Section 12.5: Equations of Planes

To determine a plane uniquely, we need:

1. \( P_0 \): point in plane \( P \)
2. \( \vec{n} \): vector normal to \( P \)

Suppose \( P_0 = (x_0, y_0, z_0) \) and \( \vec{n} = \langle a, b, c \rangle \). What is the equation of plane \( P \)?
Let \( P = (x, y, z) \) be an arbitrary point in \( P \).

Q: What is the relationship between \( \overrightarrow{P_0P} \) and \( \vec{n} \)?

A: \( \overrightarrow{P_0P} \perp \vec{n} \), or \( \overrightarrow{P_0P} \cdot \vec{n} = 0 \).

\[
\overrightarrow{P_0P} = \langle x-x_0, y-y_0, z-z_0 \rangle
\]

\[
\vec{n} = \langle a, b, c \rangle
\]

\[
0 = \overrightarrow{P_0P} \cdot \vec{n}
\]

\[
0 = a(x-x_0) + b(y-y_0) + c(z-z_0)
\]
Note: coefficients of $x, y, z$ give components of normal vector.

The plane $3x - y + 2z = 5$
has normal vector

$\vec{n} = \langle 3, -1, 2 \rangle$

In general, a plane has equation

$ax + by + cz = d$

Ex. 1

Find eq. of plane with normal vector $\vec{n} = \langle -1, 3, 2 \rangle$
containing point $P_0 = (0, 4, -5)$. 
Solution:
Plane has equation
\[-x + 3y + 2z = d\]
The point Po is on plane:
\[-0 + 3(4) + 2(-5) = d\]
\[2 = d\]
So eq. of plane is
\[-x + 3y + 2z = 2\]

Ex. 2

Suppose P has equation
\[6x - 7y + 3z = 4\]
Find an equation of the plane parallel to P but containing....
(a) the origin
(b) point \((-1, 1, 1)\)

**Solution**

Parallel planes have equivalent normal vectors.

So for (a) and (b) the plane has equation

\[ 6x - 7y + 3z = d \]

(a) Substitute \((0,0,0)\) to get \(d\):

\[ 0 - 0 + 0 = d \implies d = 0 \]

So eq. of plane is

\[ 6x - 7y + 3z = 0 \]

(b) Substitute \((-1, 1, 1)\) to get \(d\):

\[ 6(-1) - 7(1) + 3(1) = d \]
\[ \Rightarrow d = -10 \]

So eq. of plane is

\[ 6x - 7y + 3z = -10 \]

Ex. 3

Determine whether the planes are parallel.

\( \mathbf{P}_1 : \ 2x - 3y + 12z = 7 \)

\( \mathbf{P}_2 : \ -4x + 6y - 24z = 5 \)

Solution:

Question is really asking whether the planes' normal vector are parallel.

\[ \mathbf{n}_1 = \langle 2, -3, 12 \rangle \]

\[ \mathbf{n}_2 = \langle -4, 6, -24 \rangle \]
Q: Are \( \vec{n}_1 \) and \( \vec{n}_2 \) parallel?

A: Yes, since \( \vec{n}_2 = -2\vec{n}_1 \).

So the planes are parallel.

Suppose \( \mathcal{P} \) has normal vector \( \vec{n} = \langle a, b, c \rangle \). The \( \mathcal{P} \) has eq.

\[
ax + by + cz = d
\]

The equation

\[
\lambda ax + \lambda by + \lambda cz = \lambda d
\]

is algebraically equivalent \((\lambda \neq 0)\). But a normal vector is

\[
\vec{n}^* = \langle \lambda a, \lambda b, \lambda c \rangle
\]

\[
\vec{n}^* = \lambda \vec{n}
\]

So if \( \vec{n} \) is a normal vector
Find an equation of the plane containing the three points:

\[ P = (-2, 1, 1) \]
\[ Q = (3, 0, 1) \]
\[ R = (0, -2, 8) \]

**Solution:**

The cross product \( \vec{v} \times \vec{w} \) gives us a normal vector to plane \( \vec{n} \).
\[ \overrightarrow{v} = \overrightarrow{PQ} = \langle 5, -1, 0 \rangle \]
\[ \overrightarrow{w} = \overrightarrow{PR} = \langle 2, -3, 7 \rangle \]

Normal vector is 
\[ \overrightarrow{n} = \overrightarrow{v} \times \overrightarrow{w} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 5 & -1 & 0 \\ 2 & -3 & 7 \end{vmatrix} \]
\[ \overrightarrow{n} = -7\hat{i} - 35\hat{j} - 13\hat{k} \]

Eq. of plane is
\[ -7x - 35y - 13z = d \]

Point \( Q = (3, 0, 1) \) lies in plane:
\[ -7(3) - 35(0) - 13(1) = d \]
\[ -21 - 0 - 13 = d \]
\[ -34 = d \]

So eq. of plane is
\[ -7x - 35y - 13z = -34 \]
Ex. 5

Suppose the plane $\mathcal{P}$ contains the lines $\mathcal{L}_1$ and $\mathcal{L}_2$ whose parametrizations are:

$\mathcal{L}_1$: $\vec{r}_1(t) = \langle 2 + t, 1 + 2t, 3t \rangle$

$\mathcal{L}_2$: $\vec{r}_2(t) = \langle 2 + 3t, 1 + t, 8t \rangle$

Find an equation of $\mathcal{P}$.

Solution:

Can we find two vectors, $\vec{v}$ and $\vec{w}$, that lie in $\mathcal{P}$? Once we do, we know $\vec{n} = \vec{v} \times \vec{w}$. 
We can take \( \vec{v} \) and \( \vec{w} \) to be the direction vectors for \( \ell_1 \) and \( \ell_2 \).

\( \ell_1: \vec{v} = <1, 2, 3> \)

\( \ell_2: \vec{w} = <3, 1, 8> \)

The normal vector is

\[ \vec{n} = \vec{v} \times \vec{w} = 13\hat{i} + \hat{j} - 5\hat{k} \]

So eq. of \( \ell \) is ....

\[ 13x + y - 5z = d \]

Since \( \ell \) contains both lines, \( \ell \) must contain point \( \vec{r}_1(0) = <2, 1, 0> \)

So \( (2, 1, 0) \) is on \( \ell \).

\[ 13(2) + 1 - 5(0) = d \]
13(2) + (1 + 2t) - 5(3t) = d
27 = d

So e.g. of plane is

\[13x + y - 5z = 27\]

How we can check that previous answer is correct?
The line \( L \) has parametrization:

\[
\begin{align*}
x &= 2 + t \\
y &= 1 + 2t \\
z &= 3t
\end{align*}
\]

\[13(2+t) + (1+2t) - 5(3t) = \]

\[= 26 + 13t + 1 + 2t - 15t \]

\[= 27 \checkmark\]

For \( L_2 \) ....

\[x = 2 + 3t\]
\[ y = 1 + t \]
\[ z = 8t \]
\[ 13(2 + 3t) + (1 + t) - 5(8t) = \]
\[ = 26 + 34t + 1 + t - 40t \]
\[ = 27 \checkmark \]

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**Ex. 6**

Find the point of intersection of the plane

\[ x + y + z = 14 \]

and the line

\[ \vec{r}(t) = <1, 1, 0> + t <0, 2, 4> \]

**Solution**: Note that line has para.

\[ x = 1 \]
\[ y = 1 + 2t \]
\[ z = 4t \]
If line intersects plane, then
\[
\begin{align*}
\frac{x}{1} + \frac{y}{1} + \frac{z}{2} &= 14 \\
x &= 1 & y &= 1 + 2t & z &= 4t
\end{align*}
\]
\[
1 + 1 + 2t + 4t = 14
\]
\[t = 2\]
So if \(P\) is the point of intersection, then
\[
\overrightarrow{OP} = \vec{r}(2) = \langle 1, 5, 8 \rangle
\]
In other words, \(P = (1, 5, 8)\).

**Ex. 7**

Let \(L\) denote the line of intersection points of the planes
\[
P_1 : x - y - z = 1
\]
\[
P_2 : 2x + 3y + z = 2
\]
Find a parametrization of \(L\).
After a parametric form of $P$ is given.

Solution:

The line $\ell$ through page $P_1$ and $P_2$.

Direction vector $\vec{u}$ for $\ell$ is

$$\vec{u} = \vec{n}_1 \times \vec{n}_2$$

where $\vec{n}_1$ and $\vec{n}_2$ are vectors normal to $P_1$ and $P_2$ respectively.
normal to \( \vec{v}_1 \) and \( \vec{v}_2 \), respectively:

\[ \vec{n}_1 = \langle 1, -1, -1 \rangle \]

\[ \vec{n}_2 = \langle 2, 3, 1 \rangle \]

\[ \vec{u} = \vec{n}_1 \times \vec{n}_2 = \langle 2, -3, 5 \rangle \]

**Sanity check:**

\[ \vec{u} \cdot \vec{n}_1 = 0 \]

\[ \vec{u} \cdot \vec{n}_2 = 0 \]

We now need a point on \( \mathcal{L} \), which is a point common to both planes.

\[ x - y - z = 1 \]

\[ 2x + 3y + z = 2 \]

Put \( z = 0 \). Then

\[ x - y = 1 \]

\[ 2x + 3y = 2 \]

The solution is \( x = 1 \) and \( y = 0 \).
The solution is $x=1$ and $y=0$

So the point $(1,0,0)$ lies on both planes, thus lies in $L$.

Summary for $L$:

direction vector: $\mathbf{u} = \langle 2, -3, 5 \rangle$

point on $L$: $P = (1,0,0)$

So parametrization of $L$ is

$$\mathbf{r}(t) = \langle 1, 0, 0 \rangle + t \langle 2, -3, 5 \rangle$$

OR

$$x = 1 + 2t$$
$$y = -3t$$
$$z = 5t$$

Check that answer is correct.

Line $L$ should lie in both planes:

$$P_1: x - y - z = 1$$

$$P_2: 2 - x + 3y - z = 0$$
\[(1+2t) - (-3t) - 5t = \]
\[= 1 + 2t + 3t - 5t = 1 \]

\[P_2: \quad 2x + 3y + z = 2\]
\[2(1+2t) + 3(-3t) + 5t = \]
\[= 2 + 4t - 9t + 5t = 2 \]