Introduction to Deligne's category Rep(S_t) or How to cook a yummy semisimple tensor category

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[1] Happy to hear your questions or comments!

Motivation

Representations of S_n over a field F (like $F = \mathbb{C}$) form a

semisimple ← Maschke's theorem

rigid ← $*$

symmetric monoidal ← $\otimes$, $I = F$

F-linear abelian category ← $\oplus$, im, ker

(True for any finite group.)

So we have a mapping $n \mapsto \mathrm{Rep}^{\mathrm{ord.}}(S_n)$ for $n \in \mathbb{Z}^+$.

Deligne: Let's interpolate / extrapolate to $t \mapsto \mathrm{Rep}(S_t)$ for $t \in F$!

Objects in $\mathrm{Rep}^{\mathrm{ord.}}(S_n)$

Describe $\mathrm{Rep}^{\mathrm{ord.}}(S_n)$ without using the word "n".

objects:

- $\mathfrak{h} := F^n$ with basis $e_1, \ldots, e_n$, regular representation of S_n, $\sigma \cdot (x_1, \ldots, x_n) = (x_{\sigma(1)}, \ldots, x_{\sigma(n)})$
- $\mathfrak{h}^{\otimes r}$ for $r \geq 0$, $\sigma \cdot (x^{(1)} \otimes \ldots \otimes x^{((r))}) = (\sigma \cdot x^{(1)}) \otimes \ldots \otimes (\sigma \cdot x^{(r)})$
- Every object in $\mathrm{Rep}^{\mathrm{ord.}}(S_n)$ is a submodule of some $\mathfrak{h}^{\otimes r}$.

(On a side) note:

$$\mathfrak{h} = \{x : \textstyle\sum_i x_i = 0\} \oplus \{x : \textstyle\sum_i x_i \neq 0\}$$

as direct sum of S_n-modules, i.e. $\mathfrak{h}$ is reducible.

Morphisms in $\mathrm{Rep}^{\mathrm{ord.}}(S_n)$

V, W objects in $\mathrm{Rep}^{\mathrm{ord.}}(S_n)$. Then $\mathrm{Hom}_F(V, W) \cong V^* \otimes W$ and $\mathrm{Hom}_{S_n}(V, W) \cong (V^* \otimes W)^{S_n}$, the subspace of S_n-invariants.

The good news: (For any finite or compact group,) invariants correspond to orbits: $\sum_{\sigma \in S_n} \sigma \cdot x$ is an S_n-invariant for any element x, and we get all invariants this way.

$\Rightarrow$ Problem solved! up to combinatorics ;)

Morphisms between $\mathfrak{h}^{\otimes r}$ and $\mathfrak{h}^{\otimes s}$

- $P_{r,s} := \{\pi \text{ partition of } \{1,\ldots,r,1',\ldots,s'\}\}$ for $r,s \geq 0$, $FP_{r,s}$:= F-vector space with basis $P_{r,s}$
- $[r,n] := \{\mathbf{i} \text{ function } \{1,\ldots,r\} \to \{1,\ldots,n\}\}$ $\Rightarrow$ $\mathbf{i} \in [r,n], \mathbf{j} \in [s,n]$ define a function $\{1,\ldots,r,1'\ldots,s'\} \to \{1,\ldots,n\}$
- $\mathcal{H}(\pi,\mathbf{i},\mathbf{j}) := 1$ if the function defined by $\mathbf{i},\mathbf{j}$ is constant on each part of π, or 0 else, for $\pi \in P_{r,s}, \mathbf{i} \in [r,n], \mathbf{j} \in [s,n]$
- $x_{\mathbf{i}} := e_{\mathbf{i}(1)} \otimes \ldots \otimes e_{\mathbf{i}(r)} \in \mathfrak{h}^{\otimes r}$ for all $\mathbf{i} \in [r,n]$

Theorem ([CO, thm. 2.6])

The linear map $\mathcal{H} : FP_{r,s} \to \mathrm{Hom}_{S_n}(\mathfrak{h}^{\otimes r}, \mathfrak{h}^{\otimes s})$,

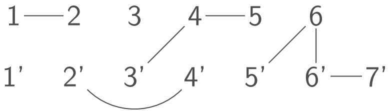
$$\pi \mapsto (x_{\mathbf{i}} \mapsto \textstyle\sum_{\mathbf{j}\in[s,n]} \mathcal{H}(\pi,\mathbf{i},\mathbf{j}) x_{\mathbf{j}})$$

is surjective, and injective if and only if $n \geq r+s$.

Morphisms as graphs

For every partition $\pi \in P_{r,s}$ there is a graph Γ with vertices $\{1, \ldots, r, 1', \ldots, s'\}$ such that π corresponds to the connected components of Γ. (Γ is not necessarily unique.)

Example [CO, Ex. 2.4 (2)]
$\pi = (1\ 2)(1')(2'\ 4')(3)(3'\ 4\ 5)(5'\ 6\ 6'\ 7') \in P_{6,7}$ corresponds to the following Γ (among others):

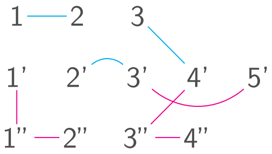


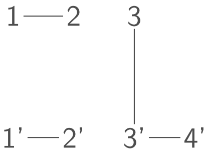
and the following map in $\mathrm{Hom}_{S_n}(\mathfrak{h}^{\otimes 6}, \mathfrak{h}^{\otimes 7})$:
$x_{\mathbf{i}} \mapsto \sum_{1 \leq j_1, j_2 \leq n} e_{j_1} \otimes e_{j_2} \otimes e_{\mathbf{i}(4)} \otimes e_{j_2} \otimes e_{\mathbf{i}(6)} \otimes e_{\mathbf{i}(6)} \otimes e_{\mathbf{i}(6)}$
if $\mathbf{i}(1) = \mathbf{i}(2)$ and $\mathbf{i}(4) = \mathbf{i}(5)$, and $x_{\mathbf{i}} \mapsto 0$ else.

Composition of morphisms

Definition by example
$\pi = (1\ 2)(1')(2'\ 3')(3\ 4')(5')$, $\mu = (1\ 1')(2\ 3)(2')(3\ 5)(3'\ 4\ 4')$

$\Rightarrow \mathcal{H}(\mu) \circ \mathcal{H}(\pi) = n^{\ell(\mu,\pi)} \mathcal{H}(\mu \cdot \pi)$,
where $\mu \cdot \pi$ is obtained graphically by the following process:



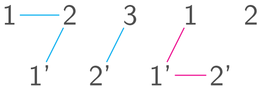


and where $\ell(\mu, \pi)$ is the number of components of the first graph involving only vertices from the middle row.
(So in this example, $\ell(\mu, \pi) = 1$.)

How to cook a yummy tensor category

$t \in F$. Define $\mathrm{Rep}_0(S_t)$ as the category with

- objects $\{[r]\}$ for $r \in \mathbb{Z}_{\geq 0}$
- morphisms $\mathrm{Hom}([r],[s]) := FP_{r,s}$
- composition $FP_{s,t} \times FP_{r,s} \to FP_{r,t}$, $\mu \circ \pi := t^{\ell(\mu,\pi)} \mu \cdot \pi$
  $\Rightarrow$ composition is associative (check diagrams)
- tensor product $\mathbf{1} := [0]$, $[r] \otimes [s] := [r+s]$,
  $\pi \otimes \mu$ defined by example:



$\in \mathrm{Hom}([5],[4])$

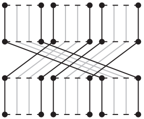
- duals $[n]^* := [n]$, evaluation $[n]^* \otimes [n] \to \mathbf{1}$, coevaluatation $\mathbf{1} \to [n] \otimes [n]^*$ given by the diagrams

Tasting our tensor category

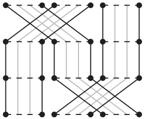
$\Rightarrow \otimes$ is associative and commutative with unit object $\mathbf{1}$, triangle axiom, pentagon axiom, hexagon axiom and rigidity hold

$\Rightarrow$ $\mathrm{Rep}_0(S_t)$ is a rigid symmetric monoidal category

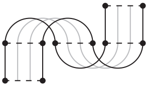
Diagrams for hexagon axiom and rigidity from [CO, sec. 2]:



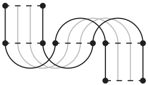
$\alpha_{m,l,n} \circ \beta_{n,m+l} \circ \alpha_{n,m,l}$



$(\mathrm{id}_m \otimes \beta_{n,l}) \circ \alpha_{m,n,l} \circ (\beta_{n,m} \otimes \mathrm{id}_l)$



$(\mathrm{id}_n \otimes ev_n) \circ (coev_n \otimes \mathrm{id}_n)$



$(ev_n \otimes \mathrm{id}_n) \circ (\mathrm{id}_n \otimes coev_n)$

Caution: $\mathrm{Rep}_0(S_t)$ is lacking directs sums and objects corresponding to proper submodules of $\mathfrak{h}^{\otimes r}$.

Caution: Since $\mathcal{H} : FP_{r,s} \to \mathrm{Hom}(\mathfrak{h}^{\otimes r}, \mathfrak{h}^{\otimes s})$ is **not injective** for $r + s > n$, there are **more** morphisms in $\mathrm{Rep}_0(S_n)$ than in $\mathrm{Rep}^{\mathrm{ord.}}(S_n)$ (identifying $[r]$ with $\mathfrak{h}^{\otimes r}$).

Adding salt and pepper

$\mathrm{Rep}_1(S_t) :=$ additive envelope of $\mathrm{Rep}_0(S_t)$ with
objects: $A_1 \oplus \cdots \oplus A_k$ for objects $A_1, \ldots, A_k$ in $\mathrm{Rep}_0(S_t)$
morphisms: matrices of morphisms in $\mathrm{Rep}_0(S_t)$
composition: matrix multiplication

$\mathrm{Rep}(S_t) :=$ Karoubi envelope of $\mathrm{Rep}_1(S_t)$ with
objects: pairs (A, e) for objects A in $\mathrm{Rep}_1(S_t)$ and
$e \in \mathrm{End}_{\mathrm{Rep}_1(S_t)}(A)$ an idempotent
morphisms: $\mathrm{Hom}_{\mathrm{Rep}_2(S_t)}((A, e), (B, f)) := f \circ \mathrm{Hom}_{\mathrm{Rep}_1(S_t)}(A, B) \circ e$
composition: as in $\mathrm{Rep}_1(S_t)$

Think of idempotents as **projections** onto submodules!
(And recall that $\mathrm{Rep}^{\mathrm{ord.}}(S_n)$ consists of submodules of $\mathfrak{h}^{\otimes r}$.)

Bon appétit!

Theorem ([CO, prop. 2.20])

- $\mathrm{Rep}(S_t)$ *is a rigid symmetric monoidal F-linear pseudo-abelian category*

  pseudo-abelian $:\Leftrightarrow$ *every idempotent (so not nec. every morphism) has a kernel and cokernel in the category*

- $\{$*indecomposable objects*$\}$ / *isomorphism*
  $\leftrightarrow \{\ ([r], e)$ *with* e *a primitive idempotent* $\}$ / *conjugacy*
  e *primitive* $:\Leftrightarrow$ e *cannot be written as sum of two non-zero idempotents*
  $e, e' \in \mathrm{End}([r])$ *conjugate* $:\Leftrightarrow \mathrm{End}([r])e \cong \mathrm{End}([r])e'$

- *every object is the finite direct sum of indecomposables, and the decomposition is unique*

Interpolation? Well, ... ($t \in \mathbb{Z}_{\geq 0}$)

Theorem ([De, thm. 6.2][CO, thm. 3.24])

$\exists$ surjective tensor functor $\mathcal{F} : \mathrm{Rep}(S_t) \to \mathrm{Rep}^{ord.}(S_t)$ sending $([r], e) \mapsto \mathcal{H}(e)(\mathfrak{h}^{\otimes r})$ with kernel $\mathcal{N}$, the tensor ideal in $\mathrm{Rep}(S_t)$ generated by the negligible morphisms.

$f \in \mathrm{Hom}(X, Y)$ negligible $:\Leftrightarrow \mathrm{Tr}(f \circ g) = 0$ for all $g \in \mathrm{Hom}(Y, X)$.

Example: $t = 1$

$$\mathrm{End}_{\mathrm{Rep}(S_1)}(([1], \mathrm{id}_{[1]})) = F(1\ 1') \oplus F((1)(1')) \cong F^2$$
$$\mathrm{End}_{\mathrm{Rep}^{\mathrm{ord.}}(S_1)}([1]) = \mathrm{End}_{S_1}(\mathfrak{h}) = \mathrm{End}_F(F) \cong F$$

$e := (1\ 1') - (1)(1') \in \mathrm{End}_{\mathrm{Rep}_0(S_1)}([1])$
$\Rightarrow e^2 = (1\ 1') - (1)(1') - (1)(1') + (1)(1') = e$,
so e is idempotent and $([1], e)$ is a non-zero object in $\mathrm{Rep}(S_1)$,
but $\mathcal{H}((1\ 1')) = \mathrm{id}_F = \mathcal{H}((1)(1'))$, so $\mathcal{F}(([1], e)) = 0$.

Upshot

Theorem ([De, thm. 6.2][CO, thm. 3.24])

$\mathrm{Rep}(S_t)/\mathcal{N} \cong \mathrm{Rep}^{ord.}(S_t)$ *is semisimple abelian for* $t \in \mathbb{Z}_{\geq 0}$.

Theorem ([De, thm. 2.18][CO, cor. 5.23])

$\mathrm{Rep}(S_t)$ *is semisimple abelian for* $t \notin \mathbb{Z}_{\geq 0}$.

For all t,

$$\left\{\begin{matrix}\text{Young}\\ \text{diagrams}\end{matrix}\right\} \leftrightarrow \left\{\begin{matrix}\text{primitive}\\ \text{idempotents}\end{matrix}\right\} / \sim \;\leftrightarrow \left\{\begin{matrix}\text{indecomposables}\\ \text{in } \mathrm{Rep}(S_t)\end{matrix}\right\} / \cong$$

(and for $t \notin \mathbb{Z}_{\geq 0}$, the indecomposables are the simple objects).

A Young diagram λ with $|\lambda| = r$ corresponds to $([r], e_\lambda)$, where $e_\lambda \in \mathrm{End}([r])$ is an idempotent determined by λ up to conjugation.

Growing tensor categories

A tensor category (essentially small rigid symmetric monoidal F-linear, $\otimes$ F-linear and exact) $\mathcal{C}$ is of subexponential growth
$:\Leftrightarrow \forall X \in \mathcal{C}\ \exists C \geq 1 : \operatorname{length} X^{\otimes n} \leq C^n\ \forall n \geq 1$

Theorem (Deligne, 2002)

$\mathcal{C}$ of subexponential growth over $\mathbb{C}$
$\Rightarrow \mathcal{C}$ is the category of "linear representations of a supergroup", i.e. representations of supercommutative Hopf algebras

[Et, p. 7] Let $(m_i)_i$ be the multiplicities of simple subobjects of an object $X \Rightarrow \operatorname{length} X = \sum m_i \geq (\sum m_i^2)^{1/2} = (\dim \operatorname{End}(X))^{1/2}$
$\Rightarrow \operatorname{length}[r] \geq |\operatorname{Part}(2r)|^{1/2} = (2r\text{-th Bell number})^{1/2}$ for $t \notin \mathbb{Z}_{\geq 0}$
$\Rightarrow$ Rep(S_t) is of superexponential growth and does not come from a supergroup! There is no tensor functor to Vec, i.e. no realization (because Vec has exponential growth)!

Was there anything special about S_n?

We used the following building blocks to interpolate / extrapolate from $\mathrm{Rep}^{\mathrm{ord.}}(S_n)$:

- the object $\mathfrak{h}$
- a distinguished basis of $\mathrm{Hom}(\mathfrak{h}^{\otimes r}, \mathfrak{h}^{\otimes s})$
- the fact that the structure constants of the composition are polynomials in n
- the additive envelope and the Karoubi envelope

Instead of S_n one can consider $GL(n)$, $O(n)$, $Sp(n)$ (Deligne), wreath products $S_n \rtimes \Gamma^n$ for Γ a finite group, $GL(n, \mathbb{F}_p)$, $O(n, \mathbb{F}_p)$, $Sp(n, \mathbb{F}_p)$ (Knop).

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