Section 14.2: Limits and Continuity

What do we mean by

\[ \lim_{(x,y) \to (a,b)} f(x,y) = L \]

In particular, what does the limit symbol mean?

In one-variable calculus, limits were much simpler...

\[ \lim_{x \to c} f(x) = L \]

The point \( x \) can approach \( c \) only from two sides. How many ways can \((x, y)\) approach \((a, b)\)?
To say that

\[ \lim_{(x, y) \to (a, b)} f(x, y) = L \]

means we get limit \( L \) along any path terminating at \((a, b)\).

*Most of the limit rules are still true.

1. \( \lim (f + g) = \lim f + \lim g \)
2. \( \lim (k f) = k \lim (f) \)

3. \( \lim (fg) = \lim (f) \cdot \lim (g) \)

4. \( \lim \left( \frac{f}{g} \right) = \frac{\lim (f)}{\lim (g)} \text{ } \text{ } \text{not } 0 \)

5. **Squeeze Theorem:**
   If \( f(x,y) \leq g(x,y) \leq h(x,y) \)
   and \( \lim (f) = \lim (h) = L \), then
   \( \lim (g) = L \) also.

**Ex. 1**

Calculate

\[ \lim_{(x,y) \to (-2,1)} \frac{2x^2}{4x+y} \]

Solution:

Direct substitution.

\[ = \frac{2(-2)^2}{4(-2)+1} = -\frac{8}{7} \]
Ex. 2
Calculate
\[ \lim_{(x,y) \to (0,0)} \frac{x^3 (1 - \cos(y))}{y^2} \]

Solution:
\[ = \lim_{(x,y) \to (0,0)} x^3 \cdot \lim_{(x,y) \to (0,0)} \frac{1 - \cos(y)}{y^2} \]
\[ = \lim_{x \to 0} x^3 \cdot \lim_{y \to 0} \frac{1 - \cos(y)}{y^2} \]
\[ = 0 \cdot \frac{1}{2} \]
\[ = 0 \]

L'Hopital's Rule

Ex. 3
Calculate
\[ \lim_{(x,y) \to P} f(x,y) \]
Contour lines of $f(x,y)$. Note that $f$ is undefined at the point $P$.

**Solution:**

1. Along this path, all values of $f$ are 1, so the limit is $1$.
2. Along this path, all values of $f$ are 5, so the limit is $5$.

Since we get different limits along two different paths, the limit $\lim_{(x,y) \to P} f(x,y)$ does not exist.

**Ex. 4**

Solve the problem.
Calculate $\lim_{(x,y) \to Q} f(x,y)$. 

(What limit is suggested by the contour lines?)

Solution:

Graph suggests limit is 4 since values of $f(x,y)$ approach 4 along any path.

(Note: the contours for this problem are not pathological as in previous example.)

Ex. 5

Show that $\lim_{(x,y) \to Q} x^2 = Q^2$. 
\[
\lim_{(x,y) \to (0,0)} \frac{x}{x^2 + y^2}
\]
does not exist.

**Solution:**

Along \( y \)-axis \((x = 0)\):
\[
\lim_{(x,y) \to (0,0)} \frac{x^2}{x^2 + y^2} = \lim_{y \to 0} \frac{0}{0 + y^2} = 0
\]
along \( y \)-axis

Along \( x \)-axis \((y = 0)\):
\[
\lim_{(x,y) \to (0,0)} \frac{x^2}{x^2 + y^2} = \lim_{x \to 0} \frac{x^2}{x^2} = 1
\]
along \( x \)-axis

We get two different limits along two different curves, so limit does not exist.

**Ex. 6**

Determine whether
\[
\lim_{(x,y) \to (0,0)} \frac{x^3 y}{x^6 + y^2}
\]
exists.

**Solution:**

Along line \( y = mx \):

\[
\lim_{{(x,y) \to (0,0)}} \frac{x^3 y}{x^6 + y^2} = \lim_{{x \to 0}} \frac{x^3 (mx)}{x^6 + (mx)^2}
\]

along \( y = mx \)

\[
= \lim_{{x \to 0}} \left( \frac{mx^2}{{x}^4 + m^2} \right) = 0
\]

Along curves \( y = mx^2 \):

\[
\lim_{{(x,y) \to (0,0)}} \frac{x^3 y}{x^6 + y^2} = \lim_{{x \to 0}} \frac{x^3 (mx^2)}{x^6 + (mx^2)^2}
\]

along \( y = mx^2 \)

\[
= \lim_{{x \to 0}} \left( \frac{m x^3}{{x}^4 + m^2} \right) = 0
\]
\[
\lim_{{x \to 0}} \left( \frac{mx}{{x^2 + m^2}} \right) = 0
\]

Along \( y = mx^3 \):
\[
\lim_{{(x,y) \to (0,0)}} \frac{x^3y}{{x^6 + y^2}} = \lim_{{x \to 0}} \left( \frac{x^3 (mx^3)}{{x^6 + (mx^3)^2}} \right)
\]
along \( y = mx^3 \)
\[
= \lim_{{x \to 0}} \left( \frac{m}{{1 + m^2}} \right) = \frac{M}{{1 + m^2}}
\]

This limit depends on the particular curve \( y = mx^3 \). So the limit does not exist.

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**Ex. 7**

Evaluate the limit
\[
\lim_{{(x,y) \to (0,0)}} \frac{xy^2}{{x^2 + y^2}}
\]
or show that it does not exist.

**Solution:**

For equations of two variables, we evaluate the limit along different paths to check for the existence of the limit.

1. **Path 1:** Along the line \( y = kx \),
\[
\lim_{{x \to 0}} \frac{xy^2}{{x^2 + y^2}} = \lim_{{x \to 0}} \frac{x(kx)^2}{{x^2 + (kx)^2}}
\]
   
2. **Path 2:** Along the parabola \( y = x^2 \),
\[
\lim_{{x \to 0}} \frac{x(x^2)^2}{{x^2 + (x^2)^2}}
\]

By evaluating these limits, we find that the limit depends on the path chosen, indicating that the limit does not exist.
To consider all possible paths, we swap to polar coordinates.

\[ x = r \cos (\theta) \]
\[ y = r \sin (\theta) \quad (r = f(\theta)) \]

If the path approaches origin, then \( r \to 0 \).

\[
\lim_{(x,y) \to (0,0)} \frac{xy^2}{x^2 + y^2} = \lim_{r \to 0} \frac{r^3 \cos(\theta) \sin(\theta)^2}{r^2}
\]

\[
= \lim_{r \to 0} \left( r \cos(\theta) \sin(\theta)^2 \right)
\]

remember that \( \theta = \Theta(r) \)

We use Squeeze Theorem. No matter how \( \Theta \) depends on \( r \),

\[ |\cos(\theta)| \leq 1 \quad \text{and} \quad |\sin(\theta)| \leq 1 \]

So we have...

\[ 0 \leq |r\cos(\theta)\sin(\theta)^2| \leq |r| \cdot 1 \cdot 1 \leq r \]
So by Squeeze Theorem,

\[
\lim_{{r \to 0}} (0) \leq \lim_{{r \to 0}} |r \cos(\theta) \sin(\theta)^2| \leq \lim_{{r \to 0}} r
\]

\[
0 \leq \lim_{{r \to 0}} |r \cos(\theta) \sin(\theta)^2| \leq 0
\]

So we get

\[
\lim_{{r \to 0}} (r \cos(\theta) \sin(\theta)^2) = 0
\]