Section 13.2: Calculus of vector-valued functions

We briefly review all of the familiar calculus operations for parametrizations of curves.

**Def:** We say \( \lim_{t \to t_0} \vec{r}(t) = \vec{u} \) if
\[
\lim_{t \to t_0} \| \vec{r}(t) - \vec{u} \| = 0
\]

**Note:** Remember \( \varepsilon-\delta \) proofs?

First note that if \( \vec{r} = \langle x, y, z \rangle \) and \( \vec{u} = \langle a, b, c \rangle \), then
\[
|x-a|^2 \leq |x-a|^2 + |y-b|^2 + |z-c|^2 = \| \vec{r} - \vec{u} \|^2
\]

So \( 0 \leq |x(t) - a| \leq \| \vec{r}(t) - \vec{u} \| \). Now use Squeeze Theorem!

If \( \lim_{t \to t_0} \| \vec{r}(t) - \vec{u} \| = 0 \), then we must also have \( \lim_{t \to t_0} |x(t) - a| = 0 \). This is equivalent to...
\[ \lim_{{t \to a}} x(t) = a \]

We get similar results for \( y \) and \( z \).

\[ \lim_{{t \to a}} y(t) = b \]

\[ \lim_{{t \to a}} z(t) = c \]

So what does this mean?

\[ \lim_{{t \to a}} \left< x(t), y(t), z(t) \right> = \left< \lim_{{t \to a}} x(t), \lim_{{t \to a}} y(t), \lim_{{t \to a}} z(t) \right> \]

Limits of vectors are computed componentwise!

So if you want to compute the limit of a vector-valued function, just compute the limit of each component separately.

What about derivatives and integrals? Since they are defined in terms of limits, they are done componentwise as well.
The derivative of \( \vec{r}(t) \) is

\[
\vec{r}'(t) = \frac{d\vec{r}}{dt} = \lim_{h \to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}
\]

\[
\vec{r}'(t) = \left< \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right>
\]

The same goes for integrals:

\[
\int_a^b \vec{r}(t) \, dt = \left< \int_a^b x(t) \, dt, \int_a^b y(t) \, dt, \int_a^b z(t) \, dt \right>
\]

(Also for antiderivatives.)

**Ex. 1**

Calculate

\[
\lim_{t \to 1} \left< \frac{t^2 - t}{t - 1}, 0, \ln(t+1) \right>
\]

Solution:

\[
= \left< \lim_{t \to 1} \frac{t^2 - t}{t - 1}, \lim_{t \to 1} (0), \lim_{t \to 1} \ln(t+1) \right>
\]
\[
\lim_{t \to 1} \frac{2t}{1} = 2, \quad 0, \quad \ln(2)
\]

\[
= \langle 2, 0, \ln(2) \rangle
\]

**Ex. 2**

Let \( \vec{r}(t) = \langle t^2, \ln(e^t + 3), t\sin(t) \rangle \)

Calculate \( \vec{r}'(t) \).

**Solution:**

\( \vec{r}'(t) = \langle 2t, \frac{e^t}{e^t + 3}, t\cos(t) + \sin(t) \rangle \)

**Ex. 3**

Calculate \( \int \vec{r}(t) \, dt \) where

\( \vec{r}(t) = \langle t^2, 0, 2te^{t^2} \rangle \)

**Solution:**

\[
\int \vec{r}(t) \, dt = \left\langle \int t^2 \, dt, \int 0 \, dt, \int 2te^{t^2} \, dt \right\rangle = \left\langle \frac{t^3}{3}, 0, e^{t^2} \right\rangle + \vec{C}
\]

can put constants of integration all in separate vector
where \( \vec{e} \) is an arbitrary constant vector.

**Product Rules**

Recall from 151,

\[
(fg)' = fg' + fg
\]

We have similar rules for vectors.
Let \( f(t) \) be scalar and \( \vec{u}(t) \) and \( \vec{w}(t) \) be vectors.

1. \[
\frac{d}{dt}(f\vec{u}) = \frac{df}{dt}\vec{u} + f\frac{d\vec{u}}{dt}
\]
2. \[
\frac{d}{dt}(\vec{u} \cdot \vec{w}) = \frac{d\vec{u}}{dt} \cdot \vec{w} + \vec{u} \cdot \frac{d\vec{w}}{dt}
\]
   *dot product*

3. \[
\frac{d}{dt}(\vec{u} \times \vec{w}) = \frac{d\vec{u}}{dt} \times \vec{w} + \vec{u} \times \frac{d\vec{w}}{dt}
\]
   *cross product*, be careful of order! \( \vec{u} \) before \( \vec{w} \)!

The vector \( \vec{f}'(t) \) has a very useful geometric interpretation.
The vector $\frac{r(t_0+h) - r(t_0)}{h}$ is some multiple of $r(t_0+h) - r(t_0)$, so points in same direction.

**IMPORTANT:**

The vector $F'(t_0)$ is a vector that

1. is tangent to curve at $t = t_0$
2. points in direction in which curve is traced.

Hence we sometimes call $F'(t)$ the tangent vector. In the next section we will see that $F'(t)$ can also be interpreted as a velocity.
to interpret as a rate of change, so we call it the **velocity vector**.

**Convention:** We usually think of vectors as based at the origin. But for tangent vectors, we think of \( \vec{r}'(t) \) based on the curve. So \( \vec{r}'(t_0) \) has base point \( \vec{r}(t_0) \).

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**Ex. 4**

Let \( C \) be the curve with parametrization

\[
\vec{r}(t) = \langle t - \sin(t), 1 - \cos(t), t \rangle
\]

Find a parametrization for the line tangent to the curve \( C \) when \( t = \pi \).

**Solution:**

The tangent line passes through the point

\[
\vec{r}(\pi) = \langle \pi, 2, \pi \rangle
\]

and has direction vector \( \vec{r}'(\pi) \).
(The tangent vector points along the tangent line.)

$$\vec{F}'(t) = \langle 1 - \cos(t), \sin(t), 1 \rangle$$

$$\vec{F}'(\pi) = \langle 2, 0, 1 \rangle$$

So the tangent line has parametrization

$$\vec{L}(t) = \langle \pi, 2, \pi \rangle + t \langle 2, 0, 1 \rangle$$

for $$-\infty < t < \infty.$$