Name (PRINT): ________________________________  ID # (last 4 digits): ____________

Signature: ________________________________

• This is a closed book exam. No notes, calculators, phones, etc. are allowed.

• Please explain and label your answers clearly and show all work. I reserve the right to give no credit for a response with no work even if the final answer is correct.

• You have 90 minutes to complete the exam. There are 100 points total.

• Please have your photo ID available. Do not start the exam until instructed to do so.

<table>
<thead>
<tr>
<th>Problem</th>
<th>Points</th>
<th>Score</th>
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<tbody>
<tr>
<td>1</td>
<td>25</td>
<td></td>
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<tr>
<td>2</td>
<td>20</td>
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<td>100</td>
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1. Let $A$ and $b$ be the matrix and vector below.

$$A = \begin{bmatrix} 1 & -2 & -10 \\ 1 & 1 & -1 \\ -1 & -2 & -2 \end{bmatrix}, \quad b = \begin{bmatrix} 5 \\ -1 \\ 5 \end{bmatrix}$$

Let $u$, $v$, and $w$ denote the first, second, and third columns of $A$, respectively.

(a) (10 pts) Compute $[R | c]$, the reduced row echelon form of the augmented matrix $[A | b]$. Show each step of your calculation and indicate each elementary row operation with the arrow notation, as in class and in the text.
(b) (5 pts) Calculate rank(A) and nullity(A).

(c) (5 pts) True or False? The set \{u, v, w\} is linearly independent. (Justify your answer.)

(d) (5 pts) True or False? The vector \(b\) is in the span of the set \{u, v, w\}. (Justify your answer.)
2. Suppose $A$ is a matrix whose reduced row echelon form is given by

$$R = \begin{bmatrix}
0 & 1 & 0 & 0 & 5 & 0 & 0 \\
0 & 0 & 1 & 0 & -3 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}$$

(Note that $R$ and $A$ may be different matrices!)

(a) (5 pts) Find the general solution to the equation $Ax = 0$. Write $x$ in vector form as a linear combination of linearly independent solutions with the free variables as coefficients.

(b) (5 pts) True or False? For every vector $b \in \mathbb{R}^3$ the equation $Ay = b$ has at least one solution $y \in \mathbb{R}^7$. (Justify your answer.)
(c) (5 pts) Suppose you are given the last three columns of $A$:

\[
a_5 = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}, \quad a_6 = \begin{bmatrix} 2 \\ 2 \\ -2 \end{bmatrix}, \quad a_7 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}
\]

Calculate the remaining columns of $A$ (column 1 through column 4).

(d) (5 pts) In part (c), suppose you are given $a_5$ and $a_6$, but you are not given $a_7$. Would you still be able to reconstruct all of $A$? Explain your answer.
3. Consider the following elementary matrices.

\[
E = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}, \quad G = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\]

(a) (3 pts) Describe the elementary row operation represented by left-multiplication by \( E \). Use this information to deduce \( E^{-1} \).

(b) (3 pts) Describe the elementary row operation represented by left-multiplication by \( F \). Use this information to deduce \( F^{-1} \).

(c) (3 pts) Describe the elementary row operation represented by left-multiplication by \( G \). Use this information to deduce \( G^{-1} \).
(d) (4 pts) Calculate $A = EFG$.

(e) (3 pts) Express $A^{-1}$ symbolically in terms of the inverses of $E$, $F$, and $G$. Then use your formula to calculate $A^{-1}$.

(f) (4 pts) Compute $GE^4F^{-1}GB$ where $B$ is the following matrix.

$$B = \begin{bmatrix} 2 & 1 \\ 0 & -5 \\ -1 & 4 \end{bmatrix}$$
4. (15 pts) Consider the matrix $A$ and vector $b$ below.

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & 0 \\ 2 & -3 & 2 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

(a) (10 pts) Calculate $A^{-1}$. 
(b) (5 pts) Use your answer from part (a) to solve the system $A^2\mathbf{x} = \mathbf{b}$.

(Note that the coefficient matrix is not $A$ but $A^2$. Hint: Recall that $A^2 = AA$ and first solve the system symbolically in terms of $A^{-1}$ and $\mathbf{b}$. Then calculate the answer using matrix multiplication.)
5. (5 pts each) Determine whether each of the following statements is true or false. You must justify your answer. If your answer is “true” explain why the statement is always true. If your answer is “false”, give an example for which the statement is false. **If you do not justify your answer, you will receive no credit!**

(a) A set of vectors in $\mathbb{R}^n$ is linearly dependent if and only if one of the vectors is a multiple of one of the others.

(b) If a subset $S$ of $\mathbb{R}^n$ spans $\mathbb{R}^n$, then $S$ must contain at least $n$ vectors.
(c) If a square matrix $A$ has a row of all zeros, then $A$ is not invertible.

(d) Let $A$ be an $m \times n$ matrix. Suppose there exists a vector $b \in \mathbb{R}^m$ for which the system $Ax = b$ has infinitely many solutions. Then there must exist a vector $c \in \mathbb{R}^m$ for which the system $Ay = c$ has no solutions.