Representations of finite groups + symmetric functions

1) Basics of group reps
2) Induced reps
3) Ring of characters of symmetric groups
4) Connect to symm. functions

A <u>representation</u> (or <u>module</u>) of G is a pair (V, ρ) where V is a vector space, $\rho : G \to GL(V)$ is a group homomorphism.

Given reps V, W, a <u>G-module homomorphism</u> is a map $\varphi : V \to W$ s.t. $\varphi(gv) = g\varphi(v)$.

$$\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[d, "g"'] & W \arrow[d, "g"] \arrow[dl, phantom, "\circlearrowleft"] \\ V \arrow[r, "\varphi"] & W \end{tikzcd}$$

The <u>character</u> of (V, ρ) is

$$\chi_V(g) = \mathrm{Tr}(\rho(g)).$$

A rep V is <u>irreducible</u> if it has no subreps. other than 0 and V.

<u>Prop</u>: Any rep of G is completely reducible:

$$V \cong \bigoplus V_i^{\oplus n_i}, \text{ where } V_i \text{ is irr.}$$

<u>Building new reps</u>. Let V, W be reps.

<u>Space</u>	<u>Action</u>	<u>Character</u>
$V \oplus W$	$g(v+w) = gv + gw$	$\chi_{V\oplus W} = \chi_V + \chi_W$.

$V \otimes W \quad g(v \otimes w) = (gv) \otimes (gw)$

$\chi_{V \otimes W} = \chi_V \chi_W$.

V^* $\quad (gf)(v) = f(g^{-1}v)$.

$= \{\text{linear } f: V \to \mathbb{C}\}$

$\chi_{V^*}(g) = \chi_V(g^{-1}) = \cancel{\chi_V(g)}$

$\text{Hom}(V, W) = \{\text{linear } f: V \to W\}$.

Define a G-rep on $\text{Hom}(V, W)$ by

$(gf)(v) = g f(g^{-1}v)$.

We have an isomorphism

$\varphi: V^* \otimes W \to \text{Hom}(V, W)$

by

$\varphi(f \otimes w)(v) = f(v)w.$

$\varphi(f \otimes w) : V \to W.$

This is an iso. of group representations!

$$\chi_{\mathrm{Hom}(V,W)}(g) = \chi_{V^* \otimes W}(g)$$

$$= \chi_{V^*}(g)\chi_W(g)$$

$$= \chi_V(g^{-1})\chi_W(g).$$

Let $V^G = \{v \in V : \forall g \in G,\ gv = v\}.$

<u>Ex</u>: $(\mathrm{Hom}(V,W))^G$ where V, W are reps of G.

$f: V \to W$ such that

$gf = f$

$g f(g^{-1}v) = f(v) \quad \forall g \in G$
$\forall v \in V$

We get $f(gv) = gf(v)$.

So

$\mathrm{Hom}(V,W)^G = \mathrm{Hom}_G(V,W)$

$= \{G\text{-module maps } V \to W\}$.

<u>Prop</u>: Let $\varphi: V \to V$ be defined by

$\varphi(v) = \frac{1}{|G|} \sum_{g \in G} gv$. This is a projection onto V^G.

<u>Sketch</u>:

$k\varphi(v) = k \frac{1}{|G|} \sum gv$

$$= \frac{1}{|G|} \sum_{g \in G} (kg) v \qquad h = kg$$

$$= \frac{1}{|G|} \sum_{h \in G} h v = \varphi(v).$$

$$\varphi^2(v) = \frac{1}{|G|} \sum_{h \in G} h \, \frac{1}{|G|} \sum_{g \in G} g v$$

$$= \cancel{\frac{1}{|G|}} \cancel{\sum_{h \in G}} \varphi(v) = \varphi(v). \qquad \square$$

$$\begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & 0 & \\ & & & & \ddots \\ & & & & & 0 \end{pmatrix} \quad \text{(matrix of a projection)}$$

$$\underline{\dim V^G} = \mathrm{Tr}(\varphi) = \mathrm{Tr}\left(\frac{1}{|G|} \sum_{g \in G} \rho(g)\right)$$

$$= \frac{1}{|G|} \sum_{g \in G} \mathrm{Tr}(\rho(g))$$

$$= \underline{\frac{1}{|G|} \sum_{g \in G} \chi_V(g)}.$$

Recall that $f: G \to \mathbb{C}$ is a <u>class function</u> if $\forall g, h \in G$,

$$f(hgh^{-1}) = f(g).$$

Define an inner product on class functions by

$$\langle \varphi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \varphi(g^{-1}) \psi(g).$$

For reps V and W,

$$\langle \chi_V, \chi_W \rangle = \frac{1}{|G|} \sum_{g \in G} \underbrace{\chi_V(g^{-1}) \chi_W(g)}$$

$$= \underbrace{\frac{1}{|G|} \sum_{g \in G} \chi_{Hom(V,W)}}$$

$$= \dim\left(Hom(V,W)^G\right)$$

$$= \dim\left(Hom_G(V,W)\right).$$

What if V and W are <u>irreducible</u>?

By Schur's lemma,

$$\langle \chi_V, \chi_W \rangle = \dim(\mathrm{Hom}_G(V, W))$$

$$= \begin{cases} 1 & : V \cong W \\ 0 & : V \not\cong W \end{cases}.$$

With this inner product, the set of characters of irreducible G-representations is an orthonormal set!

(In particular, they are independent.)

<u>Corollary</u>: Suppose $\chi_V = \chi_W$. Then $V \cong W$ as G-modules.

<u>Proof</u>: $V \cong \bigoplus V_i^{\oplus n_i}$, where

V_i are irr.

$$\chi_V = \sum \underline{n_i}\, \chi_{V_i} = \chi_W.$$

By ind, this is the unique expression for χ_W in terms of χ_{V_i}. Then

$$W \cong \bigoplus V_i^{\oplus n_i} \cong V. \quad \square$$

$$\chi_W = \sum m_j \chi_{V_j}. \quad W \cong \bigoplus V_j^{\oplus m_j}$$

<u>Prop</u>:
$\{\chi_{V_i} : V_i \text{ irr}\}$ is a basis for the space of class functions.

<u>Corollary</u>:
(# conjugacy classes of G) = dim(space of class functions) = (# of irr characters of G).

<u>Plan</u>
- Induced reps
- Ring of characters of S_n's

• Connect to symm fns

Let G be a group, H a subgroup. Let (V, ρ) of G, get a rep of H by $\rho|_H$.

Call this $\mathrm{Res}_H^G V$. (As a vector space, same as V.)

Given a rep (V, ρ) of H, can we get a rep of G?

Let $\mathbb{C}[G] = \mathrm{span}_{\mathbb{C}}\{g : g \in G\}$.

Elts are $\sum_{g \in G} c_g g$.

$(g_1 + g_2) g_3 = (g_1 g_3) + (g_2 g_3)$.

Define

$$\mathrm{Ind}_H^G V = \underline{\mathbb{C}[G]} \otimes_{\underline{\mathbb{C}[H]}} V.$$

Spanned by elements $g \otimes v$. For $g \in G$, $h \in H$,

$$(g\underline{h}) \otimes v = g \otimes (hv).$$

If H has cosets $g_1 H, \ldots, g_n H$ in G, then

$$g_i H \otimes V = g_i \otimes V. \text{ So}$$

$$\mathrm{Ind}\, V = (g_1 \otimes V) \oplus (g_2 \otimes V) \oplus \ldots \oplus (g_n \otimes V).$$

G acts by left multiplication:

$\forall k, g \in G$, $v \in V$,

$$k(g \otimes v) = (kg) \otimes v$$

Ex: $G = S_3$, $\underline{H} = \{(1), (12)\} \cong S_2$.

Let $V = \mathbb{C}$, with the H-action

$(1)v = v \quad (12)v = -v$.

$\mathrm{Ind}_{S_2}^{S_3} V$ spanned by

$(1) \otimes 1$, $(13) \otimes 1$, $(23) \otimes 1$.

How does (123) act?

$$(123)\big((1) \otimes 1\big) = \big((123)(1)\big) \otimes 1$$

$$= (123) \otimes 1 = (13)\underline{(12)} \otimes 1$$

$$= (13) \otimes \big((12)1\big)$$

$$= -(13) \otimes 1.$$

$$(123)\big((13) \otimes 1\big) = (23) \otimes 1$$

$$(123)\big((23) \otimes 1\big) = -(1) \otimes 1.$$

<u>Main Property</u>: Let V be

a rep of H. Let U be a rep of G. Then

$$\mathrm{Hom}_G(\mathrm{Ind}\, V, U)$$
$$\cong \mathrm{Hom}_H(V, \mathrm{Res}\, U)$$

<u>Sketch</u>: $\subseteq$: Let $f : \mathrm{Ind}\, V \to U$ be a G-module map. Define $f' : V \to U$ by $f'(v) = f(1 \otimes v)$. This is an H-module map.

$\supseteq$: Let $f : V \to U$ be an H-module map. Define $\bar{f} : \mathrm{Ind}\, V \to U$ by

$$\underline{\bar{f}(g \otimes v)} = \bar{f}(g(1 \otimes v))$$

$= g\,\underline{\underline{\tilde{f}(1 \otimes v)}}_{\sim} = g\,\underline{\underline{f(v)}}.$ □

Corollary (Frobenius Reciprocity)

$\langle \chi_{\mathrm{Ind}V}, \chi_U \rangle$

$= \dim \mathrm{Hom}_G(\mathrm{Ind}V, U)$

$= \dim \mathrm{Hom}_H(V, \mathrm{Res}U)$

$= \langle \chi_V, \chi_{\mathrm{Res}U} \rangle.$

We can calculate $\chi_{\mathrm{Ind}V}$ by knowing about the characters of H.

Ring of Characters of S_n's

Let R^n = abelian group generated by the irr characters of S_n.

Let $R = \bigoplus_{n=0}^{\infty} R^n$.

Want to put ring structure on R.

Let $\chi_V \in R^m$, $\chi_U \in R^n$.

$V \otimes U$ is a rep for

$S_m \times S_n$, where

$(w_1, w_2)(v \otimes u) = (w_1 v) \otimes (w_2 u)$.

View $S_m \times S_n \subseteq S_{m+n}$. Define

$\chi_V \cdot \chi_U = \chi \operatorname{Ind}_{S_m \times S_n}^{S_{m+n}} (V \otimes U)$. $\in R^{m+n}$

Can extend $\langle \cdot, \cdot \rangle$ by

$\langle R^m, R^n \rangle = 0$.

<u>Theorem</u>: Let Λ be the

ring of symmetric functions

(symmetric polynomials in arbitrarily many variables).
Can define a map
$ch : R \to \Lambda$ s.t.
ch is an isometric ring isomorphism.

<u>Applications</u>:
1) Explicit formulas for the irr characters of S_n.
2) Nice way of writing $\mathrm{Ind}_{S_n}^{S_{n+1}} V$ in terms of irr characters of S_{n+1}.

$p_r = x_1^r + x_2^r + \ldots$

$p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots p_{\lambda_k}$

$\langle p_\lambda, p_\mu \rangle = \delta_{\lambda\mu} z_\lambda$

nice combinatorial thing

Macdonald, Symmetric Functions and Hall Polynomials

1:30 wed George talk