Section 17.1: Green's Theorem

Quick Review of Notation

\[ \int_C F \cdot dr \text{ : line integral on the closed curve } C \]

\[ D \text{ : domain in } xy\text{-plane} \]

\[ \partial D \text{ : boundary of } D \]

(assumed simple, closed)

Induced Orientation

* if \( D \) is a region in \( xy\)-plane, we will say \( D \) has **positive** orientation if \( D \) is oriented by an upward (along positive \( z\)-axis) normal vector.

* Orientation of a region induces an orientation on its boundary. If \( D \) has positive orientation, then \( \partial D \) is oriented anticlockwise
Anticlockwise: If you travel along $\partial D$, then the region $D$ should always lie to your left.

**Thm.: (Green's Theorem)**
Suppose $D$ is a region in $xy$-plane with positive orientation with $\partial D$ a simple, closed curve. Then

$$\oint_C (F_1 \, dx + F_2 \, dy) = \iint_D \left( \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \, dA$$

**Ex. 1**
Verify Green's Theorem for

$$\oint_C (xy \, dx + y \, dy)$$

where $C$ is the unit circle oriented clockwise.
anticlockwise.

Solution:

(a) Direct Method

\[ C : \mathbf{r}(t) = \langle \cos(t), \sin(t) \rangle \]
\[ 0 \leq t < 2\pi \]

\[ \mathbf{F} = \langle x, y, y \rangle \]

\[ \mathbf{F}(\mathbf{r}(t)) = \langle \cos(t) \sin(t), \sin(t) \rangle \]

\[ \mathbf{r}'(t) = \langle -\sin(t), \cos(t) \rangle \]

\[ \mathbf{F} \cdot d\mathbf{r} = (-\cos(t)\sin(t)^2 + \cos(t)\sin(t)) \, dt \]

Now set up integral.

\[ \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} (-\sin(t)^2 + \sin(t)) \cos(t) \, dt \]

\[ = \left( -\frac{\sin(t)^3}{3} + \frac{\sin(t)^2}{2} \right) \bigg|_{t=0}^{t=2\pi} \]

\[ = 0 - 0 = 0 \]

(b) Green's Theorem
\[ C \quad \text{D: unit disk} \quad \text{Then } \partial D = C \]

\[ \mathbf{F} = \langle xy, y \rangle \]

\[ \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = \frac{\partial}{\partial x} (y) - \frac{\partial}{\partial y} (xy) = 0 - x = -x \]

Now use Green's Theorem.

\[ \oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_D (-x) \, dA \]

use polar coordinates to set up integral

\[ = \int_0^{2\pi} \int_0^1 (-r \cos \theta) \, r \, dr \, d\theta \]

\[ = -\int_0^{2\pi} \int_0^1 r^2 \cos (\theta) \, dr \, d\theta \]
\[ \begin{align*}
&= - \left( \int_0^1 \cos(\theta) d\theta \right) \cdot \left( \int_0^1 r dr \right) \\
&= - \left( 0 \right) \cdot \left( \frac{1}{3} \right) = 0
\end{align*} \]

**Ex. 2**

Calculate \( \int_C (e^{2x+y}\,dx + e^{-y}\,dy) \), where \( C \) is the triangle with vertices \((0,0)\), \((1,0)\), and \((1,1)\). (anticlockwise)

**Solution**: 

Let \( D \) be the interior of the triangle. Then we can use Green's Theorem.

\[ \vec{F} = \langle e^{2x+y}, e^{-y} \rangle \]

\[ \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = \frac{\partial}{\partial x} (e^{-y}) - \frac{\partial}{\partial y} (e^{2x+y}) = -e^{2x+y} \]
Now use Green's Thm.

\[ \oint_C \vec{F} \cdot d\vec{r} = \iint_D (-e^{2x+y}) \, dA \]

set up as dydx-integral

\[ = -\int_0^1 \int_0^x e^{2x+y} \, dy \, dx = -\int_0^1 (e^{2x+y} \mid_{y=0}^{y=x}) \, dx \]

\[ = -\int_0^1 (e^{3x} - e^{2x}) \, dx = \int_0^1 (e^{2x} - e^{3x}) \, dx \]

\[ = \left( \frac{e^{2x}}{2} - \frac{e^{3x}}{3} \right) \mid_{x=1}^{x=0} = \frac{e^2}{2} - \frac{e^3}{3} - \frac{1}{6} \]

**Calculating Area with Green's Theorem**

\[ \iint_D x \, dy = \iint_D \left( \frac{\partial}{\partial x} (x) - \frac{\partial}{\partial y} (0) \right) \, dA \]

\[ \vec{F} = \langle 0, x \rangle \]

\[ = \iint_D 1 \, dA = \text{area} (D) \]
Similarly, we have:

\[
\oint_{\partial D} x \, dy = -\oint_{\partial D} y \, dx = \frac{1}{2} \oint_{\partial D} (-y \, dx + x \, dy)
\]

all are equal to \( \text{area}(D) \)

Ex. 3

Use Green's Theorem to calculate the area of the polygon.

Solution:

We will use the formula

\[ \oint_{\partial D} x \, dy = \text{area}(D) \]
First parametrize each side:

\[ \mathbf{r}(t) = \langle t, 0 \rangle \quad 0 \quad 2 \]

\[ \mathbf{r}(t) = \langle t, 2t-4 \rangle \quad 2 \quad 3 \]

\[ \mathbf{r}(t) = \langle t, \frac{7}{2} - \frac{1}{2}t \rangle \quad 3 \quad 1 \]

\[ \mathbf{r}(t) = \langle t, t+2 \rangle \quad 1 \quad 0 \]

\[ \mathbf{r}(t) = \langle 0, t \rangle \quad 2 \quad 0 \]

Now set up line integral for each side:

\[ 6 \times dy \]
Now calculate the line integrals.

\[ \oint_{\mathbb{D}} x \, dy = \int_{0}^{2} 2t \, dt + \int_{2}^{3} (\frac{-1}{2}t) \, dt + \int_{3}^{1} t \, dt \]

= \left[ t^2 \right]_{2}^{3} - \frac{1}{4} t^2 \left| \right|_{3}^{1} + \frac{t^2}{2} \left| \right|_{1}^{0} \\
= (9-4) + \left( \frac{9}{4} - \frac{1}{4} \right) - \left( \frac{1}{2} - 0 \right) \\
= \frac{13}{2} \quad \text{(area of polygon)}
*What about regions whose boundaries are not connected?

*What about a curve that does not bound a closed region?

Q: If $D$ has orientation, how should each boundary curve be oriented?
Let \( C \) be the oriented curve

\[
C = Y_{in} + C_2 + Y_{out} + C_1
\]

Note the following:
- \( C \) is a closed curve
- \( C \) has \( \Theta \) orientation (Why?)

(region D always to left as we travel along \( C \))

So by Green's Theorem,

\[
\int_D \left( \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \, dA = \oint_C \mathbf{F} \cdot d\mathbf{r}
\]

\[
= \int_{Y_{in}} + \int_{C_2} + \int_{Y_{out}} + \int_{C_1}
\]

Since \( Y_{out} = -Y_{in} \), \( \int_{Y_{in}} + \int_{Y_{out}} = 0 \)

\[
= \int_{C_2} + \int_{C_1}
\]
\[ \oint_{C_2} F \cdot dr + \oint_{C_1} F \cdot dr \]

So what does all of this mean? If \( D \) has a boundary that consists of disconnected curves, then:

- outer curve should be anticlockwise
- inner curve should be clockwise

\[ \iint_D \left( \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \, dA = \oint_{C_1} \mathbf{F} \cdot d\mathbf{r} + \oint_{C_2} \mathbf{F} \cdot d\mathbf{r} \]
Ex. 4

Calculate \( \oint_{C_1} \vec{F} \cdot d\vec{r} \) where \( C_1 \) is shown below and

\[
\vec{F} = \langle x-y, x+y^3 \rangle
\]

The inner curve is the unit circle and the area of \( D \) is 8.

Solution:
Let $C_2$ be the anticlockwise unit circle. Then

$$\partial D = C_1 - C_2$$

We also have

$$\vec{F} = \langle x - y, x + y^3 \rangle$$

$$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = \frac{\partial}{\partial x}(x + y^3) - \frac{\partial}{\partial y}(x - y) = 2$$

So by Green's Theorem,

$$\iint_D 2 \, dA = \oint_{\partial D} \vec{F} \cdot d\vec{r}$$
\[ 2 \text{area}(D) = \oint_{C_1 - C_2} \vec{F} \cdot d\vec{r} \]

\[ 16 = \oint_{C_1} \vec{F} \cdot d\vec{r} - \oint_{C_2} \vec{F} \cdot d\vec{r} \]

Solving for the desired line integral gives us

\[ \oint_{C_1} \vec{F} \cdot d\vec{r} = 16 + \oint_{C_2} \vec{F} \cdot d\vec{r} \]

\( C_2 \) is anticlockwise unit circle

Now we calculate \( \oint_{C_2} \vec{F} \cdot d\vec{r} \).

\( C_2 : \vec{F}(t) = \langle \cos(t), \sin(t) \rangle \quad 0 \leq t < 2\pi \)

\[ \vec{F} = \langle x-y, x+y^3 \rangle \]

\[ \vec{F}(\vec{r}(t)) = \langle \cos(t)-\sin(t), \cos(t)+\sin(t)^3 \rangle \]

\[ \vec{F}'(t) = \langle -\sin(t), \cos(t) \rangle \]

\[ \vec{F} \cdot d\vec{r} = (1 - \cos(t)\sin(t) + \cos(t)\sin(t)^3) \, dt \]

\[ \oint \vec{F} \cdot d\vec{r} = \int_0^{2\pi} (1 - \cos(t)\sin(t) + \cos(t)\sin(t)^3) \, dt \]
\[ \int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \left( 1 - \cos(t) \sin^2(t) - \sin(t) \right) \, dt \]
\[ = \left( t - \frac{\sin(t)^2}{2} + \frac{\sin(t)^4}{4} \right) \Bigg|_{t=0}^{t=2\pi} \]
\[ = (2\pi - 0 + 0) - (0 + 0 + 0) \]
\[ = 2\pi \]

So this gives us ....

\[ \int_{C_1} \mathbf{F} \cdot d\mathbf{r} = 16 + \int_{C_2} \mathbf{F} \cdot d\mathbf{r} \]
\[ = 16 + 2\pi \]

Clarification on Multiply-Connected Regions

If we are given an orientation of the region \( D \), then this induces an orientation on \( \partial D \).

The oriented boundary of \( D \) is

\[ 2\partial D = C_1 - C_2 \]
Ex. 5

Let \( \vec{F} = \langle x^3, 4x \rangle \). Calculate \( \int_C \vec{F} \cdot d\vec{r} \) where \( C \) is the oriented curve below.

**Solution:**
Let \( \tilde{C} = C + Y \). Since \( \tilde{C} \) is closed, we can apply Green's Theorem.

\[
\oint_{\tilde{C}} \vec{F} \cdot d\vec{r} = \iint_D \left( \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dA
\]

\( D \) is polygon bounded by \( \tilde{C} \).

Now we have

\[
\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = \frac{2}{2x} (4x) - \frac{2}{2y} (x^3) = 4
\]

So then

\[
\oint_{\tilde{C}} \vec{F} \cdot d\vec{r} = \iint_D 4 \, dA = 4 \text{ area}(D) = 16
\]

Now write line integral as sum

\[
16 = \oint_{\tilde{C}} \vec{F} \cdot d\vec{r} = \int_C \vec{F} \cdot d\vec{r} + \int_Y \vec{F} \cdot d\vec{r}
\]

\[
\int_C \vec{F} \cdot d\vec{r} = 16 - \int_Y \vec{F} \cdot d\vec{r}
\]
Now calculate $\int_Y \mathbf{F} \cdot d\mathbf{r}$.

- **Parametrization of $Y$:**
  \[ \mathbf{F}(t) = \langle -1, t \rangle \quad , \quad -1 \leq t \leq 0 \]

- **Calculation of $\mathbf{F} \cdot d\mathbf{r}$:**
  \[ \mathbf{F} = \langle x^3, 4x \rangle \]
  \[ \mathbf{F}(\mathbf{r}(t)) = \langle -1, -4 \rangle \]
  \[ \mathbf{F}'(t) = \langle 0, 1 \rangle \]
  \[ \mathbf{F} \cdot d\mathbf{r} = (0 - 4) \, dt = -4 \, dt \]

- **Calculation of integral:**
  \[ \int_Y \mathbf{F} \cdot d\mathbf{r} = \int_{-1}^{0} (-4) \, dt = -4 \]

So the desired line integral is

\[ \int_C \mathbf{F} \cdot d\mathbf{r} = 16 - \int_Y \mathbf{F} \cdot d\mathbf{r} = 20 \]