Let $\mathcal{T}$ be the region bounded by the tetrahedron with vertices $(0, 0, 0)$, $(1, 0, 0)$, $(0, 2, 0)$, and $(0, 0, 4)$. Calculate $\iiint_{\mathcal{T}} 8z \, dV$.

**Solution**

The equation of the plane containing the vertices of $\mathcal{T}$ (except the origin) is

$$x + \frac{y}{2} + \frac{z}{4} = 1$$

The bottom face of $\mathcal{T}$ is the plane $z = 0$ and the top (slanted) face is the plane

$$z = 4 - 4x - 2y$$

Thus points inside $\mathcal{T}$ satisfy $0 \leq z \leq 4 - 4x - 2y$. The projection of $\mathcal{T}$ in the $xy$-plane is the triangle with vertices $(0,0)$, $(1,0)$, and $(0,2)$. The line containing $(1,0)$ and $(0,2)$ is $y = 2 - 2x$. Thus points inside $\mathcal{T}$ also satisfy $0 \leq y \leq 2 - 2x$ and $0 \leq x \leq 1$. We may now set up and compute the desired integral.

$$\iiint_{\mathcal{T}} 8z \, dV = \int_0^1 \int_0^{2-2x} \int_0^{4-4x-2y} 8z \, dz \, dy \, dx = \int_0^1 \int_0^{2-2x} (4z^2) \bigg|_{z=0}^{z=4-4x-2y} dy \, dx$$

$$= \int_0^1 \int_0^{2-2x} 4(4-4x-2y)^2 \, dy \, dx = \int_0^1 \left( -\frac{2}{3} (4-4x-2y)^3 \right) \bigg|_{y=0}^{y=2-2x} dx$$

$$= \int_0^1 \frac{2}{3} (4-4x)^3 \, dx = -\frac{1}{24} (4-4x)^4 \bigg|_{x=0}^{x=1} = \frac{4^4}{24} = \frac{32}{3}$$