Strategies For Integration

7.1: Integration by Parts

\[ \int u \, dv = uv - \int v \, du \]

Priority for choosing u:

L: logarithms
I: inverse trigonometric, inverse hyperbolic
A: algebraic
T: trigonometric, hyperbolic
E: exponential

7.2: Trigonometric Integrals

\[ \int \sin^n(x) \cos^n(x) \, dx \]

(A) m odd (n anything)

- split off factor of \( \sin(x) \)
- rewrite remaining powers of \( \sin(x) \) in terms of \( \cos(x) \) using identity \( \sin(x)^2 = 1 - \cos(x)^2 \)
• use the substitution \( u = \cos(x) \)

(B) \( n \) odd \( (m \) anything \)

• split off factor of \( \cos(x) \)

• rewrite remaining powers of \( \cos(x) \)
in terms of \( \sin(x) \) using identity
\[ \cos(x)^2 = 1 - \sin(x)^2 \]

• use the substitution \( u = \sin(x) \)

(c) \( m \) and \( n \) both even

• rewrite entire integrand in terms of \( \sin(x) \) only or \( \cos(x) \) only
  using identity \( \cos(x)^2 + \sin(x)^2 = 1 \).

• if rewritten in terms of \( \sin(x) \)...

• use integration by parts with
  \( dv = \sin(x) \, dx \)

• in resulting integral, rewrite \( \cos(x)^2 \)
as \( 1 - \sin(x)^2 \)

• algebraically solve for original integral.

• if rewritten in terms of \( \cos(x) \)...
• use integration by parts with 
  \( dv = \cos(x) \, dx \)

• in resulting integral, rewrite \( \sin(x)^2 \) as \( 1 - \cos(x)^2 \)

• algebraically solve for original integral.

\[
\int \tan(x)^m \sec(x)^n \, dx
\]

(A) Special cases (memorize)

• \( \int \tan(x) \, dx = \ln |\sec(x)| + C \)

• \( \int \sec(x) \, dx = \ln |\sec(x) + \tan(x)| + C \)

(B) \( m \) odd (\( n \) anything)

• split off factor of \( \sec(x) \tan(x) \)

• rewrite remaining powers of \( \tan(x) \) in terms of \( \sec(x) \) using identity

\( \tan(x)^2 = \sec(x)^2 - 1 \)
• use the substitution $u = \sec(x)$
  (C) $n$ even and $n \neq 2$ (m anything)

• split off factor of $\sec(x)^2$

• rewrite remaining powers of $\sec(x)$ in terms of $\tan(x)$ using identity $
\sec(x)^2 = \tan(x)^2 + 1$

• use the substitution $u = \tan(x)$

(D) $m$ even and $n$ odd

• rewrite entire integrand in terms of $\sec(x)$ only using identity
\[ \tan(x)^2 = \sec(x)^2 - 1 \]

• use integration by parts with
\[ du = \sec(x)^2 \, dx \]

• in resulting integral, rewrite $\tan(x)^2$ as $\sec(x)^2 - 1$

• algebraically solve for original integral:
\[ \int \cot(x)^m \csc(x)^n \, dx \]
(A) Special cases (memorize)
- \( \int \cot(x) \, dx = -\ln|\csc(x)| + C \)
- \( \int \csc(x) \, dx = -\ln|\csc(x) + \cot(x)| + C \)

(B) \( n \) odd (\( n \) anything)
- Split off factor of \( \csc(x) \cot(x) \)
- Rewrite remaining powers of \( \cot(x) \) in terms of \( \csc(x) \) using identity \( \cot(x)^2 = \csc(x)^2 - 1 \)
- Use the substitution \( u = \csc(x) \)

(C) \( n \) even and \( n \neq 2 \) (\( m \) anything)
- Split off factor of \( \csc(x)^2 \)
- Rewrite remaining powers of \( \csc(x) \) in terms of \( \cot(x) \) using identity \( \csc(x)^2 = \cot(x)^2 + 1 \)
- Use the substitution \( u = \cot(x) \)

(D) \( m \) even and \( n \) odd
• rewrite entire integrand in terms of \( \csc(x) \) only using identity \( \cot(x)^2 = \csc(x)^2 - 1 \)

• use integration by parts with 
  \( dv = \csc(x)^2 \, dx \)

• in resulting integral, rewrite \( \cot(x)^2 \) as \( \csc(x)^2 - 1 \)

• algebraically solve for original integral.

7.3: Trigonometric Substitution

• Use trigonometric substitution for integrands with quadratic expressions under some integer power of a square root.

• Complete the square as necessary to obtain one of the following forms:

  \[
  \begin{cases}
    \sqrt{a^2-x^2} \\
    \sqrt{a^2+x^2} \\
    \sqrt{x^2-a^2}
  \end{cases}
  \] always assume \( a > 0 \).
• Use the table below as necessary:

<table>
<thead>
<tr>
<th>Expression</th>
<th>Substitution</th>
<th>Identities</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sqrt{a^2-x^2} )</td>
<td>( x = \sin(\theta) ) (-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2})</td>
<td>( dx = a\cos(\theta),d\theta ) ( \sqrt{a^2-x^2} = a\cos(\theta) )</td>
</tr>
<tr>
<td>( \sqrt{a^2+x^2} )</td>
<td>( x = \tan(\theta) ) (-\frac{\pi}{2} &lt; \theta &lt; \frac{\pi}{2})</td>
<td>( dx = a\sec^2(\theta),d\theta ) ( \sqrt{a^2+x^2} = a\sec(\theta) )</td>
</tr>
<tr>
<td>( \sqrt{x^2-a^2} )</td>
<td>( x = \sec(\theta) ) ( \theta \leq \frac{\pi}{2} ) OR ( \pi \leq \theta \leq \frac{3\pi}{2} )</td>
<td>( dx = a\sec(\theta)\tan(\theta),d\theta ) ( \sqrt{x^2-a^2} = a\tan(\theta) )</td>
</tr>
</tbody>
</table>

7.4: Integrals with Hyperbolic Functions

Note: Be sure to go over the “Hyperbolic Functions Review Sheet”.

(A) Substitution

(B) Integration by Parts

\( \rightarrow \text{When choosing } u, \text{ treat hyperbolic and inverse hyperbolic functions} \)
as you would treat trigonometric and inverse trigonometric functions

(C) Hyperbolic Integrals

\[ \text{Treat powers of hyperbolic functions as you would treat trigonometric functions. The strategies are identical} \]

(D) Hyperbolic Substitution

\[ \text{Similar to trigonometric substitution} \]

<table>
<thead>
<tr>
<th>Expression</th>
<th>Trig. Substitution</th>
<th>Hyp. Substitution</th>
</tr>
</thead>
<tbody>
<tr>
<td>[ \sqrt{a^2 - x^2} ]</td>
<td>[ x = \sin(\theta) ]</td>
<td>[ x = \tanh^{-1}(u) ]</td>
</tr>
<tr>
<td>[ \sqrt{a^2 + x^2} ]</td>
<td>[ x = \tan(\theta) ]</td>
<td>[ x = \sinh^{-1}(u) ]</td>
</tr>
<tr>
<td>[ \sqrt{x^2 - a^2} ]</td>
<td>[ x = \sec(\theta) ]</td>
<td>[ x = \cosh^{-1}(u) ]</td>
</tr>
</tbody>
</table>

7.5: Method of Partial Fractions

- Partial fractions can be used for integrating any rational function

\[ f(x) = \frac{P(x)}{Q(x)} \]  \quad P and Q are polynomials with no common factors
• If \( \deg(P) \geq \deg(Q) \), perform long division first to write

\[
f(x) = \underbrace{R(x)} + \frac{r(x)}{Q(x)}
\]

polynomial of degree remainder term:
\( \deg(P) - \deg(Q) \quad \deg(r) < \deg(Q) \)

• Find the partial fraction decomposition (PFD) of the remainder term
  • Factor \( Q(x) \) into irreducible factors
    • \( ax + b \)  OR
    • \( ax^2 + bx + c \) (with \( b^2 - 4ac < 0 \))

• Write \( \frac{r(x)}{Q(x)} \) as a sum of simple partial fractions. If a factor of \( Q(x) \) is repeated \( m \) times, then the PFD has \( m \) terms for that repeated factor.
• The numerator of each term should be one degree less than the corresponding factor of \( Q \).

**Examples:**

\[
\frac{1}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b}
\]

\[
\frac{1}{(x-a)(x-b)^2} = \frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{(x-b)^2}
\]

\[
\frac{1}{(x-a)(x^2+b^2)} = \frac{A}{x-a} + \frac{Bx+C}{x^2+b^2}
\]

• **Find the values of the unknown coefficients using algebra.**

• **Find the desired integral using the PFD.** For quadratic factors, split into two: one part uses substitution and the other uses trig. substitution \((x = a \tan(\theta))\)
Some common integrals that are useful to memorize:

\[
\int \frac{x}{x^2 + a^2} \, dx = \frac{1}{2} \ln (x^2 + a^2) + C
\]

\[
\int \frac{1}{x^2 + a^2} \, dx = \frac{1}{a} \tan^{-1} \left( \frac{x}{a} \right) + C
\]