Name (PRINT): ___________________________  ID # (last 4 digits): ____________  
Signature: ______________________________

- This is a closed book exam. No notes, calculators, phones, etc. are allowed.
- Please explain and label your answers clearly and show all work. I reserve the right to give no credit for a response with no work even if the final answer is correct.
- You have 90 minutes to complete the exam. There are 100 points total.
- Please have your photo ID available. Do not start the exam until instructed to do so.

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1. Let $A$ and $b$ be the matrix and vector below.

$$A = \begin{bmatrix} 1 & -2 & -10 \\ 1 & 1 & -1 \\ -1 & -2 & -2 \end{bmatrix}, \quad b = \begin{bmatrix} 5 \\ -1 \\ 5 \end{bmatrix}$$

Let $u$, $v$, and $w$ denote the first, second, and third columns of $A$, respectively.

(a) (10 pts) Compute $[R|c]$, the reduced row echelon form of the augmented matrix $[A|b]$. Show each step of your calculation and indicate each elementary row operation with the arrow notation, as in class and in the text.

$$\begin{bmatrix} 1 & -2 & -10 & 5 \\ 1 & 1 & -1 & -1 \\ -1 & -2 & -2 & 5 \end{bmatrix} \xrightarrow{r_1+r_2\rightarrow r_2} \begin{bmatrix} 1 & -2 & -10 & 5 \\ 0 & 3 & 9 & -6 \\ -1 & -2 & -2 & 5 \end{bmatrix} \xrightarrow{r_1+r_3\rightarrow r_3} \begin{bmatrix} 1 & -2 & -10 & 5 \\ 0 & 3 & 9 & -6 \\ 0 & -4 & -12 & 10 \end{bmatrix}$$

$$\xrightarrow{\frac{1}{3}r_2\rightarrow r_2} \begin{bmatrix} 1 & -2 & -10 & 5 \\ 0 & 1 & 3 & -2 \\ 0 & -4 & -12 & 10 \end{bmatrix} \xrightarrow{2r_2+r_1\rightarrow r_1} \begin{bmatrix} 1 & 0 & -4 & 1 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{4r_2+r_3\rightarrow r_3} \begin{bmatrix} 1 & 0 & -4 & 1 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{\frac{1}{2}r_3\rightarrow r_3} \begin{bmatrix} 1 & 0 & -4 & 1 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{-r_3+r_1\rightarrow r_1} \begin{bmatrix} 1 & 0 & -4 & 1 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{2r_3+r_2\rightarrow r_2} \begin{bmatrix} 1 & 0 & -4 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = [R|c]$$

(b) (5 pts) Calculate rank($A$) and nullity($A$).

The rank of $A$ is the number of pivot columns of $R$, hence rank($A$) = 2. The nullity is the number of columns of $A$ minus the rank, hence nullity($A$) = 3 - 2 = 1.

(c) (5 pts) True or False? The set $\{u, v, w\}$ is linearly independent. (Justify your answer.)

False. $A$ does not have a pivot in every column.

(d) (5 pts) True or False? The vector $b$ is in the span of of the set $\{u, v, w\}$. (Justify your answer.)

False. The system $Ax = b$ is equivalent to $Ry = c$, which is inconsistent since the last row is equivalent to the equation $0 = 1$. Hence $Ax = b$ has no solution, which means $b$ is not in the span of the columns of $A$. 
2. Suppose \( A \) is a matrix whose reduced row echelon form is given by
\[
R = \begin{bmatrix}
0 & 1 & 0 & 0 & 5 & 0 & 0 \\
0 & 0 & 1 & 0 & -3 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]
(Note that \( R \) and \( A \) may be different matrices!)

(a) (5 pts) Find the general solution to the equation \( Ax = 0 \). Write \( x \) in vector form as a linear combination of linearly independent solutions with the free variables as coefficients.

The system \( Rx = 0 \) is equivalent to the system
\[
\begin{align*}
x_2 + 5x_5 &= 0 \\
x_3 - 3x_5 + x_6 &= 0 \\
x_7 &= 0
\end{align*}
\]
The variables \( x_1, x_4, x_5, \) and \( x_6 \) are free, and the other variables are given by
\[
\begin{align*}
x_2 &= -5x_5 \\
x_3 &= 3x_5 - x_6 \\
x_7 &= 0
\end{align*}
\]
Hence the vector form of the general solution is
\[
x = \begin{bmatrix}
x_1 \\
-5x_5 \\
3x_5 - x_6 \\
x_4 \\
x_5 \\
x_6 \\
0
\end{bmatrix} = x_1 \begin{bmatrix}
1 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0
\end{bmatrix} + x_4 \begin{bmatrix}
0 \\
0 \\
0 \\
1 \\
0 \\
0 \\
0
\end{bmatrix} + x_5 \begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0
\end{bmatrix} + x_6 \begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0
\end{bmatrix} + x_7 \begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0
\end{bmatrix}
\]

(b) (5 pts) True or False? For every vector \( b \in \mathbb{R}^3 \) the equation \( Ay = b \) has at least one solution \( y \in \mathbb{R}^7 \). (Justify your answer.)

True. The matrix \( A \) has a pivot in each row. Hence \( Ay = b \) is always consistent.

(c) (5 pts) Suppose you are given the last three columns of \( A \):
\[
a_5 = \begin{bmatrix}
3 \\
-1 \\
2
\end{bmatrix}, \quad a_6 = \begin{bmatrix}
2 \\
2 \\
-2
\end{bmatrix}, \quad a_7 = \begin{bmatrix}
-1 \\
0 \\
1
\end{bmatrix}
\]
Calculate the remaining columns of \( A \) (column 1 through column 4).
By the column correspondence property, we find the following relationships between the columns of $R$ (and hence also between the columns of $A$).

$$
a_1 = 0
$$
$$
a_4 = 0
$$
$$
a_5 = 5a_2 - 3a_3
$$
$$
a_6 = a_3
$$

Hence $a_3 = a_6$ and $a_2 = \frac{1}{5} (a_5 + 3a_6)$. The original matrix $A$ is therefore

$$
A = \begin{bmatrix}
0 & \frac{9}{5} & 2 & 0 & 3 & 2 & -1 \\
0 & 1 & 2 & 0 & -1 & 2 & 0 \\
0 & -\frac{4}{5} & -2 & 0 & 2 & -2 & 1
\end{bmatrix}
$$

(d) (5 pts) In part (c), suppose you are given $a_5$ and $a_6$, but you are not given $a_7$. Would you still be able to reconstruct all of $A$? Explain your answer.

No. Since $A$ has 3 pivot columns, we must be given at least 3 columns to reconstruct $A$. (The given columns must themselves also be linearly independent, but that is irrelevant. Two columns is not enough to reconstruct $A$.)
3. Consider the following elementary matrices.

\[ E = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}, \quad G = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \]

(a) (3 pts) Describe the elementary row operation represented by left-multiplication by \( E \).
Use this information to deduce \( E^{-1} \).

\[ E \text{ corresponds to } r_1 \leftrightarrow r_2, \text{ hence } E^{-1} \text{ corresponds to } r_1 \leftrightarrow r_2. \text{ Applying this row operation to } I_3 \text{ gives} \]

\[ E^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \]

(b) (3 pts) Describe the elementary row operation represented by left-multiplication by \( F \).
Use this information to deduce \( F^{-1} \).

\[ F \text{ corresponds to } -2r_2 + r_3 \rightarrow r_3, \text{ hence } F^{-1} \text{ corresponds to } 2r_2 + r_3 \rightarrow r_3. \text{ Applying this row operation to } I_3 \text{ gives} \]

\[ F^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \]

(c) (3 pts) Describe the elementary row operation represented by left-multiplication by \( G \).
Use this information to deduce \( G^{-1} \).

\[ G \text{ corresponds to } 5r_2 \rightarrow r_2, \text{ hence } G^{-1} \text{ corresponds to } \frac{1}{5}r_2 \rightarrow r_2. \text{ Applying this row operation to } I_3 \text{ gives} \]

\[ G^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{5} & 0 \\ 0 & 0 & 1 \end{bmatrix} \]

(d) (4 pts) Calculate \( A = EFG \).

\[
\begin{align*}
\begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix} & \xrightarrow{-2r_2 + r_3 \rightarrow r_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & -10 & 1 \end{bmatrix} \\
\begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & -10 & 1 \end{bmatrix} & \xrightarrow{r_1 \leftrightarrow r_2} \begin{bmatrix} 0 & 5 & 0 \\ 0 & 0 & 1 \\ 0 & -10 & 1 \end{bmatrix} = A
\end{align*}
\]
(e) (3 pts) Express \( A^{-1} \) symbolically in terms of the inverses of \( E, F, \) and \( G \). Then use your formula to calculate \( A^{-1} \).

We have \( A^{-1} = (EFG)^{-1} = G^{-1}F^{-1}E^{-1} \). Start with \( E^{-1} \) and perform the row operations corresponding to \( F^{-1} \) and then \( G^{-1} \) on \( E^{-1} \), in that order, to obtain \( A^{-1} \).

\[
\begin{bmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
\overset{2r_2+r_3\rightarrow r_3}{\longrightarrow}
\begin{bmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
2 & 0 & 1 \\
\end{bmatrix}
\overset{\frac{1}{5}r_2\rightarrow r_2}{\longrightarrow}
\begin{bmatrix}
0 & 1 & 0 \\
\frac{1}{5} & 0 & 0 \\
2 & 0 & 1 \\
\end{bmatrix}
= A^{-1}
\]

(f) (4 pts) Compute \( GE^4F^{-1}GB \) where \( B \) is the following matrix.

\[
B = \begin{bmatrix}
2 & 1 \\
0 & -5 \\
-1 & 4 \\
\end{bmatrix}
\]

First note that \( E = E^{-1} \), hence \( E^4 = I \). So we only need to calculate \( GF^{-1}GB \). Hence start with \( B \) and perform the row operations corresponding to \( G, F^{-1}, \) and then \( G \) on \( B \), in that order.

\[
\begin{bmatrix}
2 & 1 \\
0 & -5 \\
-1 & 4 \\
\end{bmatrix}
\overset{5r_2\rightarrow r_2}{\longrightarrow}
\begin{bmatrix}
2 & 1 \\
0 & -25 \\
-1 & 4 \\
\end{bmatrix}
\overset{2r_2+r_3\rightarrow r_3}{\longrightarrow}
\begin{bmatrix}
2 & 1 \\
0 & -25 \\
-1 & -46 \\
\end{bmatrix}
\overset{5r_2\rightarrow r_2}{\longrightarrow}
\begin{bmatrix}
2 & 1 \\
0 & -125 \\
-1 & -46 \\
\end{bmatrix}
\]
4. (15 pts) Consider the matrix $A$ and vector $b$ below.

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & 0 \\ 2 & -3 & 2 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

(a) (10 pts) Calculate $A^{-1}$.

We first augment $A$ with the identity to obtain $[A|I_3]$. Then we find the reduced row echelon form of this augmented matrix. If $A$ is invertible, the final form is $[I_3|A^{-1}]$.

$$[A|I_3] = \begin{bmatrix} 1 & -1 & 1 & | & 1 & 0 & 0 \\ 1 & -2 & 0 & | & 0 & 1 & 0 \\ 2 & -3 & 2 & | & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{-r_1+r_2\leftrightarrow r_2} \begin{bmatrix} 1 & -1 & 1 & | & 1 & 0 & 0 \\ 0 & -1 & -1 & | & -1 & 1 & 0 \\ 2 & -3 & 2 & | & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{-2r_1+r_3\rightarrow r_3} \begin{bmatrix} 1 & -1 & 1 & | & 1 & 0 & 0 \\ 0 & -1 & -1 & | & 0 & 1 & 0 \\ 0 & -1 & 0 & | & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{-r_2\rightarrow r_2} \begin{bmatrix} 1 & -1 & 1 & | & 1 & 0 & 0 \\ 0 & 1 & 1 & | & -1 & 1 & 0 \\ 0 & -1 & 0 & | & -2 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{r_2+r_1\rightarrow r_1} \begin{bmatrix} 1 & 0 & 2 & | & 2 & -1 & 0 \\ 0 & 1 & 1 & | & 1 & -1 & 0 \\ 0 & -1 & 0 & | & -2 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{r_2+r_3\rightarrow r_3} \begin{bmatrix} 1 & 0 & 2 & | & 2 & -1 & 0 \\ 0 & 1 & 1 & | & 1 & -1 & 0 \\ 0 & 0 & 1 & | & -1 & -1 & 1 \end{bmatrix}$$

$$\xrightarrow{-r_3+r_2\rightarrow r_2} \begin{bmatrix} 1 & 0 & 2 & | & 2 & -1 & 0 \\ 0 & 1 & 0 & | & 2 & 0 & -1 \\ 0 & 0 & 1 & | & -1 & -1 & 1 \end{bmatrix}$$

$$\xrightarrow{-2r_3+r_1\rightarrow r_1} \begin{bmatrix} 1 & 0 & 0 & | & 4 & 1 & -2 \\ 0 & 1 & 0 & | & 2 & 0 & -1 \\ 0 & 0 & 1 & | & -1 & -1 & 1 \end{bmatrix}$$

Hence we have that

$$A^{-1} = \begin{bmatrix} 4 & 1 & -2 \\ 2 & 0 & -1 \\ -1 & -1 & 1 \end{bmatrix}$$

(b) (5 pts) Use your answer from part (a) to solve the system $A^2x = b$.

(Note that the coefficient matrix is not A but $A^2$. Hint: Recall that $A^2 = AA$ and first solve the system symbolically in terms of $A^{-1}$ and $b$. Then calculate the answer using matrix multiplication.)

Multiplying the equation $A^2x = b$ on the left by $A^{-1}$ twice gives us $x = A^{-1}A^{-1}b$.
Now we calculate $x$ by matrix multiplication. Remember that matrix multiplication
is associative. So it’s easier to do the multiplication with the following grouping: $x = A^{-1}(A^{-1}b)$.

$$x = A^{-1}A^{-1}b = \begin{bmatrix} 4 & 1 & -2 \\ 2 & -1 & 1 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 4 & 1 & -2 \\ 2 & -1 & 1 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 1 & -2 \\ 2 & -1 & 1 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 9 \\ 4 \\ -3 \end{bmatrix}$$
5. (5 pts each) Determine whether each of the following statements is true or false. You must justify your answer. If your answer is “true” explain why the statement is always true. If your answer is “false”, give an example for which the statement is false. **If you do not justify your answer, you will receive no credit!**

(a) A set of vectors in $\mathbb{R}^n$ is linearly dependent if and only if one of the vectors is a multiple of one of the others.

False. Let $S = \{e_1, e_2, e_1 + e_2\}$. Then $S$ is clearly linearly dependent, but no vector is a multiple of some other vector.

(b) If a subset $S$ of $\mathbb{R}^n$ spans $\mathbb{R}^n$, then $S$ must contain at least $n$ vectors.

True. Let $S = \{a_1, \ldots, a_k\}$ be a set of vectors in $\mathbb{R}^n$ and suppose $\text{Span}(S) = \mathbb{R}^n$. Let $A$ be the matrix whose $j$th column is $a_j$, and note that $A$ is an $n \times k$ matrix. Since the columns of $A$ span $\mathbb{R}^n$, each row of $A$ must have a pivot, thus $A$ has exactly $n$ pivots. Since a given column can contain at most 1 pivot, there must be at least $n$ columns to contain the $n$ pivots. Thus $k \geq n$, whence $S$ contains at least $n$ vectors.

(c) If a square matrix $A$ has a row of all zeros, then $A$ is not invertible.

True. Suppose the $j$th row of $A$ is a row of all zeros. Then the system $Ax = e_j$ is inconsistent because the $j$th component of $Ax$ is 0 (independent of $x$) and the $j$th component of $e_j$ is 1. Since there is a vector $b$ for which the system $Ax = b$ is not consistent, $A$ cannot be invertible.

(d) Let $A$ be an $m \times n$ matrix. Suppose there exists a vector $b \in \mathbb{R}^m$ for which the system $Ax = b$ has infinitely many solutions. Then there must exist a vector $c \in \mathbb{R}^m$ for which the system $Ay = c$ has no solutions.

False. Put $A = \begin{bmatrix} 1 & 0 \end{bmatrix}$ and $b = 0 \in \mathbb{R}^1$. Then the system $Ax = b$ has infinitely many solutions, namely $x = x_2e_2$, where $x_2$ is any scalar. However, $A$ has a pivot in each row, whence the system $Ay = c$ is consistent for all $c \in \mathbb{R}^1$.

(Note: The condition that $Ax = b$ has infinitely many solutions for some $b$ requires only that $A$ have a non-pivot column. The condition that $Ay = c$ be inconsistent for some $c$ requires only that $A$ have a non-pivot row. This problem shows that the first condition does not imply the second. Indeed, it is also the case that the second condition does not imply the first: put $A = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $c = e_2$.)