Basic facts about fields

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Abstract

Self-contained definition of fields to be used in Math 350.

We start with the concise definition:

Definition 0.1. A *field* is a set $\mathbb{F}$ with two commutative associative operations $+$ and $\cdot$, each with identity, such that every element has an inverse with respect to $+$, every element except the $+$-identity has an inverse with respect to $\cdot$, and $a \cdot (b + c) = a \cdot b + a \cdot c$ for all $a, b, c \in \mathbb{F}$.

The above definition is nice because it is short, but equivalently we can state it the following way, which is probably more useful since it is explicit about all of the properties.

Definition 0.2. A *field* is a set $\mathbb{F}$ with operations $+$ and $\cdot$, usually called addition and multiplication, respectively, which satisfy:

1. $a + b = b + a$ and $a \cdot b = b \cdot a$ for all $a, b \in \mathbb{F}$ (*commutativity of* $+$ *and* $\cdot$);
2. $(a + b) + c = a + (b + c)$ and $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for all $a, b, c \in \mathbb{F}$ (*associativity of* $+$ *and* $\cdot$);
3. there exists an element $0 \in \mathbb{F}$ such that $a + 0 = a$ for all $a \in \mathbb{F}$ (*additive identity*);
4. there exists an element $1 \in \mathbb{F}$ such that $a \cdot 1 = a$ for all $a \in \mathbb{F}$ (*multiplicative identity*);
5. for all $a \in \mathbb{F}$ there exists $b \in \mathbb{F}$ such that $a + b = 0$ (all elements have an *additive inverse*);
6. for all $a \in \mathbb{F}$ with $a \neq 0$ there exists $b \in \mathbb{F}$ such that $a \cdot b = 1$ (all nonzero elements have a *multiplicative inverse*);
7. $a \cdot (b + c) = a \cdot b + a \cdot c$ for all $a, b, c \in \mathbb{F}$ (*distributive property*).

So, the most important facts here for us are items 3, 4, 5, and 6. Items 3 and 4 tell us that we always have elements called "0" and "1" which behave the way we expect (but are not necessarily the same ones from $\mathbb{R}$), and 5 and 6 essentially tell us that we can subtract any field element and divide by any field element except 0.

Traditionally, the additive inverse of $a \in \mathbb{F}$ is denoted $-a$ and, assuming $a \neq 0$, its multiplicative inverse is denoted by a^{-1}. So a computation like

$$\begin{aligned}(a + b^{-1}) \cdot b - a \cdot (b + 0) &= a \cdot b + b^{-1}b - a \cdot (b) \\ &= a \cdot b + 1 - a \cdot b \\ &= 1\end{aligned}$$

is true in *any* field, since I only used properties which hold by definition in all fields.