587 Problems

$X^\star$: open

1. Prove Farkas' Lemma (if you haven't seen it), e.g. in the form:

if $f : \mathbb{N} \to \mathbb{R}$ is *superadditive* (i.e. $f(a+b) \geq f(a) + f(b)$), then $\lim n^{-1}f(n)$ exists (it may be infinite) and is at least $m^{-1}f(m)$ for every m.

2. As in class, let $f(k) = f_3(k)$ be the maximum size of a k-uniform $\mathcal{F}$ with no *sunflower*, here meaning *of size 3*, and let $g(k)$ be the maximum size of such an $\mathcal{F}$ that is also *intersecting* (i.e. $A \cap B \neq \emptyset \ \forall A, B \in \mathcal{F}$).

(a) Show that $f(kl) \geq f(k)g(l)^k$.

(b) Conclude that for k a power of 3, $f(k) \geq 2g(k) \geq 2 \cdot 10^{(k-1)/2}$.

[Hint: Start with a 3-uniform, 10-edge hypergraph gotten by identifying antipodal points of an icosahedron.]

$3^\star$. Prove the Erdős-Szemerédi Conjecture for (any) $r \geq 4$.

4. Prove Ellenberg-Gijswijt: $g(n) < (2.75\cdots)^n$ (or just $g(n) < (3-\varepsilon)^n$).

5. Give an elementary (non-linear algebraic) proof of the C-D Theorem.

[Minor suggestion: use induction on $|B|$ (say).]

6. Prove the lemma from class that underlies the "Nullstellensatz":
If $S_1, \ldots, S_n \subseteq \mathbb{F}$ (a field) and $f \in \mathbb{F}$ vanishes on $\prod S_i$ and is "reduced" (i.e. $\deg_i(f) < s_i := |S_i| \ \forall i$), then $f \equiv 0$ (coefficientwise).

7. Prove the Alon-Nathanson-Ruzsa result stated in class:
For p prime and A, B nonempty subsets of $\mathbb{Z}_p$ with $a = |A| \neq |B| = b$,

$$|A \oplus B| \geq \min\{p, a+b-2\}$$

(where $A \oplus B = \{\alpha + \beta : \alpha \in A, \beta \in B, \alpha \neq \beta\}$).

$8^\star$. For any k and n, the "Davenport constant" of $\mathbb{Z}_k^n$ is $m := n(k-1)+1$; that is, for any $a^1, \ldots, a^m \in \mathbb{Z}_k^n$ there is some $\emptyset \neq I \subseteq [m]$ with $\sum_{i \in I} a^i = \underline{0}$.

9. Show that one can't cover $\{0,1\}^n \setminus \{\underline{0}\}$ ($\subseteq \mathbb{R}^n$) by fewer than n affine hyperplanes not containing $\underline{0}$.

[An *affine hyperplane* is $\{x \in \mathbb{R}^n : a \cdot x = b\}$ for some $\underline{0} \neq a \in \mathbb{R}^n$ and $b \in \mathbb{R}$. Of course this fails to contain $\underline{0}$ iff $b \neq 0$. You should check that n is best possible, and see what happens if we allow the hyperplanes to contain $\underline{0}$.]

10. Prove that (as stated in class) the graph on $\{k\text{-trees of } \binom{V}{r}\}$ (with $T \sim T'$ iff $T \cap T'$ is a $(k-1)$-tree) is connected.

11. Show that the Schrijver-Seymour Conjecture (which for graphs says that G satisfies

$$\text{every } \mathsf{w} : E(G) \to \mathbb{Z}_p \text{ admits a 0-tree} \tag{1}$$

provided it satisfies this for all $\mathsf{w} : E(G) \to \{0,1\} \subseteq \mathbb{Z}_p$) implies Seymour's earlier conjecture that any p-connected graph satisfies (1).

[Hint: first observe that if G is k-connected then for any $A \subseteq E = E(G)$,

$$\mathrm{rk}(A) + \mathrm{rk}(E \setminus A) \geq n + k - 1.]$$

12. Verify the formula for dual rank: $\mathrm{rk}^*(A) = |A| - \mathrm{rk}(E) + \mathrm{rk}(E \setminus A)$.

13. A theorem of A. Drisko:

> If the bipartite multigraph G is the (edge-)disjoint union of n matchings of size n, say $M_1, \ldots, M_n$, then it admits a matching M with $|M \cap M_i| = 1$ $\forall i$.

Use this to give another proof of the Erdős-Ginsburg-Ziv Theorem. (You could also try proving the theorem, but I don't know how this goes or how hard it's supposed to be.)

14. If $\mathcal{M} = (E, \mathcal{I})$ is a matroid (as usual, you can assume linear, or even graphic), $X = \{\mathbf{1}_I : I \in \mathcal{I}\}$ and $K = \mathrm{conv}(X)$ $(\subseteq \mathbb{R}^E)$, then for (distinct) $I, J \in \mathcal{I}$, $[\mathbf{1}_I, \mathbf{1}_J]$ is an edge of K iff either (a) $I \subset\!\!\cdot\, J$ (or vice versa) or (b) $|I \setminus J| = 1 = |J \setminus I|$.

[Recall that for $x, y \in X \subseteq \{0,1\}^n \subseteq \mathbb{R}^n$ and $K = \mathrm{conv}(X)$, the interval $[x, y]$ is an edge of K iff $\mathrm{conv}(X \setminus \{x, y\}) \cap [x, y] = \emptyset$. Hint (which you could skip on first pass): You may may find some use for the following fact. (You could also try proving the fact, at least in the linear case if general matroids are unfamiliar.)

> For any independent sets A, B of the same size, there are $e \in A$ and $f \in B$ such that both $A \setminus e \cup f$ and $B \setminus f \cup e$ are independent.]

15. Show that for any p and increasing $\mathcal{F} \subseteq 2^{[n]}$,

$$pI^p(\mathcal{F}) \geq \mu_p(\mathcal{F}) \log_p \mu_p(\mathcal{F}).$$

[Check: this is sharp for subcubes $\{A \subseteq [n] : A \supseteq K\}$ (with $K \subseteq [n]$).]

16. Recall that for $\mathcal{A} \subseteq \{0,1\}^n$ with $\mu(\mathcal{A}) = \alpha$ and $I_k(\mathcal{A}) = \beta_k$, we have

$$\sum \beta_k^2 > c_1 \alpha^2 (1-\alpha)^2 \log^2 n/n \tag{2}$$

and

$$\sum \beta_k / \log(1/\beta_k) > c_2 \alpha(1-\alpha) \tag{3}$$

(where the c's are positive constants). Show (3) implies (2).

17. (As in class:) Show that for any fixed t and large enough even m there is some $Y \subseteq \mathbb{Z}_m$ of size $m/2$ such that $\{Y + \gamma : \gamma \in \mathbb{Z}_m\}$ is t-wise intersecting.

18*. Prove the conjecture of Ellis and Narayanan mentioned in class: any 3-wise intersecting, symmetric $\mathcal{F} \subseteq 2^{[n]}$ has $\mu(\mathcal{F}) < \exp[-n^{\Omega(1)}]$.

19*. Prove that (as conjectured in class) any 3-wise intersecting, symmetric, *increasing* $\mathcal{F} \subseteq 2^{[n]}$ has $\mu(\mathcal{F}) < o(1)$.

20. Fill in the proof of the missing lemma from Cameron-Frankl-Kantor:

> If $\mathcal{F}$ is increasing and contained in $b(\{T\})$ (= {sets meeting T}, the "blocker" of T), then
>
> $$t^{-1} \sum_{x \in T} d_{\mathcal{F}}(x) \ (= |\mathcal{F}|^{-1} \sum_{A \in \mathcal{F}} |A|) \ \geq (1 + (2^t - 1)^{-1})|\mathcal{F}|/2$$
>
> (where $t = |T|$).

21. Show that for any star forest F, the maximum size of an F-intersecting family of spanning subgraphs of K_n is $(1/2 - o(1))2^{\binom{n}{2}}$ ...

* ... and prove there is is *some* bipartite F for which this is *not* true.

[Recall a *star forest* is a disjoint union of stars ($K_{1,m}$'s). "Spanning" just means including all vertices; so a spanning subgraph is basically a subset of $E(K_n)$. And (of course) $\mathcal{F}$ is F-*intersecting* if the intersection of any two of its members contains a copy of F.]

22. Derive Reimer's theorem from the union-closed families conjecture.

[Recall the former says that for $\mathcal{F}$ union-closed, $\sum_{A\in\mathcal{F}} |A| \geq \frac{1}{2}|\mathcal{F}| \log_2 |\mathcal{F}|$.]

23. A possible strengthening of Frankl's conjecture: If $f : 2^{[n]} \to \Re^+$ satisfies

$$f(A \cup B) \geq \min\{f(A), f(B)\} \quad \forall A, B \subseteq [n]$$

and (to avoid stupidities) $f([n]) \geq f(\emptyset)$, then there is an $x \in [n]$ with $\sum\{f(A) : A \ni x\} \geq \frac{1}{2}\sum f(A)$. Show this is *not* true.

[Remark: I do know an example (actually with just one set whose f-value isn't 0 or 1), but a simpler one would be nice.]

24. (a) Prove the following statement (conjectured by Alon, Seymour and Thomas and proved (independently) by A. Kotlov and G. Tardos, but not hard despite the pedigree).

> If X is a set of *more than half* the vertices of $\{0,1\}^n$, then some component of the subgraph (of the Hamming graph) induced by X meets every hyperplane $\{x_i = \varepsilon\}$ ($i \in [n]$, $\varepsilon \in \{0,1\}$).

[Note the lower bound on $|X|$ can't be decreased. Hint: a funny shift.]

(b) Show that $|\nabla^-(\mathcal{G})| \leq 2^{n-1}$ for any simply rooted $\mathcal{G} \subseteq 2^{[n]}$.

[This was of course the reason for (a), but it would also be interesting to see if there's a way to do it that *doesn't* use (a).]

25. Show that if the hypergraph $\mathcal{H}$ is r-uniform, intersecting and regular, then $|V(\mathcal{H})| \leq r^2 - r + 1$.

26. Here, for some graph G, $\mathcal{H}$ is the hypergraph we discussed in connection with Tuza's Conjecture: $V(\mathcal{H}) = E(G)$ and $\mathcal{H}$ is the set of triangles of G (more precisely, edge sets thereof). Show:

(a) $\tau^*(\mathcal{H}) \leq 2\nu(\mathcal{H})$ and

(b) $\tau(\mathcal{H}) \leq 2\tau^*(\mathcal{H})$.

[Note both are implied Tuza. Hint for (b): let t be an optimal fractional cover of $\mathcal{H}$ (so in G an optimal fractional cover of triangles by edges) and use different strategies for the cases $\mathrm{supp}(t) = E(G)$ and $\mathrm{supp}(t) \neq E(G)$.]

(c) Show (b) is asymptotically tight; that is, $\tau(\mathcal{H})/\tau^*(\mathcal{H})$ can be arbitrarily close to 2. (Note (a) is exact for K_4.)

[Hint: recall there are triangle-free graphs G with $\alpha(G) = o(|V(G)|)$.]

(Some of you will have seen the next two.)

27. For $\mathcal{H}$ as in the EFL Conjecture and $n \geq 3$, show $\chi'(\mathcal{H}) \leq 2n - 3$.

28. Use Motzkin's Lemma (it can also be done without Motzkin) to prove: if $\mathcal{H}$ is nearly-disjoint and intersecting, then

$$|\mathcal{H}| \leq \Delta(\mathcal{H}) := \max\{| \cup_{x \in A \in \mathcal{H}} A| : x \in V(\mathcal{H})\}.$$

[The corresponding strengthening of EFL (that is, $\chi'(\mathcal{H}) \leq \Delta(\mathcal{H})$ for nearly-disjoint $\mathcal{H}$) is also possible. Note that for graphs this is Vizing's Theorem.]

29. Show that if $\mathcal{H}$ is r-uniform and doesn't contain a $\mathcal{P}_r$ (i.e. a projective plane of order $r - 1$), then $|\mathcal{H}| \leq (r - 1)\nu(\mathcal{H})$.

[This was the missing ingredient for the second part of Füredi's theorem, stating that $\nu^*(\mathcal{H}) \leq (r - 1)\nu(\mathcal{H})$ if $\mathcal{H}$ doesn't contain a $\mathcal{P}_r$. Hint: apply Brooks' Theorem to the *line graph*, $\mathcal{L}(\mathcal{H})$, of $\mathcal{H}$ (the graph on vertex set $\mathcal{H}$ with $A \sim B$ iff $A \cap B \neq \emptyset$).]

30. Prove that (as mentioned in class) the number of sum-free subsets of $[n]$ is less than $2^{(1+o(1))n/2}$.

[Use the "nearly-regular" form of Alon's result on independent sets (here the irregularity will be very minor).]

31. Show that for Q the Hamming graph on $\{0, 1\}^n$ and $N = 2^{n-1}$,

$$i(Q) > (1 - o(1))2\sqrt{e}\, 2^N.$$

32. Refine Sapozhenko's proof of Alon's bound on $i(G)$ to show that for any d-regular, n-vertex (simple) graph G,

$$\log i(G) < (1 + d^{-1} \log^2 d)n/2.$$

[This one is tougher; possibly cryptic hint: To identify $I \in \mathcal{I}(G)$, let Z (or Z_i) be the evolving set of vertices for which membership in I is not yet known and repeat for a while: choose $x \in I \cap Z$ of maximum Z-degree to add to the evolving set X of vertices known to be in I. Track the evolution of α (or α_i ...) defined by $|Z| = (1 + \alpha)n/2$.]