Section 14.1: Functions of two or more variables.

In this section, \( f \) will denote a function

\[
f : \mathbb{R}^2 \rightarrow \mathbb{R}
\]

- Domain: set of pairs of numbers
- Range: set of one number

\[
f(x, y) = z
\]

OR

\[
f : \mathbb{R}^3 \rightarrow \mathbb{R}
\]

\[
f(x, y, z) = w
\]

Concepts from Precalculus:

1. Domain: set of all \((x, y)\) in \( \mathbb{R}^2 \) for which \( f \) is defined

2. Range: set of all \( z \) in \( \mathbb{R} \) for which \( z = f(x, y) \) for some \( (x, y) \) in \( \mathbb{R}^2 \)
Ex. 1

Find domain of \( f(x,y) = \sqrt{9-x^2-y} \) and sketch domain.

**Solution:**

Domain consists of all \((x,y)\) such that

\[
9-x^2-y \geq 0
\]

To graph this inequality, note that it is equivalent to

\[
y \leq 9-x^2
\]
So how do we examine graphs of \( f(x,y) \)? We use what are called level curves (or contour curves or contour lines).

**Def:** The level curve of \( f(x,y) \) at \( z = c \) is the curve (in \( \mathbb{R}^2 \))...
Note: Level curves show curves of constant \( z \) (height).

**Ex. 2**

Sketch level curves for \( f(x, y) = x^2 + 3y^2 \) for \( c = 0, 10, 20, \ldots, 50 \).

**Solution:**

Each level curve has the form \( x^2 + 3y^2 = c \)

So each curve is an ellipse centered at origin.
Ellipses are elongated x-axis.

Ellipses are not equally spaced.

What do the contours tell us about the shape of f?

Q: Starting from origin, in what direction does z (height)
direction does $z$ (height) increase fastest?
A: In the $y$-direction.
(\textit{In general, steepness is indicated by closely packed contour lines})

**General example of contour lines:**

- **Elliptic paraboloid**
- **Steepest in $y$-direction**

![Diagram of contour lines and graphs](image)
1. Height is constant on each black trace.
2. Closely packed contour lines correspond to steep portions of graph.

One more example of contours... Look at level curves for

\[ f(x, y) = x^2 - 3y^2 \]

Note: The curve \( x^2 - 3y^2 = c \) is a hyperbola.
So what does graph of $z = \nabla (x, y)$ actually look like?

We often want to talk about
rate of change of height with respect to distance in xy-plane.

Function does not change along the level curve

Contour interval: 100 m
Horizontal scale: 200 m

Average rate of change: \( \frac{\Delta \text{(height)}}{\Delta \text{(distance)}} \)

Avg ROC from \( \frac{200\text{m}}{} \)
A to B: \frac{200}{200} = 1

Average ROC from A to C: \frac{200}{400} = 0.5

Average ROC from A to D: 0 (since A and D have same height)

In general, rates of change depend on direction. So derivatives will be vectors, not scalars.

In future lecture . . . . .
We will examine paths of steepest ascent (or descent).

(A) Vectors pointing approximately in the direction of steepest ascent

(B) Not a path of steepest ascent