Introduction to Quantum Groups and Tensor Categories

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1Got questions or comments? Just get in touch with him.
Outline

1. Hopf Algebras and Tensor Categories
2. Quasitriangular Hopf algebras and Ribbon Hopf Algebras
3. Quantum Groups at Roots of Unity
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1. Hopf Algebras and Tensor Categories
2. Quasitriangular Hopf algebras and Ribbon Hopf Algebras
3. Quantum Groups at Roots of Unity
“A mathematican is a machine for turning coffee into theorems.”
Alfréd Rényi (often attributed to Paul Erdős)

“A comathematican is a machine for turning cotheorems into ffee.” communicated to the author by Fei Qi
(Co-)Algebras

\( k \): our favorite commutative ring/field, all maps are \( k \)-linear.

- **Algebra**: \( k \)-space \( A \) with \( \eta : k \to A, \mu : A \otimes A \to A \)
- **Coalgebra**: \( k \)-space \( C \) with \( \varepsilon : C \to k, \Delta : C \to C \otimes C \)

\[
\begin{align*}
\eta \otimes I & \quad A \otimes A & \quad I \otimes \eta & \quad A \otimes k \\
\mu & \quad A & \quad \mu & \quad A \otimes k
\end{align*}
\]

\((co-)unitarity\)

\[
\begin{align*}
\varepsilon \otimes I & \quad C \otimes C & \quad I \otimes \varepsilon & \quad C \otimes k \\
\Delta & \quad C & \quad \Delta & \quad C \otimes k
\end{align*}
\]

\((co-)associativity\)
Sweedler's Notation: \( \forall x \in C \)

\[
\Delta(x) = \sum_{i=1}^{n} x_{1,i} \otimes x_{2,i} =: x_1 \otimes x_2 \in C \otimes C
\]

Coassociativity \( \Rightarrow \) “\( x_1 \otimes x_2 \otimes x_3 \)” is well-defined

Counitarity \( \Leftrightarrow \) \( \varepsilon(x_1)x_2 = x = x_1\varepsilon(x_2) \)

Convolution: \( \forall f, g : C \to A, \ f \ast g := \mu \circ (f \otimes g) \circ \Delta, \)

i.e. \( (f \ast g)(x) := f(x_1)g(x_2) \ \forall x \in C \)

Note that \( \eta \circ \varepsilon : C \to A \) is an identity element for \( \ast : \)

\( \forall f : C \to A, x \in C, \)

\[
(f \ast (\eta \circ \varepsilon))(x) = f(x_1)\eta(\varepsilon(x_2)) = f(x_1\varepsilon(x_2))1 = f(x)
\]

\[
((\eta \circ \varepsilon) \ast f)(x) = f(\varepsilon(x_1)x_2) = f(x)
\]
Bialgebras, Hopf algebras

- **Bialgebra**: algebra and coalgebra with compatible structure maps (\(\eta, \mu\) are coalgebra maps, \(\varepsilon, \Delta\) are algebra maps.)

- **Hopf algebra**: bialgebra \(H\) with an antipode, that is a \(*\)-inverse \(S\) of \(I\) as maps \(H \to H\). For all \(x \in H\), this means

  \[x_1 S(x_2) = (I * S)(x) = \varepsilon(x)1 = (S * I)(x) = S(x_1)x_2\]

  \(\Rightarrow\) \(S\) is an antialgebra map and an anticoalgebra map, every bialgebra has at most one antipode.
Examples

- **Group algebra** $k[G]$ for a group $G$
  - basis: $\{g\}$ for $g \in G$
  - $\varepsilon g = 1$, $\Delta g = g \otimes g$, $Sg = g^{-1}$
  - ("group-like element")

- **Universal enveloping algebra** $U(\mathfrak{g})$ for a Lie group $\mathfrak{g}$
  - basis: $\{x_1^{p_1} \cdots x_n^{p_n} | p_1, \ldots, p_n \geq 0\}$ for a basis $x_1, \ldots, x_n$ of $\mathfrak{g}$
  - $\varepsilon x_i = 0$, $\Delta x_i = 1 \otimes x_i + x_i \otimes 1$, $Sx_i = -x_i$
  - ("primitive element")

$\Rightarrow$ In both cases, $S^2 = I$.

Any cocommutative Hopf algebra over $\mathbb{C}$ is generated by group-likes and primitives.\(^2\)

\(^2\)Any cocommutative Hopf algebra over $\mathbb{C}$ is the *semidirect/smash product* Hopf algebra of the group algebra of the group formed by its group-likes and the universal enveloping algebra of the Lie algebra formed by its primitives.
**Categories and their bialgebras**

“Tannaka(-Krein) duality”, “reconstruction theorems”

- $\text{Rep}(A)$: category of modules of an algebra $A$ of finite rank/dimension over $k$
- Consider categories “of $k$-modules of finite rank/dimension”.

<table>
<thead>
<tr>
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Tannaka-Krein duality

- “For $A$ an algebra and $A\text{Mod}$ its category of modules, and for $A\text{Mod} \to \text{Vect}$ the fiber functor that sends a module to its underlying vector space, we have a natural isomorphism $\text{End}(A\text{Mod} \to \text{Vect}) \cong A$ in $\text{Vect}$.”

- “The assignments

$$(C, F) \mapsto H = \text{End}(F), \quad H \mapsto (\text{Rep}(H), \text{Forget})$$

are mutually inverse bijections between (1) equivalence classes of finite tensor categories $C$ with a fiber functor $F$, up to tensor equivalence and isomorphism of tensor functors, and (2) isomorphism classes of finite dimensional Hopf algebras over $k$."

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3 https://ncatlab.org/nlab/show/Tannaka+duality
4 thm. 5.3.12 in Etingof, Gelaki, Nikshych, Ostrik: Tensor Categories.
Outline

1. Hopf Algebras and Tensor Categories

2. Quasitriangular Hopf algebras and Ribbon Hopf Algebras

3. Quantum Groups at Roots of Unity
We fix a Hopf algebra $A$ over $k$.

- $\forall V, W$ $k$-spaces, $\tau_{V,W} : V \otimes W \to W \otimes V$, $v \otimes w \mapsto w \otimes v$.
- $\forall R \in A^\otimes 2$ we define elements in $A^\otimes 3$:
  
  $R_{12} := R \otimes 1$, $R_{23} := 1 \otimes R$, $R_{13} := (I \otimes \tau)(R \otimes 1)$.

$R \in A^\otimes 2$ is called *(universal) $R$-matrix*, if

1. $R$ is invertible and $\tau \circ \Delta(a) = R\Delta(a)R^{-1}$
2. $(I \otimes \Delta)R = R_{13}R_{12}$
3. $(\Delta \otimes I)R = R_{13}R_{23}$

$\Rightarrow (\varepsilon \otimes I)R = (I \otimes \varepsilon)R = 1 \otimes 1$, $(S \otimes I)R = (I \otimes S^{-1})R = R^{-1}$

$\Rightarrow R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$ “Yang-Baxter Equation”
Scribble (some proofs)

\[ R =: R^1 \otimes R^2 =: r^1 \otimes r^2 \in A^\otimes 2, \text{ summation implied (but not a coproduct!)}. \]

\[ (\Delta \otimes \text{I})R = R_{13}R_{23} \iff R^1_1 \otimes R^2_2 \otimes R^2 = r^1 \otimes R^1 \otimes r^2 R^2 \ldots \]

\[ \ldots \Rightarrow \varepsilon(R^1_1) \otimes R^2_2 \otimes R^2 = \varepsilon(r^1) \otimes R^1 \otimes r^2 R^2 \]
\[ \Rightarrow 1 \otimes R^1 \otimes R^2 = 1 \otimes \varepsilon(r^1)R^1 \otimes r^2 R^2 \]
\[ \Rightarrow 1 \otimes 1 = \varepsilon(r^1) \otimes r^2 \]

\[ \ldots \Rightarrow S(R^1_1)R^1_2 \otimes R^2 = S(r^1)R^1 \otimes r^2 R^2 \]
\[ \Rightarrow \varepsilon(R^1) \otimes R^2 = (S(r^1) \otimes r^2)(R^1 \otimes R^2) \]
\[ \Rightarrow 1 \otimes 1 = (S(r^1) \otimes r^2)R \]
If $A$ has an $R$-matrix $R$, it is called \textit{quasitriangular}. In this case, we define maps for all pairs of objects $V, W \in \text{Rep}(A)$:

$$c_{V,W} : V \otimes W \rightarrow W \otimes V, x \mapsto \tau(Rx).$$

⇒ Then $\text{Rep}(A)$ is a braided monoidal category with braiding $c$, i.e. for any $n \geq 1$, the braid group $B_n$ acts on $n$-fold tensor products of $A$-modules via $c$.

$$u := \mu \circ (S \otimes I) \circ \tau(R) \in A$$

⇒ $u$ is invertible and $S^2(a) = uau^{-1}$, $\forall a \in A$

(compare this with our examples for Hopf algebras above)

⇒ $u^{-1} = (I \otimes S^2)\tau(R)$, $\varepsilon(u) = 1$, $\Delta u = (\tau(R)R)^{-1}(u \otimes u)$
We fix a quasitriangular Hopf algebra $A$ with R-matrix $R$.

A central invertible $\nu \in A$ is called *universal twist* or *ribbon element* if

1. $\nu^2 = uS(u)$
2. $\varepsilon(\nu) = 1$
3. $\Delta \nu = (\tau(R)R)^{-1}(\nu \otimes \nu)$
4. $S(\nu) = \nu$

Note: If $\nu = ug^{-1}$ for a group-like $g$, then (2), (3) follow directly and (1), (4) are equivalent.
If $A$ has a Ribbon element $v$, it is called *ribbon Hopf algebra*. In this case, we define maps for all objects $V \in \text{Rep}(A)$:

$$\theta_V : V \rightarrow V, x \mapsto vx.$$  

$\Rightarrow$ Then Rep$(A)$ is a Ribbon category with twist $\theta$, i.e. $\forall V, W$,

- $\theta_V \otimes W = c_W, v c_V, W(\theta_V \otimes \theta_W)$
- $(\theta_V \otimes \iota_V^*) b_V = (\iota_V \otimes \theta_V^*) b_V$, where $b_V : k \rightarrow V \otimes V^*$. 
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Definition

- **quantum group**: quantized universal enveloping algebra
- \((a_{ij})_{1 \leq i,j \leq m}\) the Cartan matrix of a simple Lie algebra \(\mathfrak{g}\) of type ADE (\(\Rightarrow a_{ii} = 2, a_{ij} = a_{ji} \in \{0, -1\}\) for \(i \neq j\))
- \(q \in \mathbb{C} \setminus \{0, \pm 1\}\)

\(U_q(\mathfrak{g})\) generated by \(\{E_i, F_i, K_i, K_i^{-1}\}_{1 \leq i \leq m}\) with relations:

\[
\begin{align*}
[K_i, K_j] &= 0, \quad K_i K_i^{-1} = 1 = K_i^{-1} K_i \\
K_i E_j &= q^{a_{ij}} E_j K_i, \quad K_i F_j = q^{-a_{ij}} F_j K_i \\
[E_i, F_j] &= \delta_{ij} \frac{K_i - K_i^{-1}}{q - q^{-1}} \\
[E_i, E_j] &= [F_i, F_j] = 0 \quad \text{if } a_{ij} = 0 \\
E_i^2 E_j - (q + q^{-1}) E_i E_j E_i + E_j E_i^2 &= 0 \\
F_i^2 F_j - (q + q^{-1}) F_i F_j F_i + F_j F_i^2 &= 0
\end{align*}
\]

if \(a_{ij} = -1\)
\( U_q(g) \) is a Hopf algebra with

\[
\begin{align*}
\Delta(E_i) &= E_i \otimes 1 + K_i \otimes E_i \quad S(E_i) = -K_i^{-1}E_i \quad \varepsilon(E_i) = 0, \\
\Delta(F_i) &= F_i \otimes K_i^{-1} + 1 \otimes F_i \quad S(F_i) = -F_iK_i \quad \varepsilon(F_i) = 0, \\
\Delta(K_i) &= K_i \otimes K_i \quad S(K_i) = K_i^{-1} \quad \varepsilon(K_i) = 1.
\end{align*}
\]

Assume \( q \) is a \( p \)-th root of unity, \( p \geq 3 \), \( p' := \begin{cases} p & p \text{ odd} \\ p/2 & p \text{ even} \end{cases} \).

\[ J := \langle E_i^{p'}, F_i^{p'}, K_i^p - 1 \rangle_i \] as ideal in \( U_q(g) \).

\[ \Rightarrow \tilde{U}_q(g) := U_q(g)/J \] is a fin.-dim. ribbon quotient Hopf algebra.
Scribble (proof ideas)

- We may verify that $U_q(g)$ is a Hopf algebra, and that $J$ is a Hopf ideal. Hence $\tilde{U}_q(g)$ is a Hopf algebra.

- It is quasitriangular, because it is the quotient of a Drinfel’d double (see following slides).

- Let $(b_{ij})_{i,j} := (a_{ij})_{i,j}^{-1}$, $b_i := \sum_j b_{ij}$, $g := K_1^{-2b_1} \cdots K_m^{-2b_m}$.
  \[ \Rightarrow g \text{ is an invertible group-like in } \tilde{U}_q(g), \]
  \[ S^2(a) = gag^{-1} \text{ for all } a \in \tilde{U}_q(g) \]

- Let $u$ be the distinguished element of the quasitriangular Hopf algebra $\tilde{U}_q(g)$.
  \[ \Rightarrow v := ug^{-1} \text{ is central invertible and we may also verify that } Sv = v. \text{ Hence, } v \text{ is a ribbon element.} \]
Drinfel’d double

Consider

- $A$ a fin.-dim. Hopf algebra with dual $A^*$
- $A^0 := A^*$ as algebra, but with $\Delta^0 := \tau \circ \Delta$, $S^0 := S^{-1}$

$\Rightarrow \exists$ Hopf algebra $D(A) \simeq A \otimes A^0$ as $k$-spaces such that the identifications $A \to A \otimes 1 \subset D(A)$ and $A^0 \to 1 \otimes A^0 \subset D(A)$, are Hopf algebra maps and such that their images generate $D(A)$ as algebra.

$D(A)$ is quasitriangular, with $R$ the identity element in $A \otimes A^0$ ($A$ has to be finite-dimensional!).

Note: $D(A)$ can be defined even if $A$ is not finite-dimensional, and even for two Hopf algebras with a suitable pairing.

Note also: $D(A)$ is the Hopf algebra corresponding to the “center” of the tensor category $\text{Mod}(A)$ by Tannaka-Krein duality.
Yetter-Drinfel’d modules, Radford’s biproduct/bosonization

For a Hopf algebra $H$, $H_H YD$ is the category of (left left) $(H, H)$-bimodules $V$ with compatibility condition

$$\delta(h \cdot v) = h_1 v_{-1} Sh_3 \otimes h_2 \cdot v_0 \quad \forall h \in H, v \in V,$$

where $\delta$ is the coaction and $\delta(v) =: v_{-1} \otimes v_0$.

$\Rightarrow H_H YD$ is a braided monoidal category

$\exists$ functor *Radford’s biproduct/bosonization*

$\{ \text{“braided” Hopf algebra in } H_H YD \} \rightarrow \{ \text{Hopf algebra} \}, A \mapsto A\#H.\textsuperscript{5}$

$A\#H$ contains $H$ as Hopf subalgebra and $A$ as subalgebra.

\textsuperscript{5}Not to be confused with the *semidirect/smash product* which is sometimes denoted identically. The latter one is a product of a Hopf algebra and a module algebra, and no comodule structure is involved.
Quantum groups revisited

\[ H := k[\mathbb{Z}^m] = k[K_1, \ldots, K_m], \ V^\pm := k^n = \bigoplus_{i=1}^{m} E_i^\pm k \]

the Yetter-Drinfel’d modules defined by \( K_i \cdot E_j^\pm = q^{\pm a_{ij}} \) and \( \delta(E_i^\pm) = K_i \otimes E_i^\pm. \)

- \( T(V^\pm) \) are braided Hopf algebras
- adding the Serre relations \( \text{ad}^{1-a_{ij}}_{E_i^\pm}(E_j^\pm) = 0 \) to \( T(V^\pm) \)
  \( \rightarrow \) braided Hopf algebras \( U(n^\pm) \)
  (“Borel part”; \( \text{ad} \) is to be taken in \( H \))
- bosonizations \( U(n^\pm) \# H \)
  \( \rightarrow \) Hopf algebras which are dual in the sense of \( A \mapsto A^0 \)
- \( U_q(\mathfrak{g}) \): Drinfel’d double \( D(U(n^+)^\# H) \) modulo identification of the two copies of \( H. \ E_i = E_i^+, \ F_i = E_i^-. \)
Quantum groups revisited / Outlook

- Drinfel’d doubles and quotients of quasitriangular Hopf algebras are quasitriangular, so $\tilde{\mathcal{U}}_q(\mathfrak{g})$ is quasitriangular.

- Generalizations of the quantum groups discussed here which are still Ribbon Hopf algebras have been defined\(^6\). The fact that quantum groups and their generalizations are ribbon Hopf algebras can be proved through general Hopf algebra theory, as well\(^7\).

- There are results on how braided tensor categories obtained from conformal field theories can be studied through quantum groups\(^8\).

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\(^6\) Majid, *Double-bosonization of braided groups and the construction of $U_q(\mathfrak{g})$*, 1996 / Heckenberger, *Nichols Algebras (Lecture Notes)*, 2008 / ...

\(^7\) Burciu, *A class of Drinfeld doubles that are ribbon algebras*, 2008.

\(^8\) see http://arxiv.org/pdf/0705.4267v2.pdf, for instance
Quantum groups are quotients of Drinfel’d doubles of bosonizations of universal enveloping algebras of Borel subalgebras of Lie algebras in a category of Yetter-Drinfel’d modules. Roughly speaking.

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<td>(*) e.g. quantum groups</td>
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For further reading


- Drinfel’d, *Quantum Groups*, 1986, [here].


- Heckenerger, *Nichols Algebras (Lecture Notes)*, 2008, [here]: section 7, see also Simon Lentner’s MO answer [here].