

(1)

## Some solutions.

P91 §2.2 #8

Solution: •  $u_{tt} = c^2 u_{xx}$ 

- let  $G(x,t) = \frac{1}{\sqrt{4\pi k t}} e^{-\frac{x^2}{4kt}}$  be the fundamental solution.

We have.  $G_t = k G_{xx}$ 

$$\cdot V(x,t) = \frac{c}{\sqrt{4\pi k t}} \int_{-\infty}^{\infty} u(s,x) e^{-\frac{s^2 c^2}{4kt}} ds$$

$$= c \int_{-\infty}^{\infty} u(s,x) G(cs,t) ds$$

$$\cdot \text{Now. } V_t = c \int_{-\infty}^{\infty} u(s,x) G_t(cs,t) ds$$

$$= kc \int_{-\infty}^{\infty} u(s,x) G_{xx}(cs,t) ds$$

~~(\*)~~  $\boxed{\frac{\partial G}{\partial s} = c G_x}$   $\rightarrow = \frac{k}{c} \int_{-\infty}^{\infty} u(s,x) G_{ss}(cs,t) ds$

$$= \frac{k}{c} \int_{-\infty}^{\infty} u_{ss}(s,x) G(cs,t) ds$$

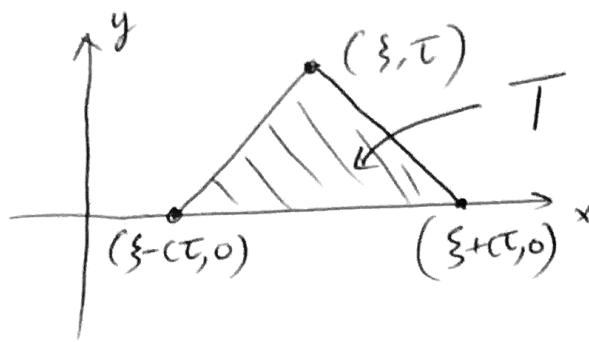
$$= ck \int_{-\infty}^{\infty} u_{xx}(s,x) G(cs,t) ds$$

$$= k V_{xx}$$

□

P92, §2.2, #9.

Solution:



- The region  $T$  is given by

$$T = \left\{ (x, y) \mid \begin{array}{l} 0 \leq y \leq \tau \\ c(y-\tau) + \xi \leq x \leq -c(y-\tau) + \xi \end{array} \right\}$$

- By the hint,

$$\begin{aligned} U(\xi, \tau) &= \frac{1}{2C} \iint_T f \, dx \, dy \\ &= \frac{1}{2C} \int_{y=0}^{\tau} \int_{x=c(y-\tau)+\xi}^{-c(y-\tau)+\xi} f(x, y) \, dx \, dy \end{aligned}$$

$$U_\xi(\xi, \tau) = \frac{1}{2C} \int_{y=0}^{\tau} \left( f(-c(y-\tau)+\xi, y) - f(c(y-\tau)+\xi, y) \right) dy$$

[by fundamental theorem of calculus]

$$U_{\xi\xi}(\xi, \tau) = \frac{1}{2C} \int_{y=0}^{\tau} \left( f_x(-c(y-\tau)+\xi, y) - f_x(c(y-\tau)+\xi, y) \right) dy$$

$$\begin{aligned} U_\tau(\xi, \tau) &= \frac{1}{2C} \int_{x=\xi}^{\xi} f(x, y) \, dx + \frac{1}{2C} \int_{y=0}^{\tau} C \left[ f_x(c(y-\tau)+\xi, y) + f(c(y-\tau)+\xi, y) \right] dy \\ &= \frac{1}{2} \int_{y=0}^{\tau} \left[ f(-c(y-\tau)+\xi, y) + f(c(y-\tau)+\xi, y) \right] dy \end{aligned}$$

$$\bullet \quad u_{tt}(\xi, \tau) = \frac{1}{2} [f(\xi, \tau) + f(\xi, -\tau)]$$

$$+ \frac{1}{2} \int_{y=0}^{\tau} [c f_x(-c(y-\tau)+\xi, y) - c f_x(c(y-\tau)+\xi, y)] dy$$

$$= f(\xi, \tau) + \frac{c}{2} \int_{y=0}^{\tau} (f_x(-c(y-\tau)+\xi, y) - f_x(c(y-\tau)+\xi, y)) dy$$

$$\Rightarrow \cancel{u_{tt} = f}$$

$$\boxed{u_{tt}(\xi, \tau) = f(\xi, \tau) + c^2 u_{xx}(\xi, \tau)}$$

$$\bullet \quad u(\xi, 0) = \frac{1}{2c} \int_{y=0}^0 \int_{x=c(y-\tau)+\xi}^{-c(y-\tau)+\xi} f(x, y) dx dy = 0$$

$$\Rightarrow \boxed{u(\xi, 0) = 0}$$

$$\bullet \quad \cancel{u_t(\xi, 0)} = \frac{1}{2c} \int_{y=0}^0 (f(-c(y-\tau)+\xi, y) - f(-c(y-\tau)+\xi, y)) dy \\ = 0$$

$$\Rightarrow \boxed{u_t(\xi, 0) = 0}$$

Therefore  $u(\xi, \tau)$  solves the wave eqn.

□

(4)

P95, §2.3, #1

Solution: We want to show the equation is unstable.

1) We consider  $u_n(x,t) = 1 + \frac{1}{n} e^{nt^2} \sin nx$ ,  $n \in \mathbb{Z}^+$ .

For  $t=0$ ,  $u_n(x,0) = 1 + \frac{1}{n} \sin nx$  is the initial condition.

2) Consider two solutions.

$u_n(x,t)$  and  $u_{2n}(x,t)$

3) at  $t=0$ ,

$$\begin{aligned}
 & |u_n(x,0) - u_{2n}(x,0)| \\
 &= \left| \frac{1}{n} e^{n^2 t} \sin(nx) - \frac{1}{2n} e^{4n^2 t} \sin(2nx) \right| \Big|_{t=0} \\
 &\leq \frac{1}{n} |\sin(nx)| + \frac{1}{2n} |\sin(2nx)| \\
 &\leq \frac{1}{n} + \frac{1}{2n} \\
 &\leq \frac{3}{2n} \quad \longrightarrow 0 \quad \text{as } n \rightarrow \infty
 \end{aligned}$$

4) Pick  $x = \frac{\pi}{2n}$ ,  $t = 1$ , Note  $|x| < \pi$

$$\begin{aligned} & \left| u_n\left(\frac{\pi}{2n}, 1\right) - u_{2n}\left(\frac{\pi}{2n}, 1\right) \right| \\ &= \left| \frac{1}{n} e^{n^2} \sin\left(\frac{\pi}{2}\right) - \frac{1}{2n} e^{4n^2} \sin(\pi) \right| \\ &= \frac{1}{n} e^{n^2} \rightarrow \infty, \quad \text{as } n \rightarrow \infty \end{aligned}$$

Combine (3) and (4), we have shown that  
the solution does not continuously depend on  
the initial data.

□

P100, §2.4, #4.

(6)

Solution: • Let  $V(x,t) = u(x,t) - 1$

• Then  $V(x,t)$  satisfies the following equation:

$$\begin{cases} V_t = KV_{xx}, & x > 0, t > 0 \\ V(0,t) = 0, & t > 0 \\ V(x,0) = -1, & x > 0 \end{cases}$$

• Then by §2.4 using the odd extension, we have

$$V(x,t) = \int_0^\infty [G(x+y,t) - G(x-y,t)] (-1) dy, \quad x > 0, t > 0$$

• Then

$$u(x,t) = V(x,t) + 1$$

$$= 1 + \int_0^\infty (G(x+y,t) - G(x-y,t)) dy.$$

□

Solution: • Consider  $w = w(x, t; \tau)$

$$\begin{cases} w_t = kw_{xx}, & x > 0, t > 0 \\ w(0, t; \tau) = 0, & t > 0 \\ w(x, 0; \tau) = f(x, \tau), & x > 0. \end{cases}$$

• By §2.4,

$$w(x, t; \tau) = \int_0^\infty (G(x-y, t) - G(x+y, t)) f(y, \tau) dy$$

• By Duhamel's principle,

$$u(x, t) = \int_0^t w(x, t-\tau; \tau) d\tau$$

$$= \int_0^t \int_0^\infty (G(x-y, t-\tau) - G(x+y, t-\tau)) f(y, \tau) dy d\tau$$