Book #1 of 1

Name: 

ID# (last 4 digits): ___________________________  Section: ___________________________

Unless stated otherwise, you must show all work clearly using proper notation and explain your reasoning in English where appropriate. Answers must be justified using techniques that have been taught in this course, and answers without such justification may receive less than full credit – or no credit at all – even if the answer is correct.

This exam is closed book. Calculators, electronic devices, notes, books, formula sheets, and other outside materials are not allowed. Phones must be turned off and put away.

Unless otherwise stated, give exact answers: e.g., write π and \( \sqrt{2} \) instead of 3.14 and 1.41. However, when an expression simplifies to a well-known value, you must use that value. For example, you must write 1 instead of \( e^0 \), and you must write \( \frac{1}{2} \) instead of \( \cos(\frac{\pi}{3}) \).

This exam has 6 questions, printed in 1 booklet(s), for a total of 100 points.

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<tr>
<th>Question</th>
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1. Consider the following vector fields.

\[ \mathbf{F}_1 = \langle 2xy^3z + z^2, 3x^2y^2z + 6yz, x^2y^3 + 2xz + 3y^2 \rangle \]
\[ \mathbf{F}_2 = \langle y^4z + 6xy, 4xy^3z - z^2, xy^4 - 2yz \rangle \]

(a) Suppose \( \mathcal{L} \) is a loop in \( \mathbb{R}^3 \). For which of the above vector fields is the line integral on \( \mathcal{L} \) necessarily 0? Explain your answer.

\( \mathbf{F}_1 \) or \( \mathbf{F}_2 \) or both? _______________________

(b) Let \( \mathbf{G} \) be the vector field that satisfies part (a) and let \( \mathcal{C} \) be a smooth curve from \( (1, 0, -1) \) to \( (2, 2, 2) \). Calculate \( \int_{\mathcal{C}} \mathbf{G} \cdot d\mathbf{r} \). If both vector fields satisfy part (a), then let \( \mathbf{G} = \mathbf{F}_1 \).

value of integral: _______________________

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2. Let $W$ be the solid region with $x \geq 0$ and bounded by the surfaces $z = 4 - \sqrt{x^2 + y^2}$ and $z = 3(x^2 + y^2)$. Find the total mass of $W$ if the mass density is $\delta(x, y, z) = z$.

mass of $W$: ________________
Consider the map \( G(u, v) = \left( \sqrt{\frac{u - v}{2}}, \frac{u + v}{2} \right) \). Let \( D \) be the region in the \( xy \)-plane bounded by the curves \( y = x^2 \), \( y = x^2 + 3 \), \( y = 4 - x^2 \), and \( y = 5 - x^2 \).

(a) Describe the image of the vertical line \( u = a \).

(b) Describe the image of the horizontal line \( v = b \).

(c) Calculate \( |\text{Jac}(G)| \).

\[ |\text{Jac}(G)| = \]
Recall that $G(u, v) = \left( \sqrt{\frac{u-v}{2}}, \frac{u+v}{2} \right)$ and $\mathcal{D}$ is the region in the $xy$-plane bounded by the curves $y = x^2$, $y = x^2 + 3$, $y = 4 - x^2$, and $y = 5 - x^2$.

(d) Find a rectangle $\mathcal{R}$ such that $G(\mathcal{R}) = \mathcal{D}$.

$\mathcal{R} =$

(e) Calculate $\iiint_{\mathcal{D}} 16x^3y \, dA$.

value of integral: 

4. Compute

\[ \int_C (x + y) \, dx + (2y - z) \, dy + (z - 3x) \, dz \]

where \( C \) is the line segment from \((1, -1, 0)\) to \((2, 3, -2)\).

value of integral: ____________________
5. Calculate \( \int_0^2 \int_0^{4-x^2} \frac{xe^{2y}}{4-y} \, dy \, dx \).

value of integral: ________________
Let $S_0$ be the sphere of radius 1 centered at the origin and let $S_1$ be the sphere of radius 1 centered at $(0, 0, 1)$. Let $W$ be the solid region that is both outside $S_0$ and inside $S_1$.

(a) Find an equation in spherical coordinates that describes $S_1$.

equation for $S_1$: ________________

(b) Use integration in spherical coordinates to find the volume of $W$.

volume: ________________
This page is for scratch work. Do not detach this sheet.
Curvature:
\[ \kappa(t) = \frac{\| \mathbf{r}'(t) \times \mathbf{r}''(t) \|}{\| \mathbf{r}'(t) \|^3} \]

Critical numbers:
\[ D = f_{xx} f_{yy} - (f_{xy})^2 \]

<table>
<thead>
<tr>
<th>sign of D</th>
<th>sign of ( f_{xx} ) or ( f_{yy} )</th>
<th>conclusion</th>
</tr>
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<tr>
<td>⊕</td>
<td>⊕</td>
<td>local minimum</td>
</tr>
<tr>
<td>⊕</td>
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<td>local maximum</td>
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<tr>
<td>⊖</td>
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<td>saddle point</td>
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Cylindrical and spherical coordinates:
\[
\begin{align*}
  x &= r \cos(\theta) \\
  y &= r \sin(\theta) \\
  z &= z \\
  dV &= r \, dz \, dr \, d\theta \\
  x &= \rho \cos(\theta) \sin(\varphi) \\
  y &= \rho \sin(\theta) \sin(\varphi) \\
  z &= \rho \cos(\varphi) \\
  dV &= \rho^2 \sin(\varphi) \, d\rho \, d\varphi \, d\theta
\end{align*}
\]

General change of variables:
\[
\text{Jac}(G) = \left| \begin{array}{cc} x_u & y_u \\ x_v & y_v \end{array} \right|, \quad \iiint_{G(R)} f(x,y) \, dx \, dy = \iiint_{R} f(x(u,v),y(u,v)) \, |\text{Jac}(G)| \, du \, dv
\]

Tangent and normal vectors:
\[
\mathbf{T}_u = \frac{\partial G}{\partial u}, \quad \mathbf{T}_v = \frac{\partial G}{\partial v}, \quad \mathbf{N} = \mathbf{T}_u \times \mathbf{T}_v
\]

Line and surface integrals:
\[
\begin{align*}
  \int_{C} f \, ds &= \int_{a}^{b} f(\mathbf{r}(t)) \| \mathbf{r}'(t) \| \, dt \\
  \iint_{S} f \, dS &= \iint_{D} f(G(u,v)) \| \mathbf{N}(u,v) \| \, dudv \\
  \int_{C} \mathbf{F} \cdot d\mathbf{r} &= \int_{a}^{b} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \, dt \\
  \iint_{S} \mathbf{F} \cdot dS &= \iint_{D} \mathbf{F}(G(u,v)) \cdot \mathbf{N}(u,v) \, dudv
\end{align*}
\]

Vector calculus theorems:

- Gradient theorem: \( f(\mathbf{r}(b)) - f(\mathbf{r}(a)) = \int_{C} \nabla f \cdot d\mathbf{r} \)
- Green’s theorem: \( \oint_{\partial D} (F_1 \, dx + F_2 \, dy) = \iint_{D} \left( \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \, dA \)
- Stokes’s theorem: \( \oint_{\partial S} \mathbf{F} \cdot d\mathbf{r} = \iint_{S} (\nabla \times \mathbf{F}) \cdot dS \)
- Divergence theorem: \( \iiint_{W} \mathbf{F} \cdot d\mathbf{S} = \iint_{W} (\nabla \cdot \mathbf{F}) \, dV \)