Let $W$ be a solid right circular cone with its vertex at the origin, with radius $R = 1$ and height $H = 2$. Using spherical coordinates, calculate the following integral.

$$\iiint_W \frac{2}{\sqrt{x^2 + y^2 + z^2}} \, dV$$

**Solution**

We will set up the integral in the order $d\rho d\varphi d\theta$. Sketching a ray from the origin, we see that the limits of $\rho$ can be described as $0 \leq \rho \leq \rho_{\text{plane}}$, where $\rho_{\text{plane}}$ is the value of $\rho$ for the upper boundary plane of the cone. This plane is described by $z = 2$, or $\rho \cos(\varphi) = 2$, whence

$$\rho_{\text{plane}} = 2 \sec(\varphi)$$

Now considering a vertical half-plane (plane of constant $\theta$), we see that the limits on $\varphi$ are $0 \leq \varphi \leq \alpha$, where $\alpha$ is the angle such that $\tan(\alpha) = R/H = 1/2$. The entire solid $W$ surrounds the $z$-axis, and so the limits on $\theta$ are $0 \leq \theta \leq 2\pi$. Hence our integral is given below.

$$\iiint_W \frac{2}{\sqrt{x^2 + y^2 + z^2}} \, dV = \int_0^{2\pi} \int_0^\alpha \int_0^{2\sec(\varphi)} 2 \rho \rho^2 \sin(\varphi) d\rho d\varphi d\theta$$

Now we calculate the integral as an iterated integral.

$$\int_0^{2\pi} \int_0^\alpha \int_0^{2\sec(\varphi)} 2\rho \sin(\varphi) \, d\rho d\varphi d\theta = \int_0^{2\pi} \int_0^\alpha \left( \rho^2 \sin(\varphi) \bigg|_{\rho=0}^{\rho=2\sec(\varphi)} \right) \, d\varphi d\theta$$

$$= \int_0^{2\pi} \int_0^\alpha 4 \sec(\varphi) \tan(\varphi) \, d\varphi d\theta$$

$$= 4 \int_0^{2\pi} \left( \sec(\varphi) \bigg|_{\varphi=0}^{\varphi=\alpha} \right) \, d\theta$$

$$= 4 \int_0^{2\pi} \left( \sec(\alpha) - 1 \right) \, d\theta$$

$$= 8\pi (\sec(\alpha) - 1) = 8\pi \left( \frac{\sqrt{5}}{2} - 1 \right)$$
17 pts

2. Let $\mathcal{D}$ be the region in the first quadrant that is inside the larger circle $x^2+y^2 = 36$ and outside the smaller circle $(x-3)^2+y^2 = 9$. Using polar coordinates, calculate the following integral.

$$\iint_{\mathcal{D}} 3\sqrt{x^2+y^2} \, dA$$

Solution

We will set up the integral in the order $dr \, d\theta$. Sketching a ray from the origin, we see that the limits of $r$ can be described as $r_{\text{small}} \leq r \leq r_{\text{large}}$ where the lower and upper limits refer, respectively, to the values of $r$ on the small and large circles. For the large circle, we have $r_{\text{large}} = 6$. For the small circle, we have

$$(x - 3)^2 + y^2 = 9 \implies x^2 + y^2 = 6x \implies r^2 = 6r \cos(\theta) \implies r = 6 \cos(\theta)$$

Hence $r_{\text{small}} = 6 \cos(\theta)$. The region $\mathcal{D}$ extends only within the first quadrant, whence the limits on $\theta$ are $0 \leq \theta \leq \pi/2$. Hence our integral is given below.

$$\iint_{\mathcal{D}} 3\sqrt{x^2+y^2} \, dA = \int_0^{\pi/2} \int_{6 \cos(\theta)}^6 3r \, rdr \, d\theta$$

Now we calculate the integral as an iterated integral.

$$\int_0^{\pi/2} \int_{6 \cos(\theta)}^6 3r^2 \, dr \, d\theta = \int_0^{\pi/2} \left( \int_{6 \cos(\theta)}^6 r^3 \, dr \right) \, d\theta = 216 \int_0^{\pi/2} (1 - \cos(\theta)^3) \, d\theta$$

To compute the integral of the term $\cos(\theta)^3$, we write

$$\cos(\theta)^3 = \cos(\theta)^2 \cos(\theta) = (1 - \sin(\theta)^2) \cos(\theta)$$
and use the substitution \( u = \sin(\theta) \).

\[
\begin{align*}
216 \int_0^{\pi/2} (1 - \cos(\theta)^3) \, d\theta &= 216 \int_0^{\pi/2} 1 \, d\theta - 216 \int_0^{\pi/2} \cos(\theta)^3 \, d\theta \\
&= 216 \cdot \frac{\pi}{2} - 216 \int_0^1 (1 - u^2) \, du \\
&= 108\pi - 216 \left( \frac{u - u^3}{3} \right) \bigg|_{u=0}^{u=1} \\
&= 108\pi - 216 \cdot \frac{2}{3} = 108\pi - 144
\end{align*}
\]

3. Let \( \mathcal{W} \) be the region in the first octant that is bounded by the surface \( z = 27 - x^3 \) and the plane \( x = 3y \). Write the integral

\[
\iiint_{\mathcal{W}} xy(x^2 + z^2) \, dV
\]

as an iterated integral in the order \( dy \, dx \, dz \). Do not evaluate your integral.

**Solution**

To find the limits for \( dy \), we consider piercing the region \( \mathcal{W} \) along an arbitrary axis parallel to the \( y \)-axis (an axis of constant \( x \) and \( z \)). The “entrance point” \( P \) is on the \( xz \)-plane, whence \( P = (x, 0, z) \). The “exit point” \( Q \) is on the vertical plane \( x = 3y \), whence \( Q = (x, x/3, z) \). Hence the limits for the \( dy \)-integral are

\[
0 \leq y \leq \frac{x}{3}
\]

Now to determine the limits for \( dx \, dz \), we project the region \( \mathcal{W} \) into the \( xz \)-plane to obtain the region \( \mathcal{D} \) below.
To find the limits for \( dx \), we slice the region \( D \) along a horizontal line (curve of constant \( z \)). The “entrance point” \( P' \) is on the \( z \)-axis, whence \( P' = (0, z) \). The “exit point” \( Q' \) is on the curve \( z = 27 - x^3 \), whence \( Q' = ((27 - z)^{1/3}, z) \). Hence the limits for the \( dx \)-integral are

\[
0 \leq x \leq (27 - z)^{1/3}
\]

To find the limits for \( dz \), we finally project the region \( D \) onto the \( z \)-axis. We immediately find that \( 0 \leq z \leq 27 \). Hence the desired integral is given below.

\[
\iiint_D xy(x^2 + z^2) \, dV = \int_0^27 \int_0^{(27-z)^{1/3}} \int_0^{x/3} xy(x^2 + z^2) \, dy \, dx \, dz
\]

4 pts

Use the method of Lagrange multipliers to find the point on the ellipse

\[ x^2 + 6y^2 + 3xy = 40 \]

with the largest \( x \)-coordinate.

Solution

We want to find the maximum value of \( f(x, y) = x \) subject to the constraint \( g(x, y) = 40 \), where

\[ g(x, y) = x^2 + 6y^2 + 3xy \]

So by the method of Lagrange multipliers, we solve the system of equations \( \nabla f = \lambda \nabla g \) (along with the constraint \( g = 40 \)). Hence we must solve the following three equations.

\[
\begin{align*}
1 &= \lambda (2x + 3y) \quad (1) \\
0 &= \lambda (12y + 3x) \quad (2) \\
40 &= x^2 + 6y^2 + 3xy \quad (3)
\end{align*}
\]

From Equation (1), we see that \( \lambda \neq 0 \). So dividing Equation (2) by \( \lambda \) gives \( 12y + 3x = 0 \), or \( x = -4y \). Now substituting \( x = -4y \) into the constraint (Equation (3)) gives the following.

\[
40 = 16y^2 + 6y^2 - 12y^2 = 10y^2 \implies y^2 = 4
\]

\[
\begin{align*}
1 &= \lambda (2x + 3y) \\
0 &= \lambda (12y + 3x) \\
40 &= x^2 + 6y^2 + 3xy
\end{align*}
\]
Hence the two solutions are $y = -2$ and $y = 2$. The corresponding points are $(8, -2)$ and $(-8, 2)$. Clearly the point for which $f(x, y) = x$ is a maximum is the point $(8, -2)$.

5. Consider the vector field $\mathbf{F}(x, y, z) = \langle 3zy^{-1}, 4x, -y \rangle$.

(a) Calculate the divergence and curl of $\mathbf{F}$.

(b) Let $\mathcal{C}$ be the curve given by $\mathbf{r}(t) = \langle e^t, e^t, t \rangle$ for $0 \leq t \leq 1$. Calculate the line integral of $\mathbf{F}$ along $\mathcal{C}$.

Solution

(a) We have the following calculations.

$$
\text{div}(\mathbf{F}) = \frac{\partial}{\partial x}(3zy^{-1}) + \frac{\partial}{\partial y}(4x) + \frac{\partial}{\partial z}(-y) = 0 + 0 + 0 = 0
$$

$$
\text{curl}(\mathbf{F}) = \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
3zy^{-1} & 4x & -y
\end{vmatrix}
$$

$$
= (-1 - 0)\mathbf{i} - \left(0 - \frac{3}{y}\right)\mathbf{j} + \left(4 - \frac{3z}{y^2}\right)\mathbf{k}
$$

$$
= -\mathbf{i} + \frac{3}{y}\mathbf{j} + \left(4 + \frac{3z}{y^2}\right)\mathbf{k}
$$

(b) First observe that we have the following.

$$
\mathbf{F}(\mathbf{r}(t)) = \left\langle \frac{3t}{e^t}, 4e^t, -e^t \right\rangle
$$

$$
\mathbf{r}'(t) = \langle e^t, e^t, 1 \rangle
$$

$$
\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = 3t + 4e^{2t} - e^t
$$

Now we set up the integral (recall that $\mathbf{F} \cdot d\mathbf{r} = \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt$) and calculate it using fundamental theorem of calculus.

$$
\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \left(3t + 4e^{2t} - e^t\right)\, dt = \left(\frac{3}{2}t^2 + 2e^{2t} - e^t\right)\bigg|_{t=0}^{t=1}
$$

$$
= \left(\frac{3}{2} + 2e^2 - e\right) - (0 + 2 - 1) = \frac{1}{2} + 2e^2 - e
$$

6. Let $\mathcal{D}$ be the region in the $xy$-plane described and shown in the figure below.

$$
\mathcal{D} : \quad x^2 \leq y \leq 3x^2, \quad 1 \leq x^2y \leq 4
$$
(a) Find a rectangle $R$ in the $uv$-plane and a map $G$ such that $G(R) = \mathcal{D}$.

You may either specify $G$ (give $x$ and $y$ in terms of $u$ and $v$) or specify $G^{-1}$ (give $u$ and $v$ in terms of $x$ and $y$).

(b) Find $|\text{Jac}(G)|$. You may give your answer in terms of $x$ and $y$ or in terms of $u$ and $v$.

(c) Calculate the following integral.

\[
\iint_{\mathcal{D}} 6x^3y \, dA
\]

**Solution**

(a) The inequalities that describe the region $\mathcal{D}$ can be written as

\[
1 \leq \frac{y}{x^2} \leq 3, \quad 1 \leq x^2y \leq 4
\]

Hence we will define the variables $u$ and $v$ as

\[
u = \frac{y}{x^2}, \quad v = x^2y
\]

Note that since we have given $u$ and $v$ in terms of $x$ and $y$, we have specified the map $G^{-1}$. That is,

\[
G^{-1}(x, y) = \left( \frac{y}{x^2}, x^2y \right)
\]

The region $\mathcal{D}$ can thus be described in terms of $u$ and $v$ by the inequalities

\[
1 \leq u \leq 3, \quad 1 \leq v \leq 4
\]

Hence the desired rectangle is $\mathcal{R} = [1, 3] \times [1, 4]$.

(b) We first calculate $\text{Jac}(G^{-1})$.

\[
\text{Jac}(G^{-1}) = \begin{vmatrix}
\frac{u_x}{u_{xx}} & \frac{u_y}{u_{yy}} \\
\frac{v_x}{v_{xx}} & \frac{v_y}{v_{yy}}
\end{vmatrix} = \begin{vmatrix}
-2x^{-3}y & x^{-2} \\
2xy & x^2
\end{vmatrix} = (-2x^{-3}y) \cdot (x^2) - (x^{-2}) \cdot (2xy) = \frac{4y}{x}
\]

Now we use the identity $\text{Jac}(G) = 1/\text{Jac}(G^{-1})$ and observe that $x, y > 0$. Hence we have the following.

\[
|\text{Jac}(G)| = \frac{x}{4y}
\]
(c) First we use the change of variables formula. (Recall that $dA = |\text{Jac}(G)| \, du \, dv$.)

$$\iint_D 6x^3y \, dA = \int_1^4 \int_1^3 6x^3y \frac{x}{4y} \, du \, dv = \frac{3}{2} \int_1^4 \int_1^3 x^4 \, du \, dv$$

Now we rewrite the integrand $x^4$ in terms of $u$ and $v$. To that end, observe that

$$\frac{v}{u} = \frac{x^2y}{y/x^2} = x^4$$

Now we may calculate our integral as an iterated integral.

$$\frac{3}{2} \int_1^4 \int_1^3 x^4 \, du \, dv = \frac{3}{2} \int_1^4 \int_1^3 \frac{v}{u} \, du \, dv = \frac{3}{2} \left( \int_1^4 v \, dv \right) \cdot \left( \int_1^3 \frac{1}{u} \, du \right)$$

$$= \frac{3}{2} \cdot \left( \frac{1}{2} v^2 \bigg|_{v=1}^{v=4} \right) \cdot \left( \ln(u) \bigg|_{u=1}^{u=3} \right) = \frac{3}{2} \cdot \frac{15}{2} \cdot \ln(3) = \frac{45}{4} \ln(3)$$