1. (The basic idea is the same for the two parts.)

(a) Choosing such $S_i$’s is the same as choosing the sets $T_j := \{i \in [k] : S_i \ni j\}$ ($j \in [n]$) to be distinct; so the number of possibilities is the number of injections $T : [n] \to 2^k$, namely $(2^k)^n$.

(b) The answer is $(2^k - 1)^n + n(2^k - 1)^{n-1}$. The first term counts choices with $\cap S_i = \emptyset$ and the second those with $|\cap S_i| = 1$. (E.g. for the second, choose $j$ with $\cap S_i = \{j\}$ in $n$ ways, and for each $l \in [n] \setminus \{j\}$ choose $\{i \in [k] : S_i \ni j\}$ in $2^k - 1$ ways.)

2. Set $M = [2n] = A \cup B$ with $|A| = |B| = n$. We claim that each side of the identity counts the number of ways to choose $Z \in (M^n)$ and a function $f : Z \cap A \to [x]$.

Left side: Choose $k := |Z \cap A| \in \{0, \ldots, n\}$; $Z \cap A$ and $Z \cap B$ (each in $\binom{n}{k} = \binom{n}{n-k}$ ways); and then $f$ (in $x^k$ ways).

Right side: Choose $j := |f^{-1}([x - 1])| \in \{0, \ldots, n\}$; $J := f^{-1}([x - 1])$ and the restriction of $f$ to $J$ (in $\binom{n}{j}(x - 1)^j$ ways); and then the rest of $Z$ (in $\binom{2n-j}{n-j} = \binom{2n-j}{j}$ ways), noting that this also completes the specification of $f$ ($f \equiv x$ on $Z \setminus J$).

[For a combinatorial proof that the r.h.s. is 0 when $x = 0$, note that in this case (and with the setup above) the r.h.s. is $\sum Z \sum Y (-1)^{|Y|}$, summed over pairs $(Z, Y)$ with $Z \subseteq A \cup B$, $|Z| = n$ and $Y \subseteq Z \cap A$; but the inner sum is 1 if $Z = B$ and 0 otherwise. (Of course once we know the identity for $x \in \mathbb{P}$, we know it for all $x$, but this is not combinatorial.)]

3. The answer is $\frac{n^{2k}}{k!2^k}$. To see this, let $\Pi$ be the set of unordered partitions of $[n]$ into $n - k$ nonempty parts, and for $l \in [k]$, let $\Pi_l$ be the set of members of $\Pi$ with exactly $l$ nonsingleton parts (the sum of whose sizes will be $k + l$). Let $T(a, b)$ be the number of unordered partitions of $[a]$ into $b$ parts of size at least 2. Then $T(2k, k) = \frac{(2k)!}{k!2^k}$ (check) and

$$|\Pi_l| = \binom{n}{k+l} T(k + l, l) = \Theta(n^{k+l})$$

(since $k$ is fixed and $l \leq k$), implying

$$S(n, n - k) = |\Pi| \sim |\Pi_k| = \binom{n}{2k} T(2k, k) \sim \frac{n^{2k}}{k!2^k}.$$
4. Let \( \{X_i\}_{i \geq 0} \) be the RW on \( \mathbb{Z}^{d+1} \) (say \( X_i = (X_{i,1}, \ldots, X_{i,d+1}) \)) conditioned on the event \( Q = \{X_1 \neq \pm(0, \ldots, 0, 1)\} \), and note that we still (i.e. even with this conditioning) have \( \mathbb{P}(\{X_i\} \text{ returns to } 0) = q_{d+1} \).

Let \( I = \{0\} \cup \{i \geq 1 : X_{i,d+1} = X_{i-1,d+1}\} = \{i_0 < i_1 < \cdots\} \), and for \( j \geq 0 \) set \( Y_j = \pi(X_{i_j}) \), where \( \pi \) is projection on the first \( d \) coordinates. Then \( \{Y_j\}_{j \geq 0} \) is RW on \( \mathbb{Z}^d \) (this uses \( \mathbb{P}(I \text{ is infinite}) = 1 \)) and returns to the origin if \( \{X_i\} \) does. (So \( q_d = \mathbb{P}(\{Y_j\} \text{ returns to } 0) \geq \mathbb{P}(\{X_i\} \text{ returns to } 0) = q_{d+1} \).)

[A little point many of you missed: why did we need the first paragraph?]

5. Write \( \{X_k\}_{k \geq 0} \) for the RW and \( \xi \) for the number of returns to the origin (so \( \xi = |\{n \geq 0 : X_n = 0\}| \)). Then \( \mathbb{E}\xi = \sum_n \mathbb{P}(X_n = 0) \) and we need to show this sum is finite. Of course \( \mathbb{P}(X_n = 0) = 0 \) if \( n \) is odd. On the other hand,

\[
\mathbb{P}(X_{2k} = 0) = 6^{-2k} \sum_{a,b,c} \binom{2k}{a,b,c} = 6^{-2k} (2k) \sum_{a,b,c} \binom{k}{a,b,c}^2 = \Theta(1/\sqrt{k}) 3^{-2k} \sum_{a,b,c} \binom{k}{a,b,c}^2,
\]

where the sums run over \( a, b, c \in \mathbb{N} \) with \( a+b+c = k \). Since \( \sum_{a,b,c} \binom{k}{a,b,c}^2 = 3^k \), we also have

\[
3^{-2k} \sum_{a,b,c} \binom{k}{a,b,c}^2 \leq 3^{-k} \max \binom{k}{a,b,c} = \Theta(1/k).
\]

Here we used the easy observation that \( \binom{k}{a,b,c} \) is maximum when \( a, b, c \) are as equal as possible, in which case Stirling gives

\[
\binom{k}{a,b,c} \sim \frac{\sqrt{2\pi k}}{(2\pi k/3)^{3/2}} \frac{k^k}{(k/3)^k} = \Theta(3^k/k).
\]

Finally, combining, we have

\[
\sum \mathbb{P}(X_{2k} = 0) = O(\sum k^{-3/2}) < \infty,
\]

as desired.