587 Problems

X*: open

1. Prove Farkas’ Lemma (if you haven’t seen it), e.g. in the form:
   if \( f : \mathbb{N} \to \mathbb{R} \) is superadditive (i.e. \( f(a+b) \geq f(a) + f(b) \)), then \( \lim_{n \to \infty} n^{-1} f(n) \) exists (it may be infinite) and is at least \( m^{-1} f(m) \) for every \( m \).

2. As in class, let \( f(k) = f_3(k) \) be the maximum size of a \( k \)-uniform \( F \) with no sunflower, here meaning of size 3, and let \( g(k) \) be the maximum size of such an \( F \) that is also intersecting (i.e. \( A \cap B \neq \emptyset \forall A, B \in F \)).
   (a) Show that \( f(kl) \geq f(k)g(l)^k \).
   (b) Conclude that for \( k \) a power of 3, \( f(k) \geq 2g(k) \geq 2 \cdot 10^{(k-1)/2} \).
   [Hint: Start with a 3-uniform, 10-edge hypergraph gotten by identifying antipodal points of an icosahedron.]

3*. Prove the Erdős-Szemerédi Conjecture for (any) \( r \geq 4 \).

4. Prove Ellenberg-Gijswijt: \( g(n) < (2.75 \cdots)^n \) (or just \( g(n) < (3 - \varepsilon)^n \)).

5. Give an elementary (non-linear algebraic) proof of the C-D Theorem.
   [Minor suggestion: use induction on \( |B| \) (say).]

6. Prove the lemma from class that underlies the “Nullstellensatz”:
   If \( S_1, \ldots, S_n \subseteq \mathbb{F} \) (a field) and \( f \in \mathbb{F} \) vanishes on \( \prod S_i \) and is “reduced” (i.e. \( \deg_S(f) < s_i := |S_i| \forall i \)), then \( f \equiv 0 \) (coefficientwise).

7. Prove the Alon-Nathanson-Ruzsa result stated in class:
   For \( p \) prime and \( A, B \) nonempty subsets of \( \mathbb{Z}_p \) with \( a = |A| \neq |B| = b \),
   \[ |A \oplus B| \geq \min\{p, a + b - 2\} \]
   (where \( A \oplus B = \{\alpha + \beta : \alpha \in A, \beta \in B, \alpha \neq \beta\} \)).

8*. For any \( k \) and \( n \), the “Davenport constant” of \( \mathbb{Z}_k^n \) is \( m := n(k-1) + 1 \); that is, for any \( a_1, \ldots, a_m \in \mathbb{Z}_k^n \) there is some \( \emptyset \neq I \subseteq [m] \) with \( \sum_{i \in I} a_i = 0 \).

9. Show that one can’t cover \( \{0,1\}^n \setminus \{0\} \) (\( \subseteq \mathbb{R}^n \)) by fewer than \( n \) affine hyperplanes not containing \( \emptyset \).
[An affine hyperplane is \( \{ x \in \mathbb{R}^n : a \cdot x = b \} \) for some \( 0 \neq a \in \mathbb{R}^n \) and \( b \in \mathbb{R} \). Of course this fails to contain \( 0 \) if \( b \neq 0 \). You should check that \( n \) is best possible, and see what happens if we allow the hyperplanes to contain \( 0 \).]

10. Prove that (as stated in class) the graph on \( \{ k \)-trees of \( \binom{V}{k} \} \) (with \( T \sim T' \) iff \( T \cap T' \) is a \( (k - 1) \)-tree) is connected.

11. Show that the Schrijver-Seymour Conjecture (which for graphs says that \( G \) satisfies

\[
\text{every } w : E(G) \to \mathbb{Z}_p \text{ admits a 0-tree (1)}
\]

provided it satisfies this for all \( w : E(G) \to \{0, 1\} \subseteq \mathbb{Z}_p \) implies Seymour’s earlier conjecture that any \( p \)-connected graph satisfies (1).

[Hint: first observe that if \( G \) is \( k \)-connected then for any \( A \subseteq E = E(G) \),

\[
\text{rk}(A) + \text{rk}(E \setminus A) \geq n + k - 1.
\]

12. Verify the formula for dual rank: \( \text{rk}^*(A) = |A| - \text{rk}(E) + \text{rk}(E \setminus A) \).

13. A theorem of A. Drisko:

If the bipartite multigraph \( G \) is the (edge-)disjoint union of \( n \) matchings of size \( n \), say \( M_1, \ldots, M_n \), then it admits a matching \( M \) with \( |M \cap M_i| = 1 \forall i \).

Use this to give another proof of the Erdős-Ginsburg-Ziv Theorem. (You could also try proving the theorem, but I don’t know how this goes or how hard it’s supposed to be.)

14. If \( M = (E, \mathcal{I}) \) is a matroid (as usual, you can assume linear, or even graphic), \( X = \{1_I : I \in \mathcal{I} \} \) and \( K = \text{conv}(X) \subseteq \mathbb{R}^E \), then for (distinct) \( I, J \in \mathcal{I} \), \( [1_I, 1_J] \) is an edge of \( K \) iff either (a) \( I \subset J \) (or vice versa) or (b) \( |I \setminus J| = 1 = |J \setminus I| \).

[Recall that for \( x, y \in X \subseteq \{0, 1\}^n \subseteq \mathbb{R}^n \) and \( K = \text{conv}(X) \), the interval \( [x, y] \) is an edge of \( K \) iff \( \text{conv}(X \setminus \{x, y\}) \cap [x, y] = \emptyset \). Hint (which you could skip on first pass): You may may find some use for the following fact. (You could also try proving the fact, at least in the linear case if general matroids are unfamiliar.)

For any independent sets \( A, B \) of the same size, there are \( e \in A \) and \( f \in B \) such that both \( A \setminus e \cup f \) and \( B \setminus f \cup e \) are independent.]
15. Show that for any \( p \) and increasing \( \mathcal{F} \subseteq 2^{[n]} \),
\[
pI_p(\mathcal{F}) \geq \mu_p(\mathcal{F}) \log_p(\mu_p(\mathcal{F})).
\]
[Check: this is sharp for subcubes \( \{A \subseteq [n] : A \supseteq K\} \) (with \( K \subseteq [n] \)).]

16. Recall that for \( \mathcal{A} \subseteq \{0, 1\}^n \) with \( \mu(\mathcal{A}) = \alpha \) and \( I_k(\mathcal{A}) = \beta_k \), we have
\[
\sum \beta_k^2 > c_1 \alpha^2 (1 - \alpha)^2 \log^2 n/n
\]
and
\[
\sum \beta_k / \log(1/\beta_k) > c_2 \alpha (1 - \alpha)
\]
(\( c_1, c_2 \) are positive constants). Show (3) implies (2).

17. (As in class:) Show that for any fixed \( t \) and large enough even \( m \) there is some \( Y \subseteq \mathbb{Z}_m \) of size \( m/2 \) such that \( \{Y + \gamma : \gamma \in \mathbb{Z}_m\} \) is \( t \)-wise intersecting.

18*. Prove the conjecture of Ellis and Narayanan mentioned in class: any 3-wise intersecting, symmetric \( \mathcal{F} \subseteq 2^{[n]} \) has \( \mu(\mathcal{F}) < \exp[-n^{\Omega(1)}] \).

19*. Prove that (as conjectured in class) any 3-wise intersecting, symmetric, \( t \)-increasing \( \mathcal{F} \subseteq 2^{[n]} \) has \( \mu(\mathcal{F}) < o(1) \).

20. Fill in the proof of the missing lemma from Cameron-Frankl-Kantor:
If \( \mathcal{F} \) is increasing and contained in \( b(\{T\}) = \{\text{sets meeting } T\} \), the “blocker” of \( T \), then
\[
t^{-1} \sum_{x \in T} d_F(x) = |F|^{-1} \sum_{A \in \mathcal{F}} |A| \geq (1 + (2^t - 1)^{-1})|F|/2
\]
(where \( t = |T| \)).

21. Show that for any star forest \( F \), the maximum size of an \( F \)-intersecting family of spanning subgraphs of \( K_n \) is \( (1/2 - o(1))2^{\binom{n}{2}} \).

* ... and prove there is is some bipartite \( F \) for which this is not true.

[Recall a star forest is a disjoint union of stars (\( K_{1,m} \)'s). “Spanning” just means including all vertices; so a spanning subgraph is basically a subset of \( E(K_n) \). And (of course) \( F \) is \( F \)-intersecting if the intersection of any two of its members contains a copy of \( F \).]

22. Derive Reimer’s theorem from the union-closed families conjecture.
23. A possible strengthening of Frankl’s conjecture: If $f : 2^{[n]} \to \mathbb{R}^+$ satisfies

$$f(A \cup B) \geq \min\{f(A), f(B)\} \quad \forall A, B \subseteq [n]$$

and (to avoid stupidities) $f([n]) \geq f(\emptyset)$, then there is an $x \in [n]$ with

$$\sum\{f(A) : x \in A\} \geq \frac{1}{2} \sum f(A).$$

Show this is not true.

[Remark: I do know an example (actually with just one set whose $f$-value isn’t 0 or 1), but a simpler one would be nice.]

24. (a) Prove the following statement (conjectured by Alon, Seymour and Thomas and proved (independently) by A. Kotlov and G. Tardos, but not hard despite the pedigree).

If $X$ is a set of more than half the vertices of $\{0, 1\}^n$, then some component of the subgraph (of the Hamming graph) induced by $X$ meets every hyperplane \(\{x_i = \varepsilon\} (i \in [n], \varepsilon \in \{0, 1\})\).

[Note the lower bound on $|X|$ can’t be decreased. Hint: a funny shift.]

(b) Show that $|\nabla^-(G)| \leq 2^{n-1}$ for any simply rooted $G \subseteq 2^{[n]}$.

[This was of course the reason for (a), but it would also be interesting to see if there’s a way to do it that doesn’t use (a).]