Section 11.3: Polar Coordinates
Section 11.4: Arc length and Area (Polar)

The \((x,y)\)-coordinates of a point are the Cartesian or rectangular coordinates of that point. But \((x,y)\) is just a label. Someone else might assign another label. What if someone uses a rotated set of axes?

The same point \(P\) may have different labels in different coordinate systems.

* You may have seen this in physics already. Common method of solving
an inclined plane problem is to orient your axes so the x-axis is parallel to the plane.

The polar coordinate system assigns different labels to the point \((x, y)\). We use the coordinates:

- \(\rho\): distance from the origin
- \(\theta\): angle from positive x-axis.

![Diagram showing polar coordinates](image)

Polar to Rectangular

\[ x = \rho \cos(\theta) \]
\[ y = \rho \sin(\theta) \]

Rectangular to Polar

\[ \rho = \sqrt{x^2 + y^2} \]
\[ \tan(\theta) = \frac{y}{x} \]

Careful when solving for \(\theta\)!
What do the “grid lines” in each coordinate system look like?

Constant $x$
Constant $y$

Constant $r$
Constant $\Theta$ (rays)

Ex. 1
Convert the following points.
(a) $P = (3, 2)$ from rect. to polar
(b) $P = (-5, 3)$ from rect. to polar
(c) $P = (3, \frac{\pi}{6})$ from polar to rect.

Solution:
(a) $P = (3, 2)$
$\rho = \sqrt{3^2 + 2^2} = \sqrt{13}$
$\tan(\theta) = \frac{2}{3}$
$\theta = \tan^{-1}\left(\frac{2}{3}\right)$
(b) \[ r = \sqrt{5^2 + 3^2} = \sqrt{34} \]
\[ \tan(\theta) = \frac{3}{-5} \]
\[ \theta \neq \tan^{-1}(-\frac{3}{5}) \]

This is an angle in \([-\frac{\pi}{2}, 0]\).

Better: find reference angle first

\[ \beta = \text{reference angle} \]
\[ \beta = \tan^{-1}\left(\frac{3}{5}\right) \]

\[ \theta = \pi - \beta = \pi - \tan^{-1}\left(\frac{3}{5}\right) \]

(c) \[ x = r \cos(\theta) = 3 \cos\left(\frac{\pi}{6}\right) = \frac{3\sqrt{3}}{2} \]
\[ y = r \sin(\theta) = 3 \sin\left(\frac{\pi}{6}\right) = \frac{3}{2} \]
Remarks

1. Since cosine and sine are $2\pi$-periodic, the $\Theta$-value is not unique. So the two ordered pairs

$$(r, \Theta) \text{ and } (r, \Theta + 2n\pi) \quad (n \in \mathbb{Z})$$

represent the same point.

2. The origin has no well-defined angular coordinate. By convention, the ordered pair $(0, \Theta)$ represents the origin for any $\Theta$.

3. We do allow negative value of $r$...

By definition, the ordered pair $(-r, \Theta)$ is the reflection of $(r, \Theta)$ through the origin. So...

$$(-r, \Theta) \iff (r, \Theta + \pi)$$
Find two polar labels of $P = (-1, 1)$ (given in rect.), one with $r \geq 0$ and one with $r < 0$.

Solution:

(a) $r = \sqrt{1^2 + 1^2} = \sqrt{2}$

$\tan(\theta) = \frac{1}{-1} \implies \theta = \frac{3\pi}{4}$ in second quadrant!
What does it mean to integrate a function $r = f(\theta)$?
Q: What does the region described by the following inequalities look like?

\[ 0 \leq r \leq f(\theta) \quad \text{"radially simple"} \]
\[ \alpha \leq \theta \leq \beta \]

A: See graph.

How do you find the area of such a region?

* Course theme: divide into smaller, easier pieces, examine one piece, then add all the pieces to get a Riemann sum.

Divide the region into circular wedges.
Adding the individual areas gives

\[ A_{\text{total}} \approx \sum_{j=1}^{N} \frac{1}{2} f(\theta_j)^2 \Delta \theta \]

Now taking \( N \to \infty \), we get

\[ A = \int_{\alpha}^{\beta} \frac{1}{2} f(\theta)^2 \, d\theta \]
Find area enclosed by the right half of the curve \( r = 4 \sin(\theta) \).

**Solution:**

First graph the curve. Let's convert to rectangular coordinates.

\[
\begin{align*}
\begin{cases}
  x = r \cos(\theta) \\
  y = r \sin(\theta)
\end{cases}
\end{align*}
\]

\[
\begin{align*}
  r &= 4 \sin(\theta) \\
  r &= 4 \left( \frac{y}{r} \right) \\
  r^2 &= 4y \\
  x^2 + y^2 &= 4y \\
  x^2 + y^2 - 4y &= 0 \\
  x^2 + y^2 - 4y + 4 &= 4 \\
  x^2 + (y - 2)^2 &= 4
\end{align*}
\]

So the curve is a circle with center \((0, 2)\) and radius 2.
This region can be described as
\[ 0 \leq \theta \leq \pi/2 \]
\[ 0 \leq r \leq 4 \sin(\theta) \]

So the area of the region is
\[
A = \int_0^{\pi/2} \frac{1}{2} (4 \sin \theta)^2 \, d\theta = \int_0^{\pi/2} 8 \sin^2 (\theta) \, d\theta \\
= \int_0^{\pi/2} 4 (1-\cos (2\theta)) \, d\theta = 2\pi
\]

**Ex. 4**

Find area enclosed by one petal of the rose \( r = \sin (3\theta) \).

**Solution:**
First graph \( r = \sin (3\theta) \). Converting to Cartesian coordinates is very difficult.
We will examine the graph of $r = \sin (3\theta)$ in the $\Theta r$-plane, then use it to graph $r = \sin (3\theta)$ in the $xy$-plane.
Now calculate the area in one petal

\[ r = \sin(3\theta) \]

\[
A_{\text{one petal}} = \frac{1}{2} \int_0^{\pi/3} \sin^2(3\theta) \, d\theta
\]

double angle: \( \sin^2(\beta) = \frac{1}{2} - \frac{1}{2} \cos(2\beta) \)

\[
= \frac{1}{4} \left[ \int_0^{\pi/3} (1 - \cos(6\theta)) \, d\theta \right]
\]

\[
= \frac{1}{4} \left( \theta - \frac{1}{6} \sin(6\theta) \right) \bigg|_0^{\pi/3}
\]

\[
= \frac{1}{4} \left( \frac{\pi}{3} - 0 \right) - \frac{1}{4} \left( 0 - 0 \right) = \frac{\pi}{12}
\]

\[
\text{Area Between Curves}
\]
Ex. 5
Find area outside the circle \( r = 1 \) and inside the circle \( r = 4 \cos \theta \).

Solution:
First graph the equations.

\[
\begin{align*}
r &= 4 \cos \theta \\
r &= 4 \left( \frac{x}{r} \right)
\end{align*}
\]
Find the points of intersection.

\[ 4 \cos(\beta) = 1 \quad \Rightarrow \quad \beta = \cos^{-1}\left(\frac{1}{4}\right) \]

Now set up integral.

\[
\text{fouter}(\theta) = 4 \cos(\theta)
\]
\( f_{\text{inner}} (\Theta) = 1 \)

\[
A = \frac{1}{2} \int_{-\beta}^{\beta} \left( 16 \cos^2 (\Theta) - 1 \right) d\Theta
\]

\[
= \frac{1}{2} \cdot 2 \cdot \int_{0}^{\beta} \left( 16 \cdot \frac{1}{2} \left( 1 + \cos (2\Theta) \right) - 1 \right) d\Theta
\]

\[
\text{symmetry double-angle}
\]

\[
= \int_{0}^{\beta} \left( 7 + 8 \cos (2\Theta) \right) d\Theta
\]

\[
= \left( 7\Theta + 4 \sin (2\Theta) \right) \bigg|_{0}^{\beta} = 7\beta + 4\sin (2\beta)
\]

\[
= 7\beta + 8 \sin (\beta) \cos (\beta)
\]

\[
\left( \sin (2\beta) = 2\sin (\beta) \cos (\beta) \right)
\]

\[
= 7 \cos^{-1} \left( \frac{1}{4} \right) + 8 \cdot \frac{\sqrt{15}}{4} \cdot \frac{1}{4}
\]

Ex. 6

Find area enclosed by lemniscate \( r^2 = \cos (2\Theta) \). (“ribboned”)
Solution:
Cartesian coordinates probably too hard

\[ r^2 = \cos(2\theta) \]
\[ r^2 = \cos(\theta)^2 - \sin(\theta)^2 \]
\[ r^2 = \left(\frac{x}{r}\right)^2 - \left(\frac{y}{r}\right)^2 \]
\[ r^4 = x^2 - y^2 \]

\[ (x^2 + y^2)^2 = x^2 - y^2 \]

We will graph this using the method of \boxed{Ex. 4}.

Now graph the equation in the xy-plane piece by piece.
\( r = \sqrt{\cos(2\theta)} \)

\[ \cos(2\theta) < 0 \text{ for } \theta \in \left(\frac{\pi}{4}, \frac{3\pi}{4}\right) \cup \left(\frac{5\pi}{4}, \frac{7\pi}{4}\right) \]

Now find area enclosed by curve

\[ A_{\text{total}} = 2 A_{\text{right loop}} = 2 \cdot \frac{1}{2} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos(2\theta) \, d\theta \]
Arc Length in Polar Coordinates

The curve \( r = f(\theta) \) has the following parametrization:

\[
x(\theta) = f(\theta) \cos(\theta) \\
y(\theta) = f(\theta) \sin(\theta)
\]

So the arc length differential \( ds \) is:

\[
ds = \sqrt{(\frac{dx}{d\theta})^2 + (\frac{dy}{d\theta})^2} \ d\theta
\]

\[
\frac{dx}{d\theta} = -f(\theta) \sin(\theta) + f'(\theta) \cos(\theta)
\]

\[
\frac{dy}{d\theta} = f(\theta) \cos(\theta) + f'(\theta) \sin(\theta)
\]

After some magic ....

\[
(\frac{dx}{d\theta})^2 + (\frac{dy}{d\theta})^2 = f(\theta)^2 + f'(\theta)^2
\]

So arc length formula for polar:
Consider the cardioid \( r = 1 - \cos(\theta) \).

(a) Graph in xy-plane.

(b) Find the total length.

(c) Find the total enclosed area.

Solution:

(a) Graph in xy-plane.
(b) Use the arc length formula.

\[ s = \int_{0}^{2\pi} \sqrt{f'(\theta)^2 + f(\theta)^2} \, d\theta \]

\[ = \int_{0}^{2\pi} \sqrt{(1-\cos(\theta))^2 + (\sin(\theta))^2} \, d\theta \]

\[ = \int_{0}^{2\pi} \sqrt{1 - 2\cos(\theta) + \cos^2(\theta) + \sin^2(\theta)} \, d\theta \]

\[ = \int_{0}^{2\pi} \sqrt{2 - 2\cos(\theta)} \, d\theta \]

\[ 2\sin^2\left(\frac{\theta}{2}\right) = 1 - \cos(\theta) \]

\[ = \int_{0}^{2\pi} \sqrt{2} \sqrt{2} \sqrt{\sin^2\left(\frac{\theta}{2}\right)} \, d\theta \]

\[ = \int_{0}^{2\pi} 2 \sin\left(\frac{\theta}{2}\right) \, d\theta = 8 \]

(c) \[ A = \int_{0}^{2\pi} \frac{1}{2} \left(1 - \cos(\theta)\right)^2 \, d\theta \]

\[ = \int_{0}^{2\pi} \frac{1}{2} \left((-2\cos(\theta) + \cos(\theta)^2)\right) \, d\theta \]
\[ = \int_0^{2\pi} \frac{1}{2} \left( 1 - 2\cos(\theta) + \frac{1}{2} + \frac{1}{2} \cos(2\theta) \right) d\theta \]

\[ = \int_0^{2\pi} \left( \frac{3}{4} - \cos(\theta) + \frac{1}{4} \cos(2\theta) \right) d\theta \]

\[ = 0 \quad = 0 \]

\[ = \frac{3}{4} \cdot 2\pi = \frac{3\pi}{2} \]