Section 14.4: Tangent Planes

YOUR TEXTBOOK IS WRONG

Opening paragraphs of 14.4 are wrong. Listen to me!
To find equation of tangent plane, we first find two non-parallel vectors in that plane.

The vector \( \mathbf{v} \) is tangent to surface and lies in plane \( y = b \).

What are components of \( \mathbf{v} \)? To stay on tangent line....

1. move 1 unit in positive \( x \)-direction
2. move 0 units in \( y \)-direction
3. move \( f_x \) units in \( z \)-direction

So this means...

\[
\mathbf{v} = \langle 1, 0, f_x \rangle
\]

Similarly, to stay on tangent line for \( \mathbf{u} \)....

1. move 0 units in \( x \)-direction
2. move 1 unit in positive \( y \)-direction
3. move \( f_y \) units in \( z \)-direction

So this means....

\[
\mathbf{u} = \langle 0, 1, f_y \rangle
\]
\[ \mathbf{u} = \langle 0, 1, f_y \rangle \]

Since \( \mathbf{u} \) and \( \mathbf{v} \) are non-parallel vectors in tangent plane, the vector \( \mathbf{v} \times \mathbf{u} \) is normal to the tangent plane.

\[
\mathbf{v} \times \mathbf{u} = \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
1 & 0 & f_x \\
0 & 1 & f_y
\end{vmatrix} = \langle -f_x, -f_y, 1 \rangle
\]

Plane passes through \((a, b, f(a, b))\). So equation of tangent plane is

\[-f_x(a, b)(x-a) - f_y(a, b)(y-b) + z - f(a, b) = 0\]

We usually write this as

\[ z = f(a, b) + f_x(a, b)(x-a) + f_y(a, b)(y-b) \]

**Equation of tangent plane at \((a, b)\).**

---

**Note:** Differentiability is defined
in terms of tangent planes in terms of MV calculus. (Beyond scope of course.)

**Thm:** If $f_x$ and $f_y$ are continuous, then $f$ is differentiable.

(Exercise 41 for counterexample.)

---

**From Calc I** (Linear Approximation)

$$f(x) \approx f(a) + f'(a)(x-a)$$

In other words, $f$ and the tangent line are approximately equal near the point of tangency.

In Calc III, tangent plane is approximation of $f$ near point of tangency.
\[ f(x, y) \approx f(a, b) + f_x(a, b) (x - a) + f_y(a, b) (y - b) \]

if \((x, y)\) is near \((a, b)\)

---

**Ex. 1**

Find the points on graph of

\[ z = 3x^2 - 4y^2 \]

at which \(\vec{n} = \langle 3, 2, 2 \rangle\) is normal to the tangent plane.

**Solution:**
The tangent plane has equation

\[ 3x + 2y + 2z = d \]

(Point of tangency is unknown.)

We rewrite this as

\[
\begin{align*}
z &= -\frac{3}{2}x - y + \bar{d} \\
f_x & f_y
\end{align*}
\]

So we identify:
\[ 6a = f_x(a, b) = -\frac{3}{2} \]
\[ -8b = f_y(a, b) = -1 \]

**Calculate partials**

**Identify coefficients in plane**

So the point of tangency is

\[ (a, b) = \left( -\frac{3}{12}, \frac{1}{8} \right) \]

---

**Ex. 2**

Use a linear approximation to estimate value of

\[ \lambda = \frac{(0.98)^2}{(2.01)^3 + 1} \]

**Solution:**

Consider the function
\[ f(x, y) = \frac{x^2}{y^2 + 1} = x^2 (y^3 + 1)^{-1} \]

and the tangent plane at \((1, 2)\).

Since we want to calculate \(f(0.98, 2.01)\), we can use tangent plane as approximation.

**Calculation of tangent plane:**

\[ f_x = \frac{2x}{y^3 + 1} \quad , \quad f_x(1, 2) = \frac{2}{9} \]

\[ f_y = \frac{-3x^2y^2}{(y^3 + 1)^2} \quad , \quad f_y(1, 2) = -\frac{4}{27} \]

\[ f(1, 2) = \frac{1}{9} \]

So the tangent plane has equation:

\[ z = \frac{1}{9} + \frac{2}{9} (x-1) - \frac{4}{27} (y-2) \]
Now substitute \((0.98, 2.01)\) to get approximation:

\[
f(0.98, 2.01) \approx \frac{1}{9} + \frac{2}{9}(-0.02) - \frac{4}{27}(0.01)
= \frac{284}{2700}
\]