Definition: We say a group $G$ acts on a set $X$ if for each $g \in G$ there is an associated operation $g \cdot \star : X \to X$ written as $g \cdot x$ sending $x$ to $g \cdot x \in X$ and this operation obeys the following rules:

- $g \cdot (h \cdot x) = (gh) \cdot x$ for all $g, h \in G$ and $x \in X$;
- $e \cdot x = x$ for all $x \in X$ (where $e$ is the identity element of $G$).

Such an operation is a group action of $G$ on $X$.

1. Observe that the following are group actions:

   (a) $S_n$ acting on $[n] = \{1, 2, \ldots, n\}$ as permutations.
   (b) $D_n$ acting on the labelled corners of an $n$-gon (note, this is a sub-action of $S_n$).
   (c) $G$ acting on $X$ by $g \cdot x = x$ for all $g \in G$ and $x \in X$.

2. Come up with (non-trivial) group actions between the following groups and sets:

   (a) $G = \mathbb{Z}$ and $X = \mathbb{Z}$.
   (b) $G = \mathbb{Z}_2$ and $X = \mathbb{Z}$.

3. Prove that if $G$ acts on $X$, there is an associated homomorphism $\phi : G \to S_X$, where $S_X$ is the symmetric group on $X$ ($S_X = \{\sigma : X \to X | \sigma \text{ is a bij.}\}$).

4. Show that $G$ acts on itself in the following ways:

   (a) Show that left multiplication forms a group action of $G$ on $X = G$. (i.e. $g \cdot x = gx$ where $g, x \in G$)
   (b) Show that conjugation forms a group action. (i.e. $g \cdot x = gxg^{-1}$ where $g, x \in G$)

5. Use 3 and 4a to prove Cayley’s theorem: Every group $G$ is isomorphic to a subgroup of $S_G$.

6. Definition: Let $x$ be an element of a set $X$ acted on by a group $G$. We define the stabilizer of $x$ as $\text{Stab}_x = \{g \in G | g \cdot x = x \} \subseteq G$. We further define the orbit of $x$ as $\text{Orb}_x = \{g \cdot x | g \in G\}$.

   (a) Prove that $\text{Stab}_x$ is a subgroup of $G$.
   (b) Prove that $g \cdot x = a \cdot x$ if and only if $g \in a \text{Stab}_x$ (the left coset of $\text{Stab}_x$ containing $a$).
   (c) Prove that $|\text{Orb}_x| = [G : \text{Stab}_x]$.

7. More on orbits:

   (a) Show that if $y \in \text{Orb}_x$, then $\text{Orb}_x = \text{Orb}_y$.
   (b) Show that if $\text{Orb}_x \cap \text{Orb}_y \neq \emptyset$, then $\text{Orb}_x = \text{Orb}_y$ (i.e. orbits are either disjoint or equal).
   (c) The orbits of $X$ are the distinct sets $\text{Orb}_x$ which partition $X$. We define the orbits of $X$ as $\mathcal{O}(X) = \{\text{Orb}_x | x \in X\}$ and observe that $X$ is the disjoint union of the $O \in \mathcal{O}(X)$. 

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(d) Observe that $\sum_{O \in \mathcal{O}(X)} |O| = |X|$

8. We wish to show that if $G$ is a group of order $p^n$ where $p$ is a prime and $n \geq 1$, then $Z(G)$ is a non-trivial subgroup (i.e. the center of $G$ has order at least $p$). To do so, we once again consider $G$ acting on $X = G$ by conjugation (i.e. $g \cdot x = gxg^{-1}$).

(a) What values can $|\text{Stab}_x|$ take?

(b) What values can $|\text{Orb}_x|$ take?

(c) For which $x$ is $|\text{Orb}_x| = 1$?

(d) Use 7d to conclude that $|Z(G)| > 1$.

9. We wish to show that any group $G$ of order $p^2$ where $p$ is a prime must be abelian. We assume for the sake of contradiction we have a non-abelian group $G$ of order $p^2$.

(a) Show that $G$ has no element of order $p^2$.

(b) Show that $|Z(G)| = p$.

(c) Show that there is some subgroup $H = \langle g \rangle$ generated by $g \in G$ such that $Z(G)H = G$.

(d) Conclude that $xy = yx$ for all $x, y \in G$, a contradiction.