1. (a) A function \( f : 2^V \to \mathbb{R} \) is submodular if
\[
f(X \cap Y) + f(X \cup Y) \leq f(X) + f(Y) \quad \forall X, Y \subseteq V.
\]
Show that for any graph \( G \) on \( V \), \( f(X) = |\nabla_G(X)| \) is submodular.

[In case we haven’t said: \( 2^X := \{ \text{subsets of } X \} \).]

(b) If \( G \) is minimally \( k \)-edge connected (i.e. \( \lambda(G) \geq k \) but \( \lambda(G - e) < k \) for every edge \( e \)), then \( \delta_G = k \) (where \( \delta \) is minimum degree).

2. Let \( x, y \) be vertices of \( G = (V, E) \) with \( d(x, y) = d \), and suppose that for any \( F \subseteq E \) of size at most \( k - 1 \), \( d_{G-F}(x, y) \) is still \( d \). Then \( G \) contains \( k \) edge-disjoint \( \{x, y\}\)-paths of length \( d \).

3. Let \( G = (V, E) \) be \( k \)-connected, \( k \geq 2 \), and \( X = \{x_1, \ldots, x_k\} \subseteq V \). Show there is a cycle of \( G \) whose vertex set contains \( X \).

4. A digraph \( (V, A) \) is strongly connected (SC) if it contains a (directed) \((s, t)\)-path for all \( s, t \in V \). Show that any multigraph \( G \) with \( \lambda(G) \geq 2 \) has a strongly connected orientation.

5. For \( n \geq 2 \), if \( G \not\sim K_n \) then \( \chi(G) \leq 2^{n-2} \).

6. If \( \chi(G) = k \), what’s the least size of a collection of \( r \)-colorable graphs whose union is \( G \)?

[Recall \( r \)-colorable means \( \chi \leq r \). It may help to start with \( r = 2 \).]

7. Improve the bound in Problem 5 when \( n = 4 \): if \( G \not\sim K_4 \) then \( \chi(G) \leq 3 \).

8. Any digraph \( D = (V, A) \) with no (directed) odd cycle has a kernel.

[Suggestion: first do it when \( D \) is strongly connected.]