Section 16.1: Vector Fields

In general, a vector field on $\mathbb{R}^n$ is a function $\vec{F}: \mathbb{R}^n \to \mathbb{R}^n$. That is, $\vec{F}$ has $n$ inputs and $n$ outputs. We will focus on $n=2$ and $n=3$.

We think of $\vec{F}(\vec{x})$ as a vector $\vec{x} = \langle x_1, y \rangle$ for $\mathbb{R}^2$

$\vec{x} = \langle x_1, y, z \rangle$ for $\mathbb{R}^3$

that is based at $\vec{x}$. $\vec{F}(\vec{x})$ represents the velocity of the wind at point $\vec{x}$.
Speed is length of vector.

\[ \mathbf{F} \cdot \hat{r} = \langle -y, x \rangle \cdot \langle x, y \rangle = 0 \]

**Notation:**

*If \( \mathbf{F} : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \), then*

\[ \mathbf{F}(\vec{r}) = \mathbf{F}(x, y) = \langle F_1(x, y), F_2(x, y) \rangle \]

\[ = F_1 \hat{i} + F_2 \hat{j} \]

*If \( \mathbf{F} : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \), then*

\[ \mathbf{F}(\vec{r}) = \mathbf{F}(x, y, z) = \langle F_1(x, y, z), F_2(x, y, z), F_3(x, y, z) \rangle \]

\[ = F_1 \hat{i} + F_2 \hat{j} + F_3 \hat{k} \]

*We will assume that \( \mathbf{F} \) has continuous
partial derivatives.

Some Important Vector Fields

Let \( r = \sqrt{x^2 + y^2} \) or \( r = \sqrt{x^2 + y^2 + z^2} \), depending on context.

1. Unit Radial Vector Field

**\( \mathbb{R}^2 \):**

\[
\hat{e}_r = \left( \frac{x}{\sqrt{x^2 + y^2}}, \frac{y}{\sqrt{x^2 + y^2}} \right)
\]

\[
\hat{e}_r = \frac{1}{r} \left< x, y \right>
\]

**\( \mathbb{R}^3 \):**

\[
\hat{e}_r = \left( \frac{x}{\sqrt{x^2 + y^2 + z^2}}, \frac{y}{\sqrt{x^2 + y^2 + z^2}}, \frac{z}{\sqrt{x^2 + y^2 + z^2}} \right)
\]

\[
\hat{e}_r = \frac{1}{r} \left< x, y, z \right>
\]

“unit”: \( ||\hat{e}_r|| = 1 \) for all \( x, y, z \)

“radial”: points radially outward at each point.
inverse-square radial vector field:
\[ \vec{F} = \frac{1}{r^2} \hat{e}_r \]
(Note: \( ||\vec{F}|| = \frac{1}{r^2} \)) \( \vec{F} = \nabla \left(-\frac{1}{r}\right) \)

2) Gradient vector field
These are vector fields \( \vec{F} \) that can be written as the gradient of a scalar function \( f \).
\[ \vec{F} = \nabla f \]
(a) Recall that \( \nabla f \) points \( \perp \) to the contour lines of \( f \).
So if we are given the graph of a vector field $\vec{F}$ which is the gradient of $f$, then we also easily draw the contour lines of $f$.

**Ex. 1**

Given the graph of $\nabla f$ below, sketch contour lines of $f$.

Solution:
(b) Gradient vector fields conserve energy. (Section 16.3: fundamental theorem of gradients)

**Def:** If there is a scalar function $f$ such that $\vec{F} = \nabla f$, then we call $\vec{F}$ a **conservative vector field**. The function $f$ is called a **potential function** for $\vec{F}$.

**Q:** Is there a way to tell whether a vector field is conservative?

**A:** Qualified “yes”. Before we answer why, we need to introduce two new operations on vector fields.

**Def:** Given a vector field $\vec{F}$, we define its **divergence** and **curl** as
\[
\text{div}(\vec{F}) = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}
\]

\[
\text{curl}(\vec{F}) = \left( \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) \hat{i} + \left( \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \right) \hat{j} + \left( \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \hat{k}
\]

"Del"-operator: \( \nabla = \hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \)

<table>
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<td>divergence</td>
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The notation is meant to be suggestive.

\[
\nabla f = \left( \hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) f
\]
\[ \nabla \times \mathbf{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} \]

\[ = \left( \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) \hat{i} + \left( \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \right) \hat{j} + \left( \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \hat{k} \]

\[ \nabla \cdot \mathbf{F} = \left( \hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) \cdot (F_1 \hat{i} + F_2 \hat{j} + F_3 \hat{k}) \]

\[ = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \]

**Ex. 2**

Let \( \mathbf{F} = <z-y^2, x+z^3, y+x^2> \)

Calculate \( \nabla \cdot \mathbf{F} \) and \( \nabla \times \mathbf{F} \).

**Solution:**

\[ \nabla \cdot \mathbf{F} = 2 \left( \frac{\partial}{\partial x} \right) + 3 \left( \frac{\partial}{\partial z} \right) + 0 \left( \frac{\partial}{\partial y} \right) \]

\[ \nabla \times \mathbf{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} \]
\[ \nabla \cdot \vec{F} = \frac{\partial}{\partial x} (z-y^2) + \frac{\partial}{\partial y} (x+z^3) + \frac{\partial}{\partial z} (y+x^2) \]

\[ = 0 + 0 + 0 \]

\[ = 0 \]

(In physics, you might be told that this implies \( \vec{F} = \nabla \times \vec{G} \).)

\[ \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z-y^2 & x+z^3 & y+x^2 \end{vmatrix} \]

\[ = \left( 1 - 3z^2 \right) \hat{i} - \left( 2x - 1 \right) \hat{j} + \left( 1 + 2y \right) \hat{k} \]

\[ \nabla \times \vec{F} = \langle 1 - 3z^2, -(2x-1), 1+2y \rangle \]

**Ex. 3**

Let \( f(x,y,z) = x^2 e^y z^3 \)
Calculate \( \vec{\nabla} \times (\vec{\nabla} f) \).

**Solution:**

\[
\vec{\nabla} f = \left( \hat{i} \frac{\partial f}{\partial x} + \hat{j} \frac{\partial f}{\partial y} + \hat{k} \frac{\partial f}{\partial z} \right) f
\]

\[
= \hat{i} \frac{\partial f}{\partial x} + \hat{j} \frac{\partial f}{\partial y} + \hat{k} \frac{\partial f}{\partial z}
\]

\[
= \langle 2xe^y z^3, xe^y z^3, 3xe^y z^2 \rangle
\]

\[
\vec{\nabla} \times (\vec{\nabla} f) = \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
2xe^{y}z^{3} & x^{2}e^{y}z^{3} & 3x^{2}e^{y}z^{2}
\end{vmatrix}
\]

\[
= (3x^2e^{y}z^2 - 3x^2e^{y}z^2) \hat{i} \\
- (6xe^{y}z^2 - 6xe^{y}z^2) \hat{j} \\
+ (2xe^{y}z^3 - 2xe^{y}z^3) \hat{k}
\]

\[
= 0 \hat{i} + 0 \hat{j} + 0 \hat{k} = \vec{0}
\]
This is not an accident.

**Thm:** Suppose \( f \) has continuous second partial derivatives. Then...

\[
\nabla \times (\nabla f) = 0
\]
\[
\nabla \cdot (\nabla \times \hat{F}) = 0
\]

This gives us one test for conservativeness.

**Thm:** If \( \nabla \times \hat{F} \neq \hat{0} \), then \( \hat{F} \) is not conservative.

Is the converse true? That is, suppose \( \nabla \times \hat{F} = \hat{0} \). Does this mean \( \hat{F} \) is conservative?

Not necessarily. Depends on domain of \( \hat{F} \). (Section 6.3.)