Sample solutions

[Meant to illustrate appropriate level of detail.]

**Problem.** A multigraph $G = (V, E)$ is *bipartite* if there’s a partition $V = X \cup Y$ so that every edge has one end in each of $X$, $Y$ (i.e. $\nabla(X, Y) = E$).

(a) Show that any bipartite multigraph with maximum degree at most $d$ is contained in a $d$-regular bipartite multigraph (possibly with additional vertices).

(b) Same for bipartite *graphs* (i.e. *simple* graphs).

**Solutions.** (Assume $G = (X \cup Y, E)$ as above.)

In either case WMA $G$ is *balanced*, i.e. $|X| = |Y|$: if it isn’t, first extend it to a balanced (bipartite) $G'$ by adding isolated vertices (vertices not contained in any edges).

(a) Let $d_G(v) = d(v)$. We proceed by induction on

$$N := \sum_{x \in X} (d - d(x)) = |X|d - |E(G)| = |Y|d - |E(G)| = \sum_{y \in Y} (d - d(y)) \quad (1)$$

(the number of “missing” edges). If $N = 0$, $G$ is already $d$-regular. If it’s not, then there are $x \in X$ and $y \in Y$ of degree less than $d$ and we can add an edge joining $x$ and $y$, decreasing $N$. (So “multi” makes this very easy.)

[Note for example: (a) this partly omitted justification for “WMA,” namely, skipped commenting on the trivial points (i) you can get to balance by adding isolates and (ii) proving the thing for $G'$ also proves it for $G$; (b) I regard (1) as not needing justification; (c) I consider the last couple lines sufficient (experience suggests some of you would feel you should say more).]

(b) If necessary add isolated vertices (the same number to each of $X, Y$) so that $N$ (as above) is at least $d - 1$. Now let $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_n\}$, and add:

- isolated vertices $u_1, \ldots, u_N$ to $X$ and $v_1, \ldots, v_N$ to $Y$;
- edges joining each $x_i$ to $d - d(x_i)$ of the $v_j$’s, with each $v_j$ used exactly once, and similarly for edges between the $y_i$’s and $u_j$’s;
- for each $j \in [N]$, edges joining $u_i$ to $v_i, \ldots, v_{i+d-2}$ (with subscripts interpreted mod $N$).
(And this does it.)

[So I’d accept (or prefer) that last assertion without the routine (but perhaps painful) justification: it’s clear the construction does what it’s supposed to, and I’m willing to believe you understand this if you’ve come up with it.]