1. Let $\mathcal{C}$ be the ellipse that is the intersection of the plane $x + 2y - z = 0$ and the cylinder $x^2 + y^2 = 9$. Suppose a particle starts at the point $(3, 0, 3)$ and traverses $\mathcal{C}$ exactly twice. (The orientation of the path is unimportant.)

(a) Find a single vector-valued function $\mathbf{r}(t)$ that describes the path of the particle. Be sure to specify the domain of $\mathbf{r}(t)$.

(b) Write down an integral whose value is the distance traveled by the particle. Do not attempt to evaluate this integral.

Solution

(a) The most general parametrization of the cylinder is $\mathbf{r}(t) = \langle 3 \cos(t), 3 \sin(t), z(t) \rangle$ for some unknown function $z(t)$. Since this curve must also lie on the plane $z = x + 2y$, we immediately find that $z(t) = 3 \cos(t) + 6 \sin(t)$. Hence our parametrization is

$$\mathbf{r}(t) = \langle 3 \cos(t), 3 \sin(t), 3 \cos(t) + 6 \sin(t) \rangle$$

Note that $\mathbf{r}(0) = \langle 3, 0, 3 \rangle$, and so our particle already starts at the correct point. If the particle is to traverse the curve exactly twice then the domain should be taken to be two periods. One such domain is $[0, 4\pi]$.

(b) The distance traveled by the particle is given by the following integral.

$$s = \int_0^{4\pi} \|\mathbf{r}(t)\| \, dt = \int_0^{4\pi} \sqrt{9 + (3 \cos(t) + 6 \sin(t))^2} \, dt$$

2. Let $\mathcal{C}$ be the curve with the following parametrization.

$$\mathbf{r}(t) = \langle \ln(2t^2 + 1), 3 \cos(t), t^3 + t^2 + 5t + 2 \rangle$$

Find a parametrization of the line tangent to $\mathcal{C}$ when $t = 0$.

Solution

The point of tangency is $\mathbf{r}(0) = \langle 0, 3, 2 \rangle$ and the direction vector is

$$\mathbf{r}'(0) = \left. \left( \frac{4t}{2t^2 + 1}, -3 \sin(t), 3t^2 + 2t + 5 \right) \right|_{t=0} = \langle 0, 0, 5 \rangle$$

Hence a parametrization of the tangent line is

$$\mathbf{L}(t) = \langle 0, 3, 2 \rangle + t \langle 0, 0, 5 \rangle = \langle 0, 3, 2 + 5t \rangle$$

(The domain is $\mathbb{R}$.)