Why Study Schemes?

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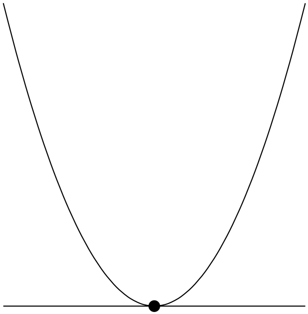
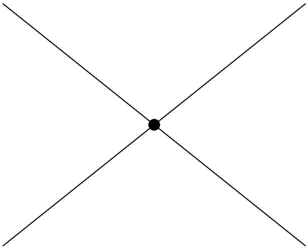
Intersection





WHAT IS SL_n?

- $\mathsf{SL}_2(\mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathsf{M}_2(\mathbb{C})\colon ad - bc = 1 \right\}$.
- For any commutative ring R and $n \geq 1$, $\mathsf{SL}_n(R)$ is the set of $n \times n$ matrices of determinant 1.
- The determinant is a polynomial in the entries $x_{11}, x_{12}, \ldots, x_{nn}$ of the matrix. So $\mathsf{SL}_n(R)$ is the set of solutions to

$$\det(x_{11}, x_{12}, \ldots, x_{nn}) = 1.$$

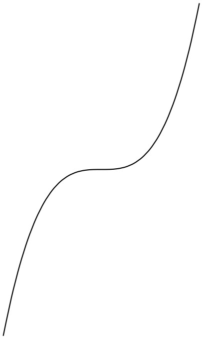
Review of Varieties

- $k = \overline{k}$
- Let $\mathbb{A}^n$ be the set k^n with the Zariski topology: for every ideal $I \subseteq k[x_1, \ldots, x_n]$, the set

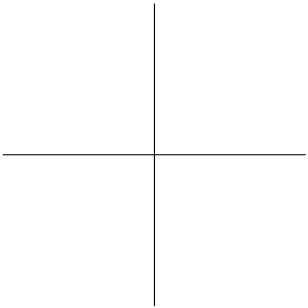
$$V(I) = \{p \in \mathbb{A}^n \colon f(p) = 0 \text{ for all } f \in I\}$$

  is closed.
- Check: this is really a topology.

Zariski Topology



$V(y - x^3) \subseteq \mathbb{A}^2$



$V(xy) \subseteq \mathbb{A}^2$

Ideals and Closed Subsets

We can map

$$\{\text{ideals of } k[x_1, \ldots, x_n]\} \to \{\text{closed subsets of } \mathbb{A}^n\}$$
$$I \mapsto V(I)$$

and

$$\{\text{subsets of } \mathbb{A}^n\} \to \{\text{ideals of } k[x_1, \ldots, x_n]\}$$
$$S \mapsto I(S)$$

where

$$I(S) = \{f \in k[x_1, \ldots, x_n] \colon f(s) = 0 \text{ for all } s \in S\}.$$

Hilbert's Nullstellensatz (Corollary)

There is a bijective correspondence

$$\{\text{radical ideals of } k[x_1, \ldots, x_n]\} \leftrightarrow \{\text{closed subsets of } \mathbb{A}^n\}$$
$$I \mapsto V(I)$$
$$I(S) \leftarrow\!\shortmid S$$

- An ideal $I \subseteq A$ is a radical ideal if $I = \sqrt{I} = \{f \in A \mid f^n \in I \text{ for some } n > 0\}$.
- Note the above correspondence is order-reversing with respect to inclusion.

Correspondence in $\mathbb{A}^n$

$$
\begin{aligned}
0 &\longleftrightarrow \mathbb{A}^n \\
\text{radical ideal} &\longleftrightarrow \text{closed subset} \\
\text{prime ideal} &\longleftrightarrow \text{irreducible closed subset} \\
\text{maximal ideal} &\longleftrightarrow \text{point} \\
k[x_1, \dots, x_n] &\longleftrightarrow \emptyset
\end{aligned}
$$

Affine Varieties

Definition. An *affine variety* is an irreducible closed subset of $\mathbb{A}^n$.

- Hartshorne's book assumes irreducible. Not all authors do.
- Example: $V(y - x^2) \subseteq \mathbb{A}^2$ is an affine variety.

A *quasi-affine variety* is an open subset of an affine variety.

Projective Varieties

Recall the definition of projective space $\mathbb{P}^n$.

- As a set, $\mathbb{P}^n = (\mathbb{A}^{n+1} \setminus \{0\})/\sim$ where

$$(a_0, \ldots, a_n) \sim (\lambda a_0, \ldots, \lambda a_n) \text{ for } \lambda \in k^\times.$$

  Equivalently, it's the set of lines through the origin in $\mathbb{A}^{n+1}$.
- Topology: the closed subsets are the zero sets of collections of homogeneous polynomials in $k[x_0, \ldots, x_n]$.
- Note, the space $\mathbb{P}^n$ has an open cover by $n+1$ copies of $\mathbb{A}^n$: identify the subset $\{(a_0, \ldots, a_n) \in \mathbb{P}^n \mid a_i = 0\}$ with $\mathbb{A}^n$ by setting $a_i = 1$.

A *projective variety* is an irreducible closed subset of $\mathbb{P}^n$. A *quasi-projective variety* is an open subset of a projective variety.

VARIETIES

Definition. A *variety* is an affine variety, quasi-affine variety, projective variety, or quasi-projective variety.

- Recall, we're working over a fixed algebraically closed field k.

The Ring $\mathcal{O}(X)$

Every variety X has an associated ring $\mathcal{O}(X)$ called the ring of regular functions.

- $X \subseteq \mathbb{A}^n$: a function $f: X \to k$ is a *regular function* if it is locally of the form $\frac{g}{h}$ for polynomials $g, h \in k[x_1, \ldots, x_n]$.
- $X \subseteq \mathbb{P}^n$: a function $f: X \to k$ is a *regular function* if it is locally of the form $\frac{g}{h}$ for homogeneous polynomials $g, h \in k[x_0, \ldots, x_n]$ of the same degree.
- Check that these form a ring.

Note: If X is a variety and $U \subseteq X$ is an open subset, there is an induced ring homomorphism $\mathcal{O}(X) \to \mathcal{O}(U)$ by $f \mapsto f|_U$.

$\mathcal{O}(X)$ and $A(X)$

Let X be an affine variety.

- The ring $A(X) := k[x_1, \ldots, x_n]/I(X)$ is called the coordinate ring of X. It is a finitely generated k-algebra and an integral domain.
    - Note: *every* finitely-generated integral domain over k is of the form $k[x_1, \ldots, x_n]/I$ for some prime ideal I.
    - If we don't assume X is irreducible, then $I(X)$ may not be prime, but it is radical. In that case, $A(X)$ may not be a domain, but it is reduced (no nontrivial nilpotents).
- One can show that $\mathcal{O}(X) = A(X)$.
    - To see $\supseteq$, check that an element $f + I \in A(X)$ uniquely determines a polynomial function $f\colon X \to k$.

$\mathcal{O}(X)$ and the Coordinate Ring

- If X is an affine variety, then on the open subset

$$X_f := \{p \in \mathbb{A}^n \colon f(p) \neq 0\},$$

  we have $\mathcal{O}(X_f) = A(X)_f$, where $A(X)_f$ is the localization of the ring $A(X)$ at f.
- In the projective case $\mathcal{O}(X)$ works differently.
- There is also a projective version of the coordinate ring.

Morphism of Varieties

A map $\varphi\colon X \to Y$ is a morphism of varieties if

- φ is continuous, and
- for every open subset $V \subseteq Y$ and every regular function $f\colon V \to k$, the function

$$f \circ \varphi\colon \varphi^{-1}(V) \to k$$

  is regular.

This allows us to define the category of varieties.

Equivalence of Categories #1

Finitely Generated k-algebras that are Integral Domains

We have an equivalence of categories

$$\left\{ \begin{array}{c} \text{finitely generated} \\ \text{integral domains over } k \end{array} \right\} \longleftrightarrow \{\text{affine varieties over } k\}$$

where

$$k[x_1, \ldots, x_n]/I \leftrightarrow V(I)$$

and

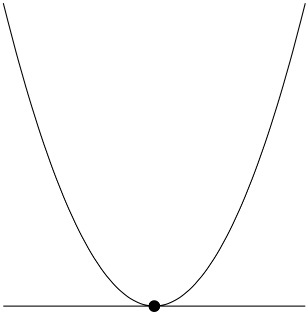
$$\varphi_* \colon \mathcal{O}(Y) \to \mathcal{O}(X) \leftrightarrow \ \varphi \colon X \to Y,$$

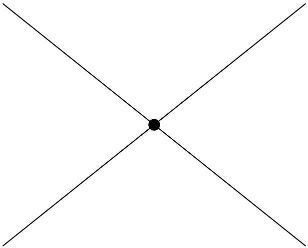
with $\varphi_*(g) = g \circ \varphi$.

Beyond Varieties

- Using the above equaivalence of categories, we can apply geometric reasoning to study a specific class of rings: finitely generated domains over an algebraically closed field.
- $\mathsf{SL}_n(\mathbb{C})$ is a variety. $\mathsf{SL}_n(\mathbb{Z})$ is not.
- We would like to apply our geometric tools to study other rings and fields, e.g. $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{Q}_p$, local rings, Dedekind domains...

Intersection





Beyond Varieties

Can we find some useful category to fill in the blank?

$$\{\text{commutative rings}\} \longleftrightarrow \{???\}$$

Schemes

A scheme is a pair

$$(X, \mathcal{O})$$

where X is a topological space and $\mathcal{O}$ is a sheaf of rings on X, satisfying certain conditions.

THE TOPOLOGICAL SPACE $\operatorname{Spec} R$

For any commutative ring R, define

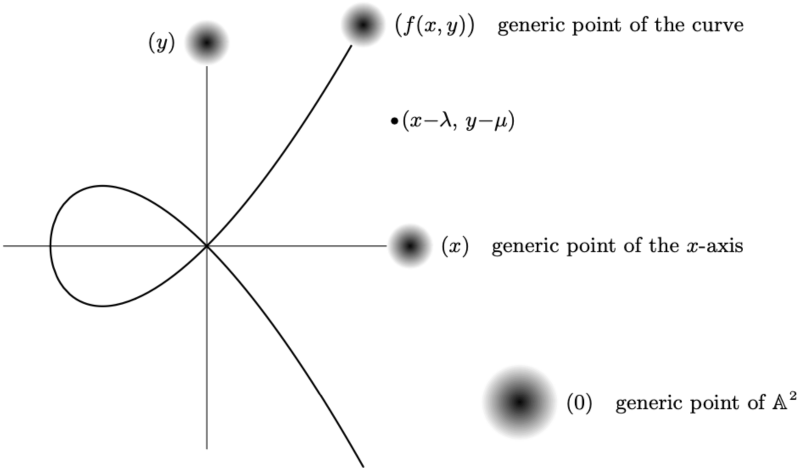
$$\operatorname{Spec} R = \{\mathfrak{p} \subseteq R \mid \mathfrak{p} \text{ is a prime ideal}\}.$$

Topology: the closed subsets of $\operatorname{Spec} R$ are the sets of the form

$$V(\mathfrak{a}) = \{\mathfrak{p} \in \operatorname{Spec} R \mid \mathfrak{a} \subseteq \mathfrak{p}\}$$

for ideals $\mathfrak{a} \subseteq R$.

Cool Picture of Spec $k[x, y]$



Sheaves

Analogy to keep in mind: “functions on an open set.”

- If X is a variety, then for each open subset $U \subseteq X$, we have a ring $\mathcal{O}(U) =$ the ring of regular functions on U.
- If $V \subseteq U \subseteq X$ then there is a “restriction” map $\mathcal{O}(U) \to \mathcal{O}(V)$ by $f \mapsto f|_V$.

Sheaves

Let X be a topological space. A *sheaf of rings* $\mathcal{O}$ on X is an assignment

$$U \mapsto \mathcal{O}(U)$$

giving a ring $\mathcal{O}(U)$ for each open subset $U \subseteq X$, together with ring homomorphisms

$$\rho_{U,V} \colon \mathcal{O}(U) \to \mathcal{O}(V) \text{ for } V \subseteq U,$$

satisfying the following conditions.

Sheaves

The "restriction" maps $\rho_{U,V}\colon \mathcal{O}(U) \to \mathcal{O}(V)$ satisfy:

1. $\rho_{U,U} = \mathrm{id}_{\mathcal{O}(U)}$.
2. $\rho_{V,W} \circ \rho_{U,V} = \rho_{U,W}$ for all $W \subseteq V \subseteq U$.
3. Locally zero implies zero: If $U = \bigcup_i V_i$ is an open cover and $s \in \mathcal{O}(U)$ such that $\rho_{U,V_i}(s) = 0$ for all i, then $s = 0$.
4. Gluing: If $U = \bigcup_i V_i$ is an open cover and $s_i \in \mathcal{O}(V_i)$ such that
$$\rho_{V_i, V_i \cap V_j}(s_i) = \rho_{V_j, V_i \cap V_j}(s_j) \text{ for all } i, j,$$
then there exists an element $s \in \mathcal{O}(U)$ such that $\rho_{U,V_i}(s) = s_i$ for all i.

The Structure Sheaf on $\operatorname{Spec} R$

Define a sheaf on $\operatorname{Spec} R$, called the *structure sheaf*, by setting

$$\mathcal{O}(U) = \left\{ s\colon U \to \coprod_{\mathfrak{p}\in U} R_{\mathfrak{p}} \;\middle|\; \begin{array}{c} s(\mathfrak{p}) \in R_{\mathfrak{p}} \text{ for all } \mathfrak{p} \\ \text{and } s \text{ is locally a quotient} \end{array} \right\}.$$

That is, for every $\mathfrak{p} \in U$ there exists a neighborhood $V \subseteq U$ of $\mathfrak{p}$ and elements $g, h \in R$ such that $s(\mathfrak{q}) = \frac{g}{h} \in R_{\mathfrak{q}}$ for every $\mathfrak{q} \in V$.

- This looks like the definition of regular functions.
- But instead of mapping to k, we map each $\mathfrak{p}$ into $R_{\mathfrak{p}}$.

Important fact: $\mathcal{O}(\operatorname{Spec} R) = R$.

The Structure Sheaf on $\operatorname{Spec} R$

Another way to think about the structure sheaf of $X = \operatorname{Spec} R$:

- The open subsets of X are generated by the subsets

$$X_f = \{\mathfrak{p} \in \operatorname{Spec} R \mid f \notin \mathfrak{p}\}.$$

- The structure sheaf is determined by $\mathcal{O}(X_f) = R_f$ and localization maps $R_f \to R_{fg} = R_g$ for $X_f \supseteq X_g$.

Affine Schemes

Definition. An *affine scheme* is a pair

$$(\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R})$$

where R is a commutative ring and $\mathcal{O}_{\operatorname{Spec} R}$ is the structure sheaf on $\operatorname{Spec} R$.

- Notation: often $\operatorname{Spec} R$ means the pair $(\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R})$.

SCHEMES

Definition-ish. A *scheme* is a pair

$$(X, \mathcal{O}_X)$$

where X is a topological space, $\mathcal{O}_X$ is a sheaf of rings on X, and $(X, \mathcal{O}_X)$ that locally looks like an affine scheme.

- That is, X is the union of some open sets U_i such that $(U_i, \mathcal{O}_X|_{U_i})$ is isomorphic to $(\operatorname{Spec} R_i, \mathcal{O}_{\operatorname{Spec} R_i})$ for some R_i.

Stalks

- X a scheme, $\mathfrak{p} \in X$
- The stalk $\mathcal{O}_{X,\mathfrak{p}}$ is the direct limit

$$\mathcal{O}_{X,\mathfrak{p}} = \varinjlim_{\mathfrak{p} \in U} \mathcal{O}_X(U).$$

  - The ring $\mathcal{O}_{X,\mathfrak{p}}$ consists of elements $s \in \mathcal{O}_X(U)$ for neighborhoods U of $\mathfrak{p}$ up to the equivalence $s_1 \sim s_2$ if $\rho_{U_1, U_1 \cap U_2}(s_1) = \rho_{U_2, U_1 \cap U_2}(s_2)$.
- By definition there's a map $\mathcal{O}_X(U) \to \mathcal{O}_{X,\mathfrak{p}}$ for any $U \ni \mathfrak{p}$.
- $\mathcal{O}_{X,\mathfrak{p}}$ is called the "local ring of X at $\mathfrak{p}$." It is a local ring.

Morphism of Schemes

A morphism of schemes is a pair $(\psi, \psi^{\#})$, where

$$\psi\colon X \to Y \text{ is continuous}$$

and $\psi^{\#}$ is a map of sheaves $\mathcal{O}_Y \to \psi_* \mathcal{O}_X$,

- that is, a collection of ring homomorphisms $\psi_U^{\#}\colon \mathcal{O}_Y(U) \to \mathcal{O}_X(\psi^{-1}U)$ for open $U \subseteq Y$ commuting with restriction maps,

such that $\psi^{\#}$ induces a local homomorphism $\mathcal{O}_{Y,\psi(\mathfrak{p})} \to \mathcal{O}_{X,\mathfrak{p}}$ on the stalks for each $p \in X$.

Morphisms of Affine Schemes

Prop. There is a one-to-one correspondence between morphisms of affine schemes and ring homomorphisms.

- A morphism $(\psi, \psi^{\#})\colon \operatorname{Spec} R \to \operatorname{Spec} T$ defines a ring homomorphism $\psi^{\#}_{\operatorname{Spec} T}\colon T \to R$.
- Conversely a ring homomorphism $\varphi\colon T \to R$ defines a continuous map $\operatorname{Spec} R \to \operatorname{Spec} T$ by $\mathfrak{p} \mapsto \varphi^{-1}(\mathfrak{p})$.
- Defining $\psi^{\#}$ from φ is a little more work.

Morphisms of Affine Schemes

Warning: a morphism of schemes is a pair $(\psi, \psi^{\#})$. It is not enough to know ψ alone!

- Example: $R = \mathbb{F}_{p^n}$.
- Ring homomorphism $\varphi\colon R \to R$ by $a \mapsto a^p$.
- The corresponding map of topological spaces is the identity:

$$\begin{aligned}\psi\colon \operatorname{Spec} R &\to \operatorname{Spec} R \\ (0) &\mapsto \varphi^{-1}((0)) = (0)\end{aligned}$$

- But $\psi^{\#}_{\operatorname{Spec} R} = \varphi$ is not the identity.

Equivalence of Categories #2

Commutative Rings

This gives us our equivalence of categories:

$$\begin{aligned} \{\text{commutative rings}\} &\leftrightarrow \{\text{affine schemes}\} \\ R &\leftrightarrow \operatorname{Spec} R \\ \varphi &\leftrightarrow (\psi, \psi^{\#}) \end{aligned}$$

$\mathrm{SL}_n(R)$

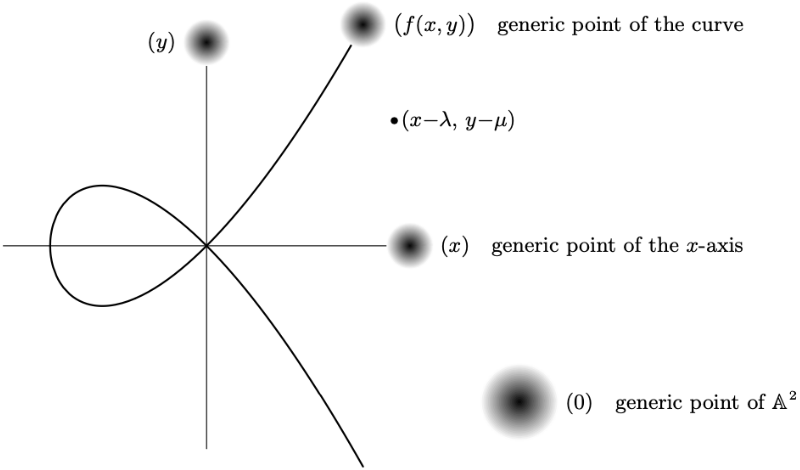
- $\mathrm{SL}_n(R) = V(\det(x_{ij}) - 1)$ where the x_{ij} are the coordinates of the matrix.
- $\mathrm{SL}_n(R) \cong \mathrm{Spec}(R[x_{11}, x_{12}, \ldots, x_{nn}]/(\det(x_{ij}) - 1))$.
- SL_n is a functor from rings to groups.

Generic Points

Example: suppose we want to show a certain property, e.g. smoothness, holds at almost every point on some variety or scheme, e.g. $V(y^2 - x^3 + x)$.

- We can try to make an argument using indeterminates that satisfy some equations.
- A "generic point" in indeterminates, like (x, y), is not really a point in the world of varieties.
- But with schemes, these can literally be points.

Generic Points



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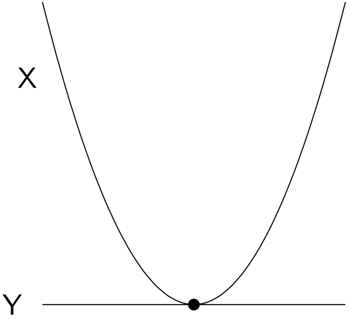
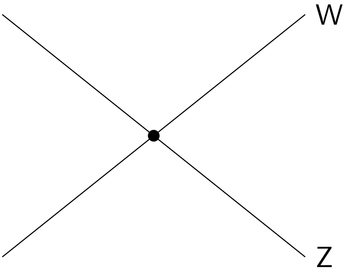
- We can try to make an argument using indeterminates that satisfy some equations.
- A "generic point" in indeterminates, like (x, y), is not really a point in the world of varieties.
- But with schemes, these can literally be points.
- We have a point $\mathfrak{p} = (y^2 - x^3 + x) \in \operatorname{Spec} k[x, y]$, and its closure is the elliptic curve defined by that equation ($\mathfrak{p}$ is dense in the curve).
- We can learn about the curve by looking at $\mathfrak{p}$.

Scheme-Theoretic Intersection



$X \cap Y = \mathsf{Spec}\, k[\varepsilon]/(\varepsilon^2)$



$Z \cap W = \mathsf{Spec}\, k$

References

- For a standard reference: *Algebraic Geometry* by R. Hartshorne (1977)
- For intuition, deeper understanding, and cool examples: *The Geometry of Schemes* by D. Eisenbud and J. Harris (2000)
- Algebra folks in our department (thanks Danny and Lev!)