RESEARCH STATEMENT

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0. OVERVIEW

My primary research interest is metric Riemannian geometry, in particular the relation between curvature and topology. My research work so far has focused on how Ricci curvature determines the fundamental group of the underlying manifold, via the method of Gromov-Hausdorff convergence.

The interplay between curvature and topology is always a core issue in Riemannian geometry. For example, a longstanding open problem is the Milnor conjecture [17], which was proposed in 1968.

Milnor Conjecture. *Let M be an open n-manifold of $\mathrm{Ric} \geq 0$, then $\pi_1(M)$ is finitely generated.*

The Milnor conjecture remains open. It is natural to ask: *on what additional conditions does the Milnor conjecture hold?* I focused my graduate work on this question.

Before stating my results, we first introduce some background information on this conjecture.

For open manifolds with non-negative sectional curvature, Toponogov's triangle comparison controls the small-scale geometry from the large one. This bounds the number of Gromov's short generators [12], and finite generation follows. Actually any open manifold with non-negative sectional curvature has finite topology [6]. However, for non-negative Ricci curvature, the manifold may have infinite topology [21]. Unlike sectional curvature, Ricci curvature lacks a strong relation between the large-scale geometry and the small-scale one, which is the main difficulty when studying Ricci curvature.

The Milnor conjecture has been proven under various additional assumptions in the past decades. For instance, for a manifold with Euclidean volume growth, Anderson and Li have independently proven that the fundamental group is finite [1, 15]. Sormani has discovered geometric properties for the shortest geodesic loop that represents the short generator [22]. With this, she has shown that the Milnor conjecture holds if the manifold has small linear diameter growth, or linear volume growth. Liu has classified open 3-manifolds of nonnegative Ricci curvature using minimal surface theory and Perelman's work on the Poincaré conjecture [16]. In particular, his result confirms the Milnor conjecture in dimension 3.

One of the results I present is a new proof of the Milnor conjecture in dimension 3 using the structure of Ricci limit spaces [18]. This paper has been accepted for publication in Journal für die reine und angewandte Mathematik.

Theorem A. *The Milnor conjecture is true in dimension* 3.

In [19], I prove a new result on the Milnor conjecture with condition on the universal cover.

Theorem B. *Let M be an open n-manifold of $\mathrm{Ric} \geq 0$. If the Riemannian universal cover of M has Euclidean volume growth and the unique tangent cone at infinity, then $\pi_1(M)$ is finitely generated.*

This paper is available on arXiv. For a more general result, see Section 2.

The key idea in proving these two theorems is that, when the fundamental group is not finitely generated, we can see its consequence from $\pi_1(M,x)$-action on $\widetilde{M}$ at infinity, where $\widetilde{M}$ is the Riemannian universal cover of M. Under additional assumptions, if one can rule out the possibility of such an impact by analyzing the geometry of M at infinity and $\pi_1(M,x)$-action at infinity, then the finite generation would follow. More explanations are included in Section 1 and 2 respectively.

In a joint work with my thesis advisor Xiaochun Rong [20], we have another result on the finite generation of fundamental groups with assumptions on $\pi_1(M,x)$-action on $\widetilde{M}$.

Theorem C. *Let (M,x) be an open n-manifold with $\mathrm{Ric}_M \geq 0$. If there are ϵ, $\eta > 0$ such that $\pi_1(M,x)$-action has no ϵ-small η-subgroup on $\widetilde{M}$ with all scales $r > 0$, then $\pi_1(M,x)$ is finitely generated.*

The no ϵ-small η-subgroup condition in Theorem C is closely related to volume and the relation among isometries on different scales. More explanations are included in Section 3.

All these proofs rely on the structure of Ricci limit spaces and equivariant Gromov-Hausdorff convergence.

Gromov-Hausdorff convergence provides a platform to study the set of all manifolds with curvature bounds as a whole instead of individual ones. For a Gromov-Hausdorff convergent sequence $M_i \overset{GH}{\to} X$, understanding the structure of limit space X, and the relations between X and M_i are crucial. Around 2000, Cheeger and Colding developed a rich theory on the limit spaces coming from manifolds with lower Ricci curvature bounds [2, 3, 4, 5]. In recent years, Cheeger, Colding, and Naber further deepened the theory of Ricci limit spaces [7, 8, 9]. These results offer powerful tools in the absence a strong relation between the large-scale geometry and the small-scale one.

To study fundamental group, it is natural to consider its action on the universal cover $\widetilde{M}$. Combined with Gromov-Hausdorff convergence, Fukaya-Yamaguchi introduced equivariant Gromov-Hausdorff convergence [11]: for a convergent sequence $X_i \overset{GH}{\to} X$, if X_i carries isometric G_i-actions, then these symmetries pass to certain limit isometric G-action on the limit space X. Together with the structure of Ricci limit spaces mentioned above, these are the main tools to prove Theorems A, B, and C.

1. 3-manifolds

For 3-manifolds, Schoen and Yau have developed minimal surface theory, and have shown that any 3-manifold of positive Ricci curvature is diffeomorphic to the Euclidean space $\mathbb{R}^3$ [23]. Based on their minimal surface theory and Perelman's work on 3-manifolds, Liu has proven that for a 3-manifold with non-negative Ricci curvature, either it is diffeomorphic to $\mathbb{R}^3$, or its universal cover splits off a line [16]. A direct consequence is the Milnor conjecture in dimension 3.

My proof in [18] is completely different and more self-contained as Riemannian geometry. As indicated above, the key is investigating the tangent cones of M and $(\widetilde{M}, \pi_1(M,x))$ at infinity via equivariant Gromov-Hausdorff convergence ($r_i \to \infty$):

$$\begin{array}{ccc} (r_i^{-1}\widetilde{M}, \tilde{x}, \pi_1(M,x)) & \xrightarrow{GH} & (\widetilde{Y}, \tilde{y}, G) \\ \downarrow \pi & & \downarrow \pi \\ (r_i^{-1}M, x) & \xrightarrow{GH} & (Y = \widetilde{Y}/G, y). \end{array} \quad (1)$$

One observation I note is that, if $\pi_1(M)$ is not finitely generated, then using the lengths of Gromov's short generators $r_i \to \infty$, we can obtain a tangent cone of $(\widetilde{M}, \pi_1(M,x))$ at infinity $(\widetilde{Y}, \tilde{y}, G)$, such that the orbit $G \cdot \tilde{y}$ is not connected [18]. In this way, we can see the consequence of non-finite generation from $\pi_1(M,x)$-action at infinity. One may compare this phenomenon with a result by Sormani [22]. She has shown that if $\pi_1(M)$ is not finitely generated, then with the same $r_i \to \infty$, the lengths of short generators, the corresponding tangent cone of M at infinity (Y, y) satisfies that y is not a pole of Y. Both results see the consequences of non-finite generation, through the geometry of M at infinity, or through $\pi_1(M,x)$-action at infinity. They are crucial in my proof of Theorem A.

In dimension 3, in terms of the dimension in the Colding-Naber sense [9] of Y and $\widetilde{Y}$ in (1), the possibilities are very limited:
Case 1. $\dim \widetilde{Y} = 3$ (with $\dim Y = 1, 2$ or 3);
Case 2. $\dim Y = \dim \widetilde{Y} = 2$;
Case 3. $\dim Y = 1$ (with $\dim \widetilde{Y} = 1, 2$ or 3).
Moreover, we can further assume that $\pi_1(M)$ is abelian and torsion free due to the reductions by Wilking and Evans-Moser [24, 10]. Using the structure of Ricci limit spaces, I carefully rule out these possibilities when $\pi_1(M)$ is not finitely generated. This leads to Theorem A.

2. Equivariant stability at infinity

My proof of Theorem A convinces me that more partial results on the Milnor conjecture are within reach by investigating the tangent cones of $(\widetilde{M}, \pi_1(M,x))$ at infinity (1). In [19] I consider the case that the maximal Euclidean factor of $\widetilde{M}$ at infinity is stable. More precisely, for an open n-manifold of $\mathrm{Ric} \geq 0$ and an integer k, we say that M is *k-Euclidean at infinity*, if any tangent cone of M at infinity is a metric cone, whose maximal Euclidean factor has dimension k. For example, if M has Euclidean volume growth and the unique tangent cone at infinity, then it is k-Euclidean at infinity for some k. In general, being k-Euclidean at infinity allows M to have different tangent cones at infinity, even with different dimensions.

For a tangent cone of $\widetilde{M}$ at infinity as a metric cone $(\widetilde{Y}, \tilde{y}) = (\mathbb{R}^k \times C(Z), (0, z))$, where $C(Z)$ has $\mathrm{diam}(Z) < \pi$ and vertex z, due to Cheeger-Colding's splitting theorem [2], the isometry group $\mathrm{Isom}(\widetilde{Y})$ splits as $\mathrm{Isom}(\mathbb{R}^k) \times \mathrm{Isom}(Z)$. Hence for a limit G-action on such a space $(\widetilde{Y}, \tilde{y})$, we can consider the projected G-action on $\mathbb{R}^k$-factor, that is, $(\mathbb{R}^k, 0, p(G))$, where $p : \mathrm{Isom}(\widetilde{Y}) \to \mathrm{Isom}(\mathbb{R}^k)$ is the projection. As the main technical result in [19], I prove that if the universal cover $\widetilde{M}$ is k-Euclidean at infinity, then $\pi_1(M)$-action on $\widetilde{M}$ has equivariant stability at infinity, in terms of the projected action on $\mathbb{R}^k$-factor.

Theorem 2.1. *Let M be an open n-manifold with abelian fundamental group and* $\mathrm{Ric} \geq 0$*, whose universal cover $\widetilde{M}$ is k-Euclidean at infinity. Then there exist a closed abelian subgroup K of $O(k)$ and some integer $l \in [0, k]$ such that for any tangent cone of $(\widetilde{M}, \pi_1(M,x))$ at infinity $(\widetilde{Y}, \tilde{y}, G) = (\mathbb{R}^k \times C(Z), (0, z), G)$, its projected G-action on $\mathbb{R}^k$-factor $(\mathbb{R}^k, 0, p(G))$ satisfies $p(G) = K \times \mathbb{R}^l$, where K-action fixes 0 and the subgroup $\{e\} \times \mathbb{R}^l$ acts as translations in $\mathbb{R}^k$.*

Theorem 2.1 says that stability of Euclidean factors at infinity implies stability of projected $\pi_1(M,x)$-action on these Euclidean factors at infinity. A direct consequence of Theorem 2.1 is that the orbit $G \cdot \tilde{y}$ is always connected. Thus Theorem 2.1 confirms that the Milnor conjecture is true for manifolds whose universal covers are k-Euclidean at infinity. This also implies Theorem B.

Theorem 2.2. *Let M be an open n-manifold of* $\mathrm{Ric} \geq 0$*. If the Riemannian universal cover of M is k-Euclidean at infinity, then $\pi_1(M)$ is finitely generated.*

To prove Theorem 2.1, I develop a critical rescaling argument to understand the equivariant stability at infinity. I believe that this method has further applications on other conditions (see Section 4).

3. Actions with no small almost subgroups

Cheeger and Colding have shown that, the isometry group of any non-collapsing Ricci limit space, $(X, p) \in \mathcal{M}(n, -1, v)$, is a Lie group [4]. More precisely, they have shown that there are no arbitrarily small subgroups of $\mathrm{Isom}(X)$ in terms of their displacements. In a joint work with Xiaochun Rong [20], we prove a quantitative version of this result: there is a uniform lower bound $\delta(n, v)$ on the displacement of any non-trivial subgroup on the unit metric ball.

Theorem 3.1. *Given $n, v > 0$, there exists a positive constant $\delta(n, v)$ such that for any Ricci limit space $(X, x) \in \mathcal{M}(n, -1, v)$ and any nontrivial subgroup H in* $\mathrm{Isom}(X)$*, we have $D_{r,x}(H) \geq r\delta$ for all $r \in (0, 1]$, where $D_{r,x}(H) = \sup_{q \in B_r(x), h \in H} d(hq, q)$.*

For an equivariant convergent sequence and a subgroup in the limit group G, we may not find a sequence of subgroups converging to this subgroup. Hence it is necessary to consider subsets that are very close to subgroups. The concept of almost groups describes how a subset of G is close to being a subgroup, in terms of G-action. Note that for a subset A of G with $A^{-1} = A$, it is a subgroup if and only if $A^2 = A$, where $A^{-1} = \{a^{-1} | a \in A\}$ and $A^2 = \{aa' | a, a' \in A\}$. This inspires us to define a subset A with $A^{-1} = A$ as an *η-subgroup* at q, if the scaling invariant

$$\frac{d_H(Aq, A^2 q)}{\mathrm{diam}(Aq)} \leq \eta,$$

where d_H is the Hausdorff distance.

Definition 3.2. Let $\epsilon, \eta, r > 0$ and (M, x) be a complete n-manifold. For a subgroup G of $\mathrm{Isom}(M)$ acting freely and isometrically on M, we say that G-action has *no ϵ-small η-subgroup* at $q \in M$ with scale r, if for any nontrivial η-subgroup A at q, $D_{r,q}(A) \geq r\epsilon$ holds.

Comparing Definition 3.2 with Theorem 3.1, one may expect that $\pi_1(M)$-action has no ϵ-small η-subgroup with scale $r \in (0, 1]$ if $\mathrm{vol}(B_1(\tilde{x})) \geq v > 0$. However, this is much harder. It turns out that this is closely related to isometries on different scales.

Question 3.3. *Let (M_i, p_i) be a sequence of complete n-manifolds with* $\mathrm{Ric}_{M_i} \geq -(n-1)$ *and* $\mathrm{vol}(B_1(p_i)) \geq v > 0$*. Let f_i be a sequence of isometries on M_i. Suppose that the following sequences converge ($r_i \to \infty$):*

$$(M_i, x_i, f_i) \xrightarrow{GH} (X, x, f_\infty), \quad (r_i M_i, x_i, f_i) \xrightarrow{GH} (X', x', \mathrm{id}),$$

where id *means the identity map. Is it true that $f_\infty = \mathrm{id}$ always holds?*

Question 3.3 asks whether there is a control between the large-scale geometry and the small one, in terms of isometries that are very close to the identity map. There are counter-examples if the volume lower bound is dropped. Also, it can be shown that when the sectional curvature has a lower bound (with no restrictions on volume), the answer is yes [20].

If Question 3.3 has an affirmative answer, then we show that there are positive constants $\epsilon(n,v)$ and $\eta(n,v)$ such that $\pi_1(M,x)$-action has no ϵ-small η-subgroups at $\tilde{x}$ with scale $r \in (0,1]$, given that $\mathrm{vol}(B_1(\tilde{x})) \geq v > 0$ [20]. Together with Theorem C, this would imply that the Milnor conjecture holds for manifolds whose universal covers have Euclidean volume growth.

The technical part in proving Theorem C is studying the relation between an equivariant convergent sequence and its rescaling sequence:

$$\begin{array}{ccccccc} (\widetilde{M}_i,\tilde{x}_i,\Gamma_i) & \xrightarrow{GH} & (\widetilde{X},\tilde{x},G) & \qquad & (r_i\widetilde{M}_i,\tilde{x}_i,\Gamma_i) & \xrightarrow{GH} & (\widetilde{X}',\tilde{x}',G') \\ \big\downarrow \pi_i & & \big\downarrow \pi & & \big\downarrow \pi_i & & \big\downarrow \pi' \\ (M_i,x_i) & \xrightarrow{GH} & (X=\widetilde{X}/G,x), & & (r_iM_i,x_i) & \xrightarrow{GH} & (X'=\widetilde{X}'/G',x'), \end{array} \tag{2}$$

where $\Gamma_i = \pi_1(M_i,x_i)$. When M_i has uniform lower sectional curvature bound, it is known that X and X' satisfies $\dim(X') \geq \dim(X)$. If $\dim(\widetilde{X}') = \dim(\widetilde{X}) = n$, then it is equivalent to study the dimension of limit symmetries: $\dim(G)$ and $\dim(G')$. For Ricci curvature, there is no such dimension monotonicity in general. We prove that if Γ_i-action has no small almost subgroups, then dimension monotonicity on symmetries holds:

Theorem 3.4. *Let (M_i,x_i) be a sequence of complete n-manifolds with abelian fundamental groups Γ_i and $\mathrm{Ric}_{M_i} \geq -(n-1)$. Consider the convergent sequence and its rescaling as in (2). If there are $\epsilon,\eta > 0$ such that for each i, Γ_i-action has no ϵ-small η-subgroups on $B_1(\tilde{x}_i) \subseteq \widetilde{M}$ with all scales $r \in (0,1]$, then $\dim(G') \leq \dim(G)$.*

With Theorem 3.4, we bound the number of short generators at x by an induction argument on $\dim(G)$, which implies Theorem C.

Theorem 3.5. *Given $n,R,\epsilon,\eta > 0$, there exists a constant $C(n,R,\epsilon,\eta)$ such that for any complete n-manifold M with $\mathrm{Ric}_M \geq -(n-1)$, if $\pi_1(M,x)$ is abelian and $\pi_1(M,x)$-action has no ϵ-small η-subgroup on $B_1(\tilde{x})$ with all scales $r \in (0,1]$, then the number of short generators at x is bounded by $C(n,R,\epsilon,\eta)$.*

Kapovitch-Wilking proved that with the Ricci curvature condition alone, there is $q \in B_1(x)$ such that the number of short generators at q is bounded by a constant $C(n,R)$ [14]. To prove the Milnor conjecture by bounding number of short generators, it is essential to bound exactly at x.

4. Research Agenda

In the future, I intend to continue my work on the interplay between curvature and fundamental groups. I would also like to extend my research to related areas.

Regarding the Milnor conjecture, as is seen in my work [18, 19], the investigation of tangent cones of $(\widetilde{M},\pi_1(M,x))$ at infinity brings new insights into the Milnor conjecture, and I believe that the critical rescaling argument in [19] has more applications on other conditions. The equivariant stability at infinity I discovered not only is closely related to the Milnor conjecture, but also addresses this question: *does stability of the space at infinity implies stability of its symmetries at infinity?*. In [19], I consider the case that the maximal Euclidean factor of tangent cones of $\widetilde{M}$ at infinity is stable. It would be interesting to consider other conditions and answer the above question. From this perspective, [19] is a starting point in understanding equivariant stability at infinity. I would expect certain equivariant stability holds with the following assumptions:

- *Any tangent cone of $\widetilde{M}$ at infinity is a metric cone that is sufficiently close to some fixed metric* *cone* $C(Z)$. A special case is when $\widetilde{M}$ has almost maximal volume growth so that any tangent cone of $\widetilde{M}$ at infinity is very close to the Euclidean space $\mathbb{R}^n = C(S^{n-1})$. In general, this condition allows $\widetilde{M}$ to have tangent cones with different dimensions or different Euclidean factors.

• *$\widetilde{M}$ has Euclidean volume growth.* With this assumption, any tangent cone of $\widetilde{M}$ at infinity is a metric cone of Hausdorff dimension n [2]. Moreover, the volume is stable at infinity, in the sense that, for any tangent cone of $\widetilde{M}$ at infinity $(\widetilde{Y}, \tilde{y})$, the volume of $B_1(\tilde{y})$ is invariant.

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