1. Let $G$ be a group with $H \leq G$ and $N \trianglelefteq G$.
   (a) Show that $HN = \{hg \mid h \in H, g \in N\}$ is a subgroup of $G$.
   (b) Suppose $K$ is also normal in $G$. Prove that $KN \leq G$.
   (c) Show that $K \cap N \leq G$.
   (d) Suppose that $KN = G$. Prove that $G/(K \cap N) \cong G/K \times G/N$.

2. **Definition:** We say that $N \leq G$ is **characteristic** in $G$ if, for any automorphism $\phi$ of $G$, we have $\phi(N) = N$.
   (a) Prove that if $N$ is characteristic in $G$ then $N$ is normal (i.e. show being characteristic is stronger than being normal).
   (b) Can you give an example of an $N \leq G$ which isn’t characteristic? (Hint: try looking at $Q_8$, the Quaternions)
   (c) Give an example of $H \trianglelefteq N \trianglelefteq G$ such that $H$ is not normal in $G$ (i.e. show that normality is not transitive).
   (d) Prove that if $H$ is characteristic in $N$ and $N \trianglelefteq G$, then $H \trianglelefteq G$.

3. Let $N$ be a normal subgroup of $G$ with order $|N| = n$ such that $(n, [G : N]) = 1$.
   (a) Show that if $g \in G$ has order $|g|$ which divides $n$, then $g \in N$.
   (b) Use this to prove that $N$ is the unique subgroup of $G$ of order $n$.

4. **Definition:** Let $g, h \in G$. We define the **commutator** of $g$ and $h$ as $[g, h] := ghg^{-1}h^{-1}$.
   (a) Show that $g$ and $h$ commute if and only if $[g, h] = e$.
   (b) The **commutator** subgroup of $G$ is defined as $[G, G] := \langle \{ghg^{-1}h^{-1} \mid g, h \in G\} \rangle$.
      i.e. it is the subgroup generated by all commutators of elements of $G$ ($[g, h]$ for $g, h \in G$).
      Prove that $[G, G] \leq G$.
   (c) It turns out, $K = [G, G]$ is the “smallest” normal subgroup of $G$ such that $G/K$ is abelian. Prove that $G/N$ is abelian if and only if $[G, G] \leq N \leq G$. 