Determine whether each series converges or diverges. Justify your answer. That means you should name any test you use explicitly and verify any relevant hypotheses.

(a) \[\sum_{n=1}^{\infty} \frac{n!}{n^n}\]  
(b) \[\sum_{n=1}^{\infty} \left(\frac{2 + 3n}{1 + 4n}\right)^n\]  
(c) \[\sum_{n=1}^{\infty} \frac{(-2)^n}{n^{2/3}}\]  
(d) \[\sum_{n=1}^{\infty} \sin\left(\frac{1}{n^2}\right)\]  
(e) \[\sum_{n=1}^{\infty} \frac{\sin(n)}{n^2}\]  
(f) \[\sum_{n=2}^{\infty} \frac{1}{n \ln(n)^{3/2}}\]  
(g) \[\sum_{n=1}^{\infty} (-1)^n \cos\left(\frac{1}{n}\right)\]  
(h) \[\sum_{n=1}^{\infty} \frac{n^2 3^n}{(2n + 1)!}\]  
(i) \[\sum_{n=1}^{\infty} \frac{n^{2/3}}{n^2 + \sqrt{n}}\]  
(j) \[\sum_{n=1}^{\infty} \frac{3^n + 5^n}{7^n + 8^n}\]  
(k) \[\sum_{n=1}^{\infty} \frac{n^3 + 3^n}{\sqrt{n} + 9^n}\]  
(l) \[\sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n \ln(n)^2}}\]  

Tests you can use:
- geometric series test
- $N$th term divergence test
- integral test
- $p$-test
- direct comparison test
- limit comparison test
- absolute convergence test
- alternating series test
- ratio test
- root test
Sketch of Solutions

(a) Converges by Ratio Test

\[ \frac{a_{n+1}}{a_n} = \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} = \frac{(n+1)!}{n!} \cdot \frac{n^n}{(n+1)^{n+1}} \]

\[ = (n+1) \cdot \frac{n^n}{(n+1)^{n+1}} = \frac{n^n}{(n+1)^n} = \left(1 + \frac{1}{n}\right)^{-n} \]

So \( p = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^{-n} = e^{-1} < 1 \)

(b) Converges by Root Test.

\[ S = \lim_{n \to \infty} \left[ \left( \frac{2+3n}{1+4n} \right)^n \right]^{1/n} = \frac{3}{4} < 1 \]

(c) Diverges by Nth term divergence test or by Ratio Test.

\[ \lim_{n \to \infty} \frac{(-2)^n}{n^{3/2}} \text{ does not exist or} \]

\[ S = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = 2 > 1 \]

(d) Converges by LCT, compare to \( \sum_{n=1}^{\infty} \frac{1}{n^2} \)

\[ \lim_{n \to \infty} \frac{\sin \left( \frac{1}{n^2} \right)}{1/n^2} = \lim_{\theta \to 0} \frac{\sin(\theta)}{\theta} = 1 \]
(e) Converges by Abs. Convergence + DCT

\[ 0 \leq \frac{|\sin(n^2)|}{n^2} \leq \frac{1}{n^2}, \text{ so } \sum \frac{|\sin(n^2)|}{n^2} \text{ converges} \]

So series absolutely convergent.

(f) Converges by Integral Test

\[ f(x) = \frac{1}{x \ln(x)^{3/2}} \leq \text{ positive, decreasing, continuous} \]

\[ \int_2^\infty \frac{1}{x \ln(x)^{3/2}} \, dx = \int_{\ln(2)}^\infty u^{-3/2} \, du = -2u^{-1/2} \bigg|_{\ln(2)}^\infty \]

\[ = (0) - (-2 \ln(2)^{-1/2}) = \frac{2}{\sqrt{\ln(2)}} < \infty \]

(g) Diverges by Nth term divergence test.

\[ \lim_{n \to \infty} (-1)^n \cos \left( \frac{1}{n} \right) \text{ does not exist.} \]

(h) Converges by Ratio Test.

\[ \frac{a_{n+1}}{a_n} = \frac{(n+1)^2 3^{n+1}}{(2n+3)!} \cdot \frac{(2n+1)!}{n^2 3^n} = \left( \frac{n+1}{n} \right)^2 \cdot 3 \cdot \frac{1}{(2n+3)(2n+2)} \]

\[ p = \lim_{n \to \infty} \left( \ldots \right) = 1.3 \cdot 0 = 0 < 1 \]
(i) Converges by DCT, compare to $\sum \frac{1}{n^{4/3}}$

\[0 \leq \frac{n^{2/3}}{n^2 + \sqrt{n}} \leq \frac{n^{2/3}}{n^2} = \frac{1}{n^{4/3}}\]

(i) Converges by LCT, compare to $\sum \frac{5^n}{8^n}$

Or converges by DCT, compare to $\sum \frac{5^n}{7^n}$

LCT: $\lim_{n \to \infty} \frac{3^n + 5^n}{7^n + 8^n} = \lim_{n \to \infty} \frac{(\frac{3}{5})^n + 1}{(\frac{7}{8})^n + 1} = \frac{0+1}{0+1} = 1$

DCT: $0 \leq \frac{3^n + 5^n}{7^n + 8^n} < \frac{5^n}{7^n + 8^n} < \frac{5^n}{7^n} = (\frac{5}{7})^n$

(k) Diverges by Nth term divergence test:

\[\lim_{n \to \infty} \frac{n^3 + 3^n}{\sqrt{n^9 + 9^n}} = \lim_{n \to \infty} \frac{n^3}{\sqrt{n^9}} + 1 = 0+1 = 1 \neq 0\]

(l) Converges by alternating series test.

Let $a_n = \frac{1}{\sqrt{n} \ln(n)^2}$, positive decreasing limit 0

So $\sum (-1)^n a_n$ converges.