x) \) is divided by \( x^2 + 1 \). This is well-defined, and in \( \mathbb{R}_1[x] \), by the division algorithm in \( \mathbb{R}[x] \).

   a) Calculate \( \varphi(3x^2 + 4x + 7) \) and \( \varphi(x^8) \).

   b) Show that \( \varphi(f + g) = \varphi(f) + \varphi(g) \) (i.e. it preserves additive structure).

   c) \( \mathbb{R}_1[x] \) is not closed under the ordinary multiplication of polynomials, so our only hope for \( \varphi \) to be a ring homomorphism is to re-define multiplication in \( \mathbb{R}_1[x] \) to make it a ring in the first place. Find a definition of multiplication in \( \mathbb{R}_1[x] \) that makes \( \varphi \) a ring homomorphism.

   d) What is the kernel of \( \varphi ? \) (ker(\( \varphi \)) := \( \varphi^{-1}(0) = \{ f(x) \in \mathbb{R}[x] \mid \varphi(f) = 0 \} \))

   e) Show that \( \mathbb{R}_1[x] \) forms a field with your chosen multiplication rule.

2. Matrix Rings. Let \( R = \mathbb{Z}[\sqrt{2}] = \{ a + b\sqrt{2} : a, b \in \mathbb{Z} \} \), and define \( S \subseteq M_2(\mathbb{R}) \) by

   \[
   S = \left\{ \begin{pmatrix} a & 2b \\ b & a \end{pmatrix} : a, b \in \mathbb{Z} \right\}.
   \]

   a) Show that \( R \cong S \).

   b) Show that \( R \) is a homomorphic image of \( \mathbb{Z}[x] \).

   c) What is the smallest ring containing \( \mathbb{Z} \) and \( 3\sqrt{2} ? \) Call this \( \mathbb{Z}[3\sqrt{2}] \).

   d) Find a ring of matrices isomorphic to \( \mathbb{Z}[3\sqrt{2}] \).

3. Idempotents.

   a) Prove that \( x \mapsto ax \) defines a ring homomorphism from \( \mathbb{Z} \) to \( \mathbb{Z}_n \) if and only if \( a \) is an idempotent in \( \mathbb{Z}_n \).

   b) What other restriction on \( a \) exists for ring homomorphisms from \( \mathbb{Z}_m \) to \( \mathbb{Z}_n \)?

   c) For any \( n \), 0 and 1 are idempotent elements in \( \mathbb{Z}_n \). Show that any other idempotents come in pairs (for example, 4 and 9 are such a pair modulo 12) and that these elements are zero-divisors.

   d) There are no zero-divisors in \( \mathbb{Z}_p \), for \( p \) prime. What conclusion follows for ring homomorphisms from \( \mathbb{Z} \) to \( \mathbb{Z}_p \)?