Part I

**Definition:** We say a group $G$ acts on a set $X$ if for each $g \in G$ there is an associated operation $g \cdot x : X \to X$ written as $g \cdot x$ sending $x$ to $g \cdot x \in X$ and this operation obeys the following rules:

- $g \cdot (h \cdot x) = (gh) \cdot x$ for all $g, h \in G$ and $x \in X$;
- $e \cdot x = x$ for all $x \in X$ (where $e$ is the identity element of $G$).

Such an operation is a **group action** of $G$ on $X$.

1. **Definition:** Let $x$ be an element of a set $X$ acted on by a group $G$. We define the **stabilizer** of $x$ as $\text{Stab}_x = \{ g \in G \mid g \cdot x = x \} \subseteq G$. We further define the **orbit** of $x$ as $\text{Orb}_x = \{ g \cdot x \mid g \in G \}$.
   
   (a) Prove that $\text{Stab}_x$ is a subgroup of $G$.
   
   (b) Prove that $g \cdot x = a \cdot x$ if and only if $g \in a \text{Stab}_x$ (the left coset of $\text{Stab}_x$ containing $a$).
   
   (c) Prove that $|\text{Orb}_x| = [G : \text{Stab}_x]$.

2. More on orbits:
   
   (a) Show that if $y \in \text{Orb}_x$, then $\text{Orb}_x = \text{Orb}_y$.
   
   (b) Show that if $\text{Orb}_x \cap \text{Orb}_y \neq \emptyset$, then $\text{Orb}_x = \text{Orb}_y$ (i.e. orbits are either disjoint or equal).
   
   (c) The orbits of $X$ are the distinct sets $\text{Orb}_x$ which partition $X$. We define the orbits of $X$ as $\mathcal{O}(X) = \{ \text{Orb}_x \mid x \in X \}$ and observe that $X$ is the disjoint union of the $O \in \mathcal{O}(X)$.
   
   (d) Observe that $\sum_{O \in \mathcal{O}(X)} |O| = |X|$

3. We wish to show that if $G$ is a group of order $p^n$ where $p$ is a prime and $n \geq 1$, then $Z(G)$ is a non-trivial subgroup (i.e. the center of $G$ has order at least $p$). To do so, we once again consider $G$ acting on $X = G$ by conjugation (i.e. $g \cdot x = g x g^{-1}$).
   
   (a) What values can $|\text{Stab}_x|$ take?
   
   (b) What values can $|\text{Orb}_x|$ take?
   
   (c) For which $x$ is $|\text{Orb}_x| = 1$?
   
   (d) Use 2d to conclude that $|Z(G)| > 1$.

4. We wish to show that any group $G$ of order $p^2$ where $p$ is a prime must be abelian. We assume for the sake of contradiction we have a non-abelian group $G$ of order $p^2$.
   
   (a) Show that $G$ has no element of order $p^2$.
   
   (b) Show that $|Z(G)| = p$.
   
   (c) Show that there is some subgroup $H = \langle g \rangle$ generated by $g \in G$ such that $Z(G)H = G$.
   
   (d) Conclude that $xy = yx$ for all $x, y \in G$, a contradiction.
Part II

1. Let $S_n$ be the symmetric group on $n$ letters, and let $\text{sgn}: S_n \to \mathbb{Z}_2$ be defined by

$$\text{sgn}(\sigma) = \begin{cases} 
0 & \text{if } \sigma \text{ is even} \\
1 & \text{if } \sigma \text{ is odd}
\end{cases}$$

(a) Prove that $\text{sgn}$ is a surjective homomorphism with kernel $A_n$.

(b) Let $G$ be a subgroup of $S_n$. Prove that either $G \leq A_n$ or $G$ consists of half even permutations and half odd permutations.

(c) Prove that any subgroup of $S_n$ containing at least one odd permutation contains a normal subgroup of index two.

2. Let $p$ be the least prime dividing $|G|$. We wish to show that if $H \leq G$ is of index $p$, then $H \trianglelefteq G$.

(a) (Warm-up) Suppose $[G : H] = 2$. Prove that $H$ is a normal subgroup of $G$ by proving that $gH = Hg$ for all $g \in G$.

(b) The Strong Cayley Theorem is an analogue of the Cayley Theorem. Let $G/H$ be the set of left cosets of $H$ (note: $G/H$ might not be a group if $H$ is not a normal subgroup). Show that $G$ acts naturally via left multiplication on $G/H$ and that this action induces a homomorphism $\phi : G \to S_n$ where $n = [G : H]$.

(c) Let $[G : H] = p$ be the smallest prime dividing $|G|$. Let $K = \ker(\phi)$ where $\phi$ is the above map. What can you say about $[G : K]$, the order of $G/K$?

(d) Use the definition of $\phi$ to prove that $K \leq H$.

(e) Observe that $[G : K] = [G : H][H : K]$. What does this tell you about $[H : K]$?

(f) Why can we conclude that $H$ is normal?

3. Let $G$ act on itself via conjugation. We call the orbit of $g$ under this action the conjugacy class of $g$ and denote it as $K(g) = K_G(g)$. Similarly, the stabilizer of $g$ under this action is also known as the centralizer and denoted $C(g) = C_G(g) = \{h \in G \mid hg = gh\}$.

In 3 of Part I, we used 2d in order to show what is known as the Class Equation:

$$|G| = \sum |K(g)| = |Z(G)| + \sum_{K(g) \neq \{g\}} |K(g)|$$

(a) Let $N \subseteq G$. Observe that for each $g \in G$, either $K(g) \cap N = \emptyset$ or $K(g) \subseteq N$. Thus, $N$ is a union of conjugacy classes.

(b) Confirm that $C(g)$ is in fact $\text{Stab}_g$ and use 1c from Part I to show that the size of the conjugacy class of $g$ is $[G : C(g)]$, the index of the centralizer of $g$ in $G$. Use to restate the Class Equation.

(c) What is the class equation for the symmetric group $S_n$?

(d) What are the centralizers $C_{S_n}(\sigma)$ in $S_n$?

(e) Use $|C_{A_n}(\sigma)|$ to find the sizes of all the conjugacy classes in $A_5$.

(f) Use 3a and Lagrange’s Theorem to show that $A_5$ does not have a non-trivial normal subgroup.