Problem Set 3 (second installment)

1. For $n \geq 2$, find the least $m = m(n)$ such that each $n$-element poset is (isomorphic to) a subposet of $2^m$.

2. Show that there is a fixed $C > 1$ such that $|\text{End}(P)| > C^n$ for each $n > 1$ and poset $P$ of size $n$ (i.e. with ground set of size $n$).

[For simplicity let’s restrict to $P$’s in which each element is comparable to at least one other. (Of course “isolated” elements just make it easier, right?)]

3. Let $A_1, \ldots, A_m \in \binom{V}{a}$ and $B_1, \ldots, B_m \in \binom{V}{b}$ satisfy $A_i \cap B_i = \emptyset$ for all $i$ and $(A_i \cap B_j) \cup (A_j \cap B_i) \neq \emptyset$ for all $i \neq j$. Then $m \leq (a + b)^a a^{-a} b^{-b}$.

4. For $P$ a poset and $X \subseteq P$, let (as in Perles’ proof of Dilworth)

$$A(X) = \{ y \in P : \exists x \in X, y \leq x \},$$
$$B(X) = \{ y \in P : \exists x \in X, y \geq x \}.$$  

Show that any maximal antichain $X$ of $2^n$ admits a partition $X = X_1 \cup X_2$ such that $A(X_1) \cup B(X_2) = 2^n$.

[Hint (for one way to do it): fix an ordering “$<$” of $X$ with $|x| < |y| \Rightarrow x < y$. Try to use elements that come early in $<$ for $X_2$, and those that come late for $X_1$.]