Please see the homework guidelines on the course page. If something seems wrong, please ask before wasting a lot of time on it. All problem parts have equal weight.

1. In each case find the number of ways to choose $S_1, \ldots, S_k \subseteq [n]$ satisfying the stated restriction:
   
   (a) no two elements of $[n]$ are in precisely the same $S_i$’s;
   
   (b) $| \cap S_i | \leq 1$.

   [Of course here—and similarly elsewhere—answers are to be justified.]

2. Show combinatorially that for any $n \in \mathbb{N}$ and $x \in \mathbb{P}$,

   $$\sum_{k=0}^{n} \binom{n}{k}^2 x^k = \sum_{j=0}^{n} \binom{n}{j} \left( \binom{2n-j}{n} (x-1)^j \right).$$

   (As in class, “combinatorial” means bijective; so you should find either a bijection between two objects whose sizes are the two sides of the identity, or a single object whose size is given by each of the two expressions.)

   [Possibly (or possibly not) helpful: for $x = 1$ this follows from Vandermonde. You can also try it for $x = 0$, where it’s still true but as far as I know needs a slightly different (combinatorial) proof. This needn’t be handed in.]

3. For fixed $k$ and $n \to \infty$, find (and justify) an asymptotic formula for $S(n, n-k)$ (the Stirling number of the 2nd kind).

4. Let $q_d$ be the probability that random walk (RW) on $\mathbb{Z}^d$ returns to the origin. Show that $q_d \geq q_{d+1}$.

   [Suggestion: find a way to directly compare the two processes. This is a soft argument, but maybe a challenge to write decently. For notational simplicity you can just do it for $d = 2$, of course without using the fact that $q_2 = 1$.]

5. Show that (as mentioned in class) random walk on $\mathbb{Z}^d$ is transient.

   [My current write-up for this takes a little more than half a page.]