Solutions to the practice final exam

Problem 3 Solve the following Laplace equation
$$
\begin{cases}
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, & 0 < x < 1, 0 < y < 1, \\
u(0,y) = 10y, \ \left.\frac{\partial u}{\partial x}\right|_{x=1} = -1, \\
u(x,0) = 0, \ u(x,1) = 0.
\end{cases}
$$

Solution:

(1) Let $u(x,y) = F(x)G(y)$. Then we have
$$
\frac{F''}{F} = -\frac{G''}{G} = -\lambda.
$$

(2) Solve for G first. We have $\lambda = -n^2\pi^2$ and $G(x) = \sin(n\pi y)$ using the second boundary value condition.
   Then we solve for $F(x)$. $F(x) = c_1 e^{n\pi x} + c_2 e^{-n\pi x}$.

(3) The general solution is given by

$$
u(x,y) = \sum_{n=1}^{\infty} \left(a_n e^{n\pi x} + b_n e^{-n\pi x} \right) \sin(n\pi y).
$$

(4) We will use the first boundary value condition to determine a_n and b_n. First we have $u(0,y) = 10y = \sum_{n=1}^{\infty} (a_n + b_n)\sin(n\pi y)$ and so

$$
a_n + b_n = 2 \int_0^1 10y \sin(n\pi y) dy = 20(n\pi)^{-1}(-1)^{n+1}.
$$

   Then wehave $u_x(1,y) = -1 = \sum_{n=1}^{\infty} \left(n\pi a_n e^{n\pi} - n\pi b_n e^{-n\pi} \right) \sin(n\pi y)$ and so

$$
n\pi e^{n\pi} a_n - n\pi e^{-n\pi} b_n = 2 \int_0^1 (-1)\sin(n\pi y) dy = 2(n\pi)^{-1}((-1)^n - 1).
$$

   Now we can solve for a_n and b_n.

$$
a_n = \left((1 + e^{2n\pi}) n^2 \pi^2 \right)^{-1} \left(20(-1)^{n+1} n\pi + 2 e^{n\pi}((-1)^n - 1) \right),
$$

$$
b_n = 2 \int_0^1 10y \sin(n\pi y) dy = 20(n\pi)^{-1}(-1)^{n+1} - a_n.
$$

Problem 4 Solve the following boundary value problem.

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - 2u, & 0 < x < \pi, \ t > 0, \\ u(0,t) = 0, \ u(\pi,t) = -1, & t > 0, \\ u(x,0) = 0, & 0 < x < \pi. \end{cases}$$

Solution:

(1) We first make the boundary value condition homogenous by letting

$$v(x,t) = u(x,t) + \pi^{-1}x.$$

Then

$$\begin{cases} \frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} - 2v + 2\pi^{-1}x, & 0 < x < \pi, \ t > 0, \\ v(0,t) = 0, \ v(\pi,t) = 0, & t > 0, \\ v(x,0) = \pi^{-1}x, & 0 < x < \pi. \end{cases}$$

(2) Now consider the homogeneous equation $\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} - 2v$. Let $v(x,t) = F(x)G(t)$. Then

$$\frac{F''}{F} = \frac{G'}{G'} + 2 = \lambda.$$

Since $F(0,t) = F(\pi,t) = 0$, the eigenfunctions are given by

$$F(x,t) = \sin(nx), \ \lambda = -n^2.$$

(3) We now consider the equation

$$\begin{cases} \frac{\partial W}{\partial t} = \frac{\partial^2 W}{\partial x^2} - 2W + 2\pi^{-1}x, & 0 < x < \pi, \ t > 0, \\ W(0,t) = 0, \ W(\pi,t) = 0, & t > 0, \\ W(x,0) = 0, & 0 < x < \pi. \end{cases}$$

The general solution is given by

$$W(x,t) = \sum_{n=1}^{\infty} G_n(t)\sin(nx).$$

Plugging the expression $W(x,t)$ into the equation, we have

$$\sum_{n=1}^{\infty} G_n'(t)\sin(nx) = \sum_{n=1}^{\infty}(-n^2)G_n(t)\sin(nx) - 2\sum_{n=1}^{\infty} G_n(t)\sin(nx) + 2\pi^{-1}x.$$

On the other hand, the Fourier series of $2\pi^{-1}x$ over $[0,\pi]$ is given by

$$2(\pi)^{-1}x = \sum_{n=1}^{n} a_n\sin(nx), \ a_n = 4(\pi)^{-1}\int_0^{\pi} \pi^{-1}x\sin(nx)dx = 4(n\pi)^{-1}(-1)^{n+1}.$$

Therefore $G_n(t)$ satisfies

$$G_n' = -(n^2 + 2)G_n + a_n, \ G_n(0) = 0.$$

Therefore

$$G_n(t) = \int_0^{t} a_n e^{-(n^2+2)(t-\tau)}d\tau = (n^2 + 2)^{-1}a_n \left(e^{-(n^2+2)(t-\tau)} - e^{-(n^2+2)t}\right).$$

So

$$W(x,t) = \sum_{n=1}^{\infty}(n^2+2)^{-1}a_n \left(e^{-(n^2+2)(t-\tau)} - e^{-(n^2+2)t}\right)\sin(nx), \ a_n = 4(n\pi)^{-1}(-1)^{n+1}.$$

(4) We now consider the equation

$$\begin{cases} \frac{\partial Y}{\partial t} = \frac{\partial^2 Y}{\partial x^2} - 2Y, & 0 < x < \pi, \ t > 0, \\ Y(0,t) = 0, \ Y(\pi,t) = 0, & t > 0, \\ Y(x,0) = \pi^{-1}x, & 0 < x < \pi. \end{cases}$$

The general solution is given by

$$Y(x,t) = \sum_{n=1}^{\infty} b_n e^{-(n^2+2)t} \sin(nx).$$

Using the initial value condition, we have

$$Y(x,0) = \pi^{-1}x = \sum_{n=1}^{\infty} b_n \sin(nx).$$

So $b_n = a_n/2 = 2(n\pi)^{-1}(-1)^{n+1}$.

(5) Finally, we have

$$V(x,t) = W(x,t) + Y(x,t)$$

and

$$u(x,t) = W(x,t) + Y(x,t) - \pi^{-1}x.$$

Problem 5 Solve the following boundary value problem.

$$\begin{cases} \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0, & 0 \le \theta \le 2\pi, \ 0 < a < r < b, \\ u(a,\theta) = f(\theta), \ u(b,\theta) = g(\theta), & 0 \le \theta \le 2\pi. \end{cases}$$

Solution:

(1) Write $u(r,\theta) = F(r)G(\theta)$. Then we have

$$\frac{G''}{G} = -\frac{r^2 F'' + rF'}{F} = \lambda.$$

(2) Solve for $G(\theta)$ first. Since G is periodic in 2π, we have $\lambda = -n^2$ and

$$G(\theta) = a_n \cos(n\theta) + b_n \sin(n\theta).$$

F satisfies

$$r^2 F'' + rF' = -\lambda F = n^2 F.$$

So

$$F(r) = r^n, \ r^{-n}, \ \text{or} \ \ln r.$$

(3) The general solution is given by

$$u(r,\theta) = K_0 + K_1 \ln r + \sum_{n=1}^{\infty} r^n (A_n \cos(n\theta) + B_n \sin(n\theta)) + \sum_{n=1}^{\infty} r^{-n}(C_n \cos(n\theta) + D_n \sin(n\theta)).$$

(4)

$$u(a,\theta) = K_0 + K_1 \ln a + \sum_{n=1}^{\infty} a^n (A_n \cos(n\theta) + B_n \sin(n\theta)) + \sum_{n=1}^{\infty} a^{-n}(C_n \cos(nx) + D_n \sin(n\theta))$$

$$u(b,\theta) = K_0 + K_1 \ln b + \sum_{n=1}^{\infty} b^n (A_n \cos(n\theta) + B_n \sin(n\theta)) + \sum_{n=1}^{\infty} b^{-n}(C_n \cos(n\theta) + D_n \sin(n\theta))$$

Now suppose the Fourier series of f and g are given by

$$f(\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos(n\theta) + b_n \sin(n\theta)), \ g(\theta) = \frac{c_0}{2} + \sum_{n=1}^{\infty}(c_n \cos(n\theta) + d_n \sin(n\theta))$$

with

$$a_n = \pi^{-1}\int_0^{2\pi} f(\theta)\cos(n\theta)d\theta, \ c_n = \pi^{-1}\int_0^{2\pi} g(\theta)\cos(n\theta)d\theta, \ n = 0,1,...$$

$$b_n = \pi^{-1}\int_0^{2\pi} f(\theta)\sin(n\theta)d\theta, \ d_n = \pi^{-1}\int_0^{2\pi} g(\theta)\sin(n\theta)d\theta, \ = 1,2,...$$

Then we have

$$K_0 + K_1 \ln a = a_0/2, \ K_0 + K_1 \ln b = c_0/2$$

$$a^n A_n + a^{-n}C_n = a_n, a^n B_n + a^{-n}D_n = b_n, \ n = 1,2,...$$

$$b^n A_n + b^{-n}C_n = c_n, b^n B_n + b^{-n}D_n = d_n, \ n = 1,2,...$$

Then solve for A_n, B_n, C_n, D_n. For example,

$$A_n = (b^n c_n - a^n a_n)/(b^{2n} - a^{2n}).$$

Problem 6 Solve the following boundary value problem.

$$\begin{cases} u_t = u_{xx}, & 0 < x < \pi, \\ u(0,t) = e^{-t}, \ u(\pi,t) = t, & t > 0, \\ u(x,0) = x, & 0 < x < \pi. \end{cases}$$

Solution:

(1) First we let

$$v(x,t) = u(x,t) - \left((1-x)e^{-t} + xt\right).$$

Then $v(x,t)$ satisfies

$$\begin{cases} v_t = v_{xx} + e^{-t} - (1+e^{-t})x, & 0 < x < \pi, \\ v(0,t) = v(\pi,t) = 0, & t > 0, \\ v(x,0) = 2x - 1, & 0 < x < \pi. \end{cases}$$

(2) We first solve

$$\begin{cases} W_t = W_{xx} + e^{-t} - (1+e^{-t})x, & 0 < x < \pi, \\ W(0,t) = W(\pi,t) = 0, & t > 0, \\ W(x,0) = 0, & 0 < x < \pi. \end{cases}$$

The eigenfunctions for the homogenous equation are given by $\sin(nx)$. Therefore we write

$$W(x,t) = \sum_{n=1}^{n} G_n(t)\sin(nx).$$

Then

$$\sum_{n=1}^{n} G_n'(t)\sin(nx) = \sum_{n=1}^{n}(-n^2)G_n(t)\sin(nx) + e^{-t} - (1+e^{-t})x$$

The Fourier series of $e^{-t} - (1+e^{-t})x$ is given by

$$e^{-t} - (1+e^{-t})x = \sum_{n=1}^{n} a_n(t)\sin(nx),$$

where

$$a_n(t) = 2\pi^{-1}\int_0^\pi (e^{-t} - (1+e^{-t})x)\sin(nx)dx = 2(n\pi)^{-1}(1+e^{-t}).$$

So

$$G_n'(t) = -n^2 G_n(t) + a_n(t), \ G_n(0) = 0.$$

Solving G_n, we have

$$G_n(t) = \int_0^t a_n(\tau)e^{-n^2(t-\tau)}d\tau.$$

When $n \geq 2$,

$$G_n(t) = \frac{2e^{-n^2t}}{n\pi}\left(n^{-2}(e^{n^2t} - 1) + (n^2-1)^{-1}(e^{(n^2-1)t} - 1)\right).$$

When $n = 1$,

$$G_1(t) = \frac{2e^{-t}}{\pi}\left(n^{-2}(e^{n^2t} - 1) + t\right).$$

(3) We now solve

$$\begin{cases} Y_t = Y_{xx}, & 0 < x < \pi, \\ Y(0,t) = Y(\pi,t) = 0, & t > 0, \\ Y(x,0) = 2x - 1, & 0 < x < \pi. \end{cases}$$

The general solution is given by

$$Y(x,t) = \sum_{n=1}^{\infty} b_n e^{-n^2 t} \sin(nx).$$

So

$$Y(x,0) = 2x - 1 = \sum_{n=1}^{\infty} b_n \sin(nx)$$

$$b_n = 2\pi^{-1} \int_0^{\pi} (2x - 1) \sin(nx) dx = 2(n\pi)^{-1} (1 + (-1)^{n+1} (2\pi - 1)).$$

(4) Finally, we have

$$u(x,t) = v(x,t) + \left((1-x)e^{-t} + xt\right) + W(x,t) + Y(x,t) + \left((1-x)e^{-t} + xt\right).$$