1. Let $S$ be the portion of the paraboloid $z = x^2 + y^2$ cut out by the planes $z = 1$ and $z = 4$, and suppose $S$ is oriented by a downward-pointing normal vector. Let $F = (4x, 4y, 2)$.

(a) Find a parametrization of $S$. Indicate the parameter domain and indicate whether your parametrization preserves the orientation of $S$. Explain your answer.

(b) Calculate the flux of $F$ across the oriented surface $S$.

Solution

(a) A parametrization of the surface using cylindrical coordinates is given by:

$$G(r, \theta) = (r \cos(\theta), r \sin(\theta), r^2)$$

The projection of the surface in the $xy$-plane is the annular region bounded by the circles $r = 1$ and $r = 2$. Hence the parameter domain is $(r, \theta) \in [1, 2] \times [0, 2\pi]$. The normal vector induced by $G$ is

$$N = T_r \times T_\theta = \begin{vmatrix} i & j & k \\ \cos(\theta) & \sin(\theta) & 2r \\ -r \sin(\theta) & r \cos(\theta) & 0 \end{vmatrix} = \langle -2r^2 \cos(\theta), -2r^2 \sin(\theta), r \rangle$$

Note that $N$ has a positive $z$-component, and so $N$ reverses the orientation of $S$.

(b) We have the following calculations.

$$F(G(r, \theta)) = (4r \cos(\theta), 4r \sin(\theta), 2)$$

$$F \cdot N = -8r^3 + 2r$$

The flux is thus given by the following integral. (Note the leading minus sign since $G$ parametrizes $-S$ (i.e., $S$ with the opposite orientation).

$$\iint_S F \cdot dS = - \int_0^{2\pi} \int_1^2 (2r - 8r^3) \, dr \, d\theta$$

$$= - \left( \int_0^{2\pi} d\theta \right) \left( \int_1^2 (2r - 8r^3) \, dr \right) = -2\pi(-27) = 54\pi$$