Representations of Finite Groups: The Basics

September 9, 2020

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Big Picture

G gp finite.

$$\begin{tikzcd} G \arrow[r, squiggly] & X \arrow[l, bend left] \end{tikzcd} \quad X \text{ set.}$$

$$\mathrm{id}_G \cdot x = x$$

$$g \cdot (h \cdot x) = (gh) \cdot x$$

- orbit-stabilizer
- the centers of p-groups are nontrivial.

Changing X $\to$ V v.s.

V v.s. / $\mathbb{C}$ and

$g : V \to V$

$v \mapsto g \cdot v$ linear

Ansth

Another View

V v.s. $\mathbb{F}$

$$\rho : G \longrightarrow GL(V)$$

group hom

(V, ρ) is a rep of G.

Examples

- Permutation Representation

$G \curvearrowright X$ set

V w/ basis.

$\{e_x \mid x \in X\}$.

$g \cdot e_x = e_{gx}$

$\sum_{x \in X} a_x e_x$, $a_x \in \mathbb{C}$

R R_G.

- Regular Representation

$X = G$ $G \curvearrowright G$

left mult

V w/ basis

$\{e_g \mid g \in G\}$.

$\sum_{g \in G} a_g e_g$, $a_g \in \mathbb{C}$

The Regular Representation is Special

regular rep. $\Rightarrow$ V v.s. on which G acts linearly.

also have algebra structure on V.

$$\left(\sum_g a_g e_g\right)\left(\sum_g b_g e_g\right) = \sum_g \left(\sum_{hk=g} a_h a_k\right) e_g$$

$$e_g e_h = e_{gh}$$

$\mathbb{C}G$, $\mathbb{C}[G]$ group-algebra.

$\{\text{Grep over } \mathbb{C}\} \longleftrightarrow \{\text{module of } \mathbb{C}[G]\}$.

Even More Examples (better ones)

$G = S_3$

① Trivial rep.

$U = \mathbb{C}$, $\rho : G \to GL(U)$

$g \mapsto [1]$

② Alternating rep.

$U' = \mathbb{C}$, $\rho : G \to GL(U)$

$g \mapsto [\mathrm{sgn}(g)]$.

(1), (123)

(12) ← not trivial.

S_3 Examples Continued...

③ Permutation rep. $S_3 \curvearrowright X = \{1,2,3\}$

$W = \mathbb{C}^3$ basis $\{e_1, e_2, e_3\}$.

$$\rho : G \to GL(W)$$

$$g(e_1) = e_{g(1)}$$

$$g \cdot (z_1, z_2, z_3) = (z_{g^{-1}(1)}, z_{g^{-1}(2)}, z_{g^{-1}(3)}),$$

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Motivating Problem

What are "all" rep of a group G?

- up to isomorphism.

V, W

$$\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[d, "g"'] \arrow[dr, phantom, "G"] & W \arrow[d, "g"] \\ V \arrow[r, "\varphi"] & W \end{tikzcd}$$

linear.

$\varphi(g \cdot v) = g \cdot \varphi(v)$.

- $\ker\varphi$, $\operatorname{Im}\varphi$, $\operatorname{coker}\varphi$. all rep of G as well.

Still Some Redundancy

Consider our previous example, the trivial representation of S_3:

$$G = S_3, \;\; U = \mathbb{C}, \;\; \rho : G \to GL(U) \text{ where } \rho(g) = \begin{bmatrix} 1 \end{bmatrix}.$$

- $V_1 = \mathbb{C}^2$ and $\rho : G \to GL(V_1)$

  $g \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

  $V_1 = U \oplus U$.

- $V_2 = \mathbb{C}^2$ and $\rho : G \to G(V_2)$

  $g \mapsto \begin{bmatrix} 1 & 0 \\ 0 & \operatorname{sgn}(g) \end{bmatrix}$.

  $V_2 = U \oplus U'$

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Definitions

V rep of G.

- subrep $W \subset V$ ① subspace
  ② Invariant under action of G.

$gW \subset W$

- V is irred. if $V \neq 0$ and has no sub rep besides 0 and V,

$U \subsetneq V_1$ and V_2.

Schur's Lemma

If irred rep V and W (of G) and

$\varphi : V \to W$ is a G-linear,

1) φ is either an isomorphism or the zero map.

2) if $V = W$ then $\varphi = \lambda \cdot \mathrm{Id}$ for some $\lambda \in \mathbb{C}$.

1) $\ker \varphi$ subrep of V, $\mathrm{Im}\,\varphi$ subrep of W.

2). λ eigenvalue. $\ker(\varphi - \lambda \mathrm{Id})$ subrep of V.

Irreducible Representations as Building Blocks

f.d.

A representation V of G is **completely reducible**, i.e. given a subrepresentation U of V, there is a subrepresentation W of V such that $V = U \oplus W$.

Pf: $V = U \oplus W'$ as a v.s.

$\pi_0 : V \to U$ projection.

"averaging trick"

$$\pi : V \to U$$

$$v \mapsto \frac{1}{|G|} \sum_{g \in G} g \cdot (\pi_0(g^{-1} \cdot v))$$

Proof Continued

$V = U \oplus W$ $\quad \pi_0 : V \to U.$

$$\pi(v) = \frac{1}{|G|} \sum_{g} g \cdot (\pi_0(g^{-1} \cdot v))$$

Can check that $W = \ker \pi$ works.

(1) π is a projection onto U.

$$\pi(u) = \frac{1}{|G|} \sum_{g \in G} g \cdot (\underbrace{\pi_0(g^{-1} \cdot u)}_{g^{-1} \cdot u})$$

$g \cdot (g^{-1} \cdot u)$

$(g \cdot g^{-1}) \cdot u$,

$$= \frac{1}{|G|} \sum_{g \in G} u.$$

$$= \frac{|G|\, u}{|G|}$$

② $W = \ker \pi$ invariant under G.

$w \in \ker \pi$, $h \cdot w \in \ker \pi$, $\forall h \in G$.

$$\pi(h \cdot w) = \frac{1}{|G|} \sum_{g \in G} \underbrace{h h^{-1}}_{\text{id}} g \cdot \left(\pi_0 \left(g^{-1} (h \cdot w) \right) \right)$$

$$= h \cdot \frac{1}{|G|} \sum_{g \in G} (h^{-1} g) \cdot \left(\pi_0 \left((h^{-1} g)^{-1} \cdot w \right) \right)$$

$$= h \cdot \underbrace{\frac{1}{|G|} \sum_{k \in G} k \left(\pi_0 \left(k^{-1} \cdot w \right) \right)}$$

$= h \cdot 0 = 0$

$V = W \oplus U$.

Putting it all together (and gossing over some details)

Theorem

Any representation V is a direct sum of irreducible representations and can write

$$V = \bigoplus_i V_i^{\oplus n_i}$$

.

V_i are distinct irred. rep.

Schur's lemma,

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Question: Is the following representation irreducible?

$G = S_3$, $V = \mathbb{C}^3$ with basis $\{e_1, e_2, e_3\}$, (Permutation representation)

$v = e_1 + e_2 + e_3$

$U = \text{span}\{v\}$.

$G \curvearrowright U$ trivial.

$V = U \oplus W$

3 dim'l

1 dim'l

2 dim'l

*. is W irred?

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Definition

Def: V a rep of G. χ_V function

$$\chi_V : G \longrightarrow \mathbb{C}$$
$$g \longmapsto Tr(g|_V).$$

Why? what are eigenvalues of "g"?

observation: $\chi_V(g^{-1}hg) = \chi_V(h)$

Class function

Let's compute some characters

$\rho(g) = [1]$

Let $G = S_3$

1. $U = \mathbb{C}$ with trivial action.

   $\chi_u(g) = 1 \quad \forall g \in G,$

2. $U' = \mathbb{C}$ with alternating action.

   $\rho(g) = [\operatorname{sgn}(g)]$

   $\chi_{u'}(g) = \operatorname{sgn}(g). \quad \forall g \in G,$

Character of Permutation Representation

$G \curvearrowright X$, $\quad V = \{ e_x \mid x \in X \}$.

$\chi_V(g) = \#$ of elts in X fixed by g

$V = \mathbb{C}^3$ basis $\{e_1, e_2, e_3\}$, $G = S_3$.

$\chi_V((1)) = 3$, $\chi_V((12)) = 1$, $\chi_V((123)) = 0$

$V = \underbrace{U}_{\text{trivial}} \oplus W$ $\qquad \chi_V(g) = \chi_U(g) + \chi_W(g)$.

$\chi_W((1)) = 2$, $\chi_W((12)) = 0$, $\chi_W((123)) = -1$

All of this information in a table

	1	3	2	← # of elts in each conjugacy class.
→ $\mathfrak{S}_3$	1	(12)	(123)	←
trivial U	1	1	1	
alternating U'	1	−1	1	
standard K V	2	0	−1	

- $\chi_V(\mathrm{id}_G) = \dim V$.

A Hermitian Inner Product

$\mathbb{C}_{class}(G)$

$$(\alpha, \beta) = \frac{1}{|G|} \sum_{g \in G} \overline{\alpha(g)} \beta(g)$$

<u>Ex</u>: Trivial rep U of S_3.

$$(\chi_u, \chi_u) = \frac{1}{6}\big((1)(1)(1) + 3(1)(1) + 2(1)(1)\big)$$

$$= 1$$

Where is this coming from?

One cool consequence

V is irred as rep of G iff

$$(\chi_V, \chi_V) = 1.$$

$$\chi_W((1)) = 2, \quad \chi_W((12)) = 0, \quad \chi_W((123)) = -1$$

$$(\chi_W, \chi_W) = \frac{1}{6}\left((1)(2)^2 + 3(0)^2 + 2(-1)^2\right)$$

$$= 1$$

W is irred.