Section 6.4: Method of Cylindrical Shells

*This is not volume by slicing. Instead the volume is approximated by nested shells. So consider one thin shell:

inner radius: \( r_i \)
outer radius: \( r_2 \)
average of inner and outer radius: \( \bar{r} = \frac{r_i + r_2}{2} \)

When we say the shell is “thin” we mean the following:
1. thickness of shell \( r_2 - r_1 = dr \)

\( \rightarrow \) \( dr \) is an arbitrarily small number
2. Even though the radius $r$ varies across the ring, we approximate $r$ as constant.

→ so $r = \bar{r}$ for entire shell.

So now what is the volume of the shell?

Volume = (larger cylinder) - (smaller cylinder)

$\begin{align*}
dV &= (\pi r_2^2 h) - (\pi r_1^2 h) \\
dV &= \pi h \cdot (r_2 + r_1)(r_2 - r_1) \\
dV &= \pi h \cdot \left(\frac{r_2 + r_1}{2}\right) \cdot (r_2 - r_1) \\
&= \bar{r} = dr
\end{align*}$

$dV = 2\pi h \bar{r} dr$

The total volume of the solid is the "sum" of all the small volumes. But the "sum" of all $dV$'s is an integral.

Total volume: $V = \int dV = \int_0^R 2\pi h \bar{r} dr$

(We will have $dr = dx$ or $dr = dy$, and $h$ and $\bar{r}$ depend on $x$ or $y$ accordingly.)
This is our formula for volume of revolution by **Method of Shells**:

**Summary:**
- \( h \) = height of shell
- \( R \) = radius of shell
- \( dr \) = thickness of shell

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**Ex. 1**

Let \( R \) be the region bounded by coordinate axes, \( x=1 \), and

\[
f(x) = 1 - 2x + 3x^2 - 2x^3
\]

Find the volume of the solid obtained by rotating \( R \) about the y-axis.

**Solution:**

Why is method of washers very difficult?
\[ y = f(x) \]

\[ P = (0, y) \]
\[ C = (0, y) \]
\[ Q = (f^{-1}(y), y) \]

* Finding \( f^{-1}(y) \) is essentially impossible. So we have to use shells.

Shells uses a strip parallel to the axis of rotation.

\[ P = (x, 0) \]
\[ Q = (x, f(x)) = (x, 1 - 2x + 3x^2 - 2x^3) \]
\[ h = |PQ| = 1 - 2x + 3x^2 - 2x^3 \]

\[ F = \text{“distance from axis to shell wall”} \]

\[ = x_{\text{wall}} - x_{\text{axis}} = x - 0 = x \]

(Generally, \( F = a \pm x \) or \( F = a \pm y \))

\[ dr = dx \]

(Vertical strip \( \Rightarrow dx \), Horizontal strip \( \Rightarrow dy \))

So the volume is

\[ V = 2\pi \int_0^1 (1 - 2x + 3x^2 - 2x^3) \cdot dx \cdot h \cdot F \cdot dr \]

\[ = 2\pi \left( \frac{1}{2}x^2 - \frac{2}{3}x^3 + \frac{3}{4}x^4 - \frac{2}{5}x^5 \right) \bigg|_0^1 = \frac{11\pi}{30} \]

**Ex. 2**

Let \( R \) be region bounded by \( y = \sqrt{x} \) and \( y = x^2 \). Find volume of solid obtained by rotating \( R \) about \( y \)-axis.

**Solution:**
\( P = (x, \sqrt{x}) \)

\( Q = (x, x^2) \)

\[ h = |PQ| = P_y - Q_y = \sqrt{x} - x^2 \]

\( r = x \)

\( dr = dx \)

\[ V = 2\pi \int_{0}^{1} h r \, dr \]

\[ = 2\pi \int_{0}^{1} (\sqrt{x} - x^2) x \, dx \]

\[ = 2\pi \int_{0}^{1} (x^{3/2} - x^3) \, dx \]

\[ = 2\pi \left( \frac{2}{5} x^{5/2} - \frac{1}{4} x^4 \right) \bigg|_{0}^{1} = \frac{3\pi}{10} \]
P = ( y^2, y )  \quad y = \sqrt{x} \implies x = y^2
Q = ( \sqrt{y}, y )  \quad y = x^2 \implies x = \sqrt{y}
C = ( 0, y )

R_{out} = |QC| = Q_x - C_x = \sqrt{y}
R_{in} = |PC| = P_x - C_x = y^2

V = \pi \int_0^1 (R_{out}^2 - R_{in}^2) \, dy = \pi \int_0^1 (y - y^4) \, dy
= \pi \left( \frac{1}{2}y^2 - \frac{1}{5}y^5 \right) \bigg|_0^1 = \frac{3\pi}{10}

Should you use shells or washers?
Really you should determine whether $dx$ or $dy$ is easier. Then use shells or washers appropriately.

Washers: slice \perp rotation axis
Shells: strip \parallel rotation axis

So you decide orientation of slice or strip. Then you are forced to use one method.

**Ex. 3**

Let $R$ be the region below:

Let $S_x$ be the solid obtained by rotating $R$ about $x$-axis.
(a) washers
(b) shells

Let $S_y$ be the solid obtained by rotating $R$ about the $y$-axis.

(a) washers
(b) shells

**Solution:**

$x$-axis, Washers

Point $P = (x, 1)$
Point $Q = (x, \sin(x))$
Point $C = (x, 0)$

$R_{out} = |PC| = 1$
\[ R_{in} = |Qc| = \sin(x) \]

\[ V = \pi \int_{0}^{\pi/2} (R_{out}^2 - R_{in}^2) \, dx \]

\[ V = \pi \int_{0}^{\pi/2} (1 - \sin(x)^2) \, dx \]

\[ V = \pi \int_{0}^{\pi/2} \cos(x)^2 \, dx \]

\[ u = \cos(x) \quad dv = \cos(x) \, dx \]

\[ du = -\sin(x) \, dx \quad v = \sin(x) \]

\[ V = \pi \left( \cos(x) \sin(x) \bigg|_{0}^{\pi/2} + \int_{0}^{\pi/2} \sin(x)^2 \, dx \right) \]

\[ V = 0 \]

\[ V = \pi \int_{0}^{\pi/2} \sin(x)^2 \, dx = \pi \int_{0}^{\pi/2} (1 - \cos(x)^2) \, dx \]

\[ V = \pi \int_{0}^{\pi/2} 1 \, dx - \pi \int_{0}^{\pi/2} \cos(x)^2 \, dx \]

\[ 2V = \pi \left( \frac{\pi}{2} \right) \quad \Rightarrow \quad V = \frac{\pi^2}{4} \]
\[ P = (0, y) \]
\[ Q = (\sin^{-1}(y), y) \]
\[ h = |\overline{PQ}| = Q_x - P_x = \sin^{-1}(y) \]
\[ F = y \]
\[ dr = dy \]
\[ V = \int_0^1 2\pi hF \, dr = \int_0^1 2\pi y \sin^{-1}(y) \, dy \]

Substitution \( y = \sin(u) \), \( dy = \cos(u) \, du \) is one option, but let's use IBP.
\[ u = \sin^{-1}(y) \quad du = 2\pi y \, dy \]
\[ dv = \frac{1}{\sqrt{1-y^2}} \, dy \quad v = \pi y^2 \]

\[ = \pi y^2 \sin^{-1}(y) \bigg|_0^1 - \int_0^1 \frac{\pi y^2}{\sqrt{1-y^2}} \, dy \]

\[ (\pi \cdot 1 \cdot \frac{\pi}{2} - 0 = \frac{\pi^2}{2}) \]

\[ = \frac{\pi^2}{2} - \int_0^1 \frac{\pi y^2}{\sqrt{1-y^2}} \, dy \]

\[ \text{Method 1: Trig sub with } y = \sin(\theta) \]
\[ \text{Method 2: Integration by parts} \]

Look at integral separately.

\[ \int_0^1 \frac{\pi y^2}{\sqrt{1-y^2}} \, dy = \int_0^1 \pi y \cdot \frac{y}{\sqrt{1-y^2}} \, dy \]

\[ u = \pi y \quad du = \pi \, dy \quad dv = \frac{y}{\sqrt{1-y^2}} \, dy \quad v = -\sqrt{1-y^2} \]
\[
\begin{align*}
\text{area of a quarter-circ}\text{le of radius 1} \quad &\quad \pi y \sqrt{1-y^2} \bigg|_0^1 + \int_0^1 \pi \sqrt{1-y^2} \, dy \\
= &\quad 0
\end{align*}
\]

Going back to (\(\star\))
\[
V = \frac{\pi^2}{2} - \int_0^1 \frac{\pi y^2}{\sqrt{1-y^2}} \, dy
\]
\[
= \frac{\pi^2}{2} - \left( 0 + \pi \cdot \frac{\pi}{4} \right) = \frac{\pi^2}{4}
\]

**y-axis, washers**

\[C = P, \quad Q = (\sin^{-1}(y), y), \quad C = (0, y), \quad \text{R}_{\text{out}} = |Q\overline{C}| = Q_x - C_x = \sin^{-1}(y)\]
\[ R_{\text{in}} = |P_{\text{C}}| = P_x - C_x = 0 \]

\[ V = \pi \int_0^1 (R_{\text{out}}^2 - R_{\text{in}}^2) \, dy = \pi \int_0^1 \sin^{-1}(y)^2 \, dy \]

There are a few ways to proceed:

**Method 1:**

Substitute \( u = \sin^{-1}(y) \), or \( y = \sin(u) \)
(so \( \, dy = \cos(u) \, du \) )

\[ V = \pi \int_0^1 \sin^{-1}(y) \, dy = \pi \int_0^{\pi/2} u^2 \cos(u) \, du \]

Now use IBP twice or tabular IBP

\[
\begin{align*}
  u^2 &\quad \leftrightarrow \quad \cos(u) \\
  2u &\quad \leftrightarrow \quad \sin(u) \\
  2 &\quad \leftrightarrow \quad -\cos(u) \\
  0 &\quad \leftrightarrow \quad -\sin(u)
\end{align*}
\]

\[ = \pi \left( u^2 \sin(u) + 2u \cos(u) - 2 \sin(u) \right) \bigg|_0^{\pi/2} 
\]

\[ = \pi \left( \left( \frac{\pi^2}{4} \cdot 1 + 0 - 2 \right) - (0 + 0 - 0) \right) = \frac{\pi^3}{4} - 2\pi \]

**Method 2:**
Use IBP twice immediately

\[ u = (\sin^{-1}y)^2 \quad dv = dy \]
\[ du = \frac{2 \sin^{-1}y}{\sqrt{1-y^2}} dy \quad v = y \]

\[ V = \pi \int_0^1 \sin^{-1}(y)^2 dy = \]
\[ = \pi y \sin^{-1}(y)^2 \bigg|_0^1 - \int_0^1 \frac{2\pi y \sin^{-1}y}{\sqrt{1-y^2}} dy \]
\[ = (\pi \cdot 1 \cdot \frac{\pi^2}{4} - 0) = \frac{\pi^3}{4} \quad \text{Integration by parts again} \]

\[ u = 2\pi \sin^{-1}(y) \quad dv = \frac{y}{\sqrt{1-y^2}} dy \]
\[ du = \frac{2\pi}{\sqrt{1-y^2}} dy \quad v = -\sqrt{1-y^2} \]

\[ = \frac{\pi^3}{4} - \left( \left(-2\pi \sqrt{1-y^2} \sin^{-1}(y) \right|_0^1 + \int_0^1 2\pi dy \right) \]
\[ = 0 - 0 = 0 \quad \text{Integration by parts again} \quad = 2\pi \]

\[ = \frac{\pi^3}{4} - 2\pi \quad (\text{Phew! Same as Method 1}) \]

y-axis, shells
\[ P = (x, 1) \]
\[ Q = (x, \sin(x)) \]
\[ h = |PQ| = P_y - Q_y = 1 - \sin(x) \]
\[ \bar{r} = x_{\text{strip}} - x_{\text{axis}} = x - 0 = x \]
\[ dr = dx \]
\[ V = 2\pi \int_0^{\pi/2} h \bar{r} \, dr = 2\pi \int_0^{\pi/2} (1 - \sin(x)) \, x \, dx \]

Use IBP twice or tabular IBP:
\[
\begin{align*}
  x & \quad + \quad 1 - \sin(x) \\
  1 & \quad \rightarrow \quad x + \cos(x) \\
  0 & \quad \rightarrow \quad \frac{1}{2} x^2 + \sin(x)
\end{align*}
\]
\[
V = 2\pi \left[ x \left( x + \cos(x) \right) - 1 \left( \frac{1}{2} x^2 + \sin(x) \right) \right] \bigg|_0^{\pi/2} = 2\pi \left[ \frac{1}{2} x^2 + x \cos(x) - \sin(x) \right] \bigg|_0^{\pi/2}
\]
Let $R$ be the region shown below. Find the volume of the solid obtained by rotating $R$ about $y = 8$.

Solution: **Washers**
\[ P = (x, 8 - 2x^2) \]
\[ Q = (x, 4 - x^2) \]
\[ C = (x, 8) \]

\[ R_{\text{out}} = |\overline{QC}| = C_y - Q_y = 4 + x^2 \]
\[ R_{\text{in}} = |\overline{PC}| = C_y - P_y = 2x^2 \]

\[ V = \pi \int_0^2 (R_{\text{out}}^2 - R_{\text{in}}^2) \, dx \]
\[ = \pi \int_0^2 \left( (4 + x^2)^2 - (2x^2)^2 \right) \, dx \]
\[ = \pi \int_0^2 \left( -3x^4 + 8x^2 + 16 \right) \, dx = \frac{512\pi}{15} \]

\[ \text{Shells} \]

\[ y = 8 - 2x^2 \]
\[ y = 4 - x^2 \]
We need two integrals, one for rotating \( R_1 \) and one for rotating \( R_2 \).

**Integral for region \( R_1 \),**

\[
\begin{align*}
P &= (\sqrt{4-y}, y) \quad y = 4 - x^2 \\
Q &= (\sqrt{\frac{8-y}{2}}, y) \quad y = 8 - 2x^2 \\
\end{align*}
\]

\[
h = |PQ| = Q_x - P_x = \sqrt{\frac{8-y}{2}} - \sqrt{4-y}
\]

\[
\bar{r} = 8 - y
\]

\[
dr = dy
\]

\[
V_{R_1} = 2\pi \int_0^4 \left( \sqrt{\frac{8-y}{2}} - \sqrt{4-y} \right) (8-y) \, dy
\]
This integral is a lot of coefficients and arithmetic. Use substitution

\[ u = 4 - y, \quad -du = \, dy \]

\[ V_{R_1} = 2\pi \int_4^0 \left( \frac{\sqrt{u+4}}{\sqrt{2}} - \sqrt{u} \right) (u+4) (-du) \]

\[ = 2\pi \int_0^4 \left( \frac{(u+4)^{3/2}}{\sqrt{2}} - u^{3/2} - 4u^{1/2} \right) du = \cdots \]

(straightforward from there.)

Integral for region \( R_1 \),

\[ P = (0, y) \]

\[ Q = \left( \sqrt{\frac{8-y}{2}}, \, y \right) \quad y = 8 - 2x^2 \]
\[ h = |PQ| = Q_x - P_x = \sqrt{\frac{8-y}{2}} \]
\[ r = 8 - y \]
\[ dr = dy \]
\[ V_{R_2} = 2\pi \int_{4}^{8} \left( \sqrt{\frac{8-y}{2}} \right) (8-y) \, dy \]

This integral is easier than the other.

\[ V_{R_2} = 2\pi \int_{4}^{8} \frac{(8-y)^{3/2}}{\sqrt{2}} \, dy = \pi \sqrt{2} \int_{4}^{8} (8-y)^{3/2} \, dy \]
\[ = -\frac{2}{5} \pi \sqrt{2} (8-y)^{5/2} \bigg|_{4}^{8} = \ldots \]

With some arithmetic we get

\[ V_{R_1} + V_{R_2} = \frac{512\pi}{15} \]

(Washers is a lot easier!)