Section 10.6: Power Series

Definition #1:
A power series with center \( c \) is an infinite series of the form

\[
F(x) = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \ldots
\]

\[
F(x) = \sum_{n=0}^{\infty} a_n (x-c)^n \quad \text{(convention: } 0^0 = 1)\]

Theorem #1:
For any power series \( \sum_{n=0}^{\infty} a_n (x-c)^n = f(x) \), exactly one of the following is true:

1. There is a positive number \( R \), called the radius of convergence, such that \( f(x) \) converges absolutely for \( |x-c| < R \) and \( f(x) \) diverges for \( |x-c| > R \).

2. \( f(x) \) converges for all \( x \). (radius of convergence is \( R = \infty \))
In other words, the convergence of a power series looks like:

- **Convergence**: \( c - R \) to \( c + R \)
- **Divergence**: anything can happen at \( x = c \pm R \).

Primary goal for a given series is to find the **interval of convergence**.

**Ex. 1**

Determine **ROC** (radius of conv.) and **IOC** (interval of conv.) for the series

\[
f(x) = \sum_{n=0}^{\infty} \frac{x^n}{2^n}
\]

**Solution**: 

Note the center is $x = 0$. Find the ROC using Ratio or Root Test.

**Ratio Test**

\[
\rho = \lim_{n \to \infty} \left| \frac{b_{n+1}}{b_n} \right| = \lim_{n \to \infty} \left| \frac{x^{n+1}}{2^{n+1}} \cdot \frac{2^n}{x^n} \right| = \lim_{n \to \infty} \left| \frac{x}{2} \cdot \frac{2^n}{2^{n+1}} \right| = \lim_{n \to \infty} \frac{|x|}{2} = \frac{|x|}{2}
\]

So we conclude:

- $\rho < 1$: \((\frac{|x|}{2} < 1), \text{ or } (|x| < 2)\) \text{ convergence}
- $\rho > 1$: \((\frac{|x|}{2} > 1), \text{ or } (|x| > 2)\) \text{ divergence}
- $\rho = 1$: \((\frac{|x|}{2} = 1), \text{ or } x = \pm 2\) ????

So right now we know that the ROC is $R = 2$ and the I0C is at least as large as $(-2, 2)$, but it may include one or both endpoints. We test the two endpoints separately.
x = -2: \[ \sum_{n=0}^{\infty} \frac{(-2)^n}{2^n} = \sum_{n=0}^{\infty} (-1)^n \]

This series diverges by Nth term div. test since \( \lim_{n \to \infty} (-1)^n \neq 0 \).

x = 2: \[ \sum_{n=0}^{\infty} \frac{2^n}{2^n} = \sum_{n=0}^{\infty} 1 \]

This series diverges by Nth term div. test since \( \lim_{n \to \infty} (-1)^n \neq 0 \).

So our final answer is:

**ROC**: 2

**IOC**: (-2, 2)

---

**Ex. 2**

Determine ROC and IOC for the series

\[ f(x) = \sum_{n=0}^{\infty} \frac{(-5)^n}{n!} (x+1)^n \]

**Solution**: (Center: \( c = -1 \)) Use Ratio Test to get ROC.
\[ p = \lim_{n \to \infty} \left| \frac{b_{n+1}}{b_n} \right| = \lim_{n \to \infty} \left| \frac{(-5)^{n+1} (x+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{(-5)^n (x+1)^n} \right| \]

\[ = \lim_{n \to \infty} \left| \frac{(-5)^{n+1} \cdot n!}{(-5)^n (n+1)!} \cdot \frac{(x+1)^{n+1}}{(x+1)^n} \right| \]

\[ = \lim_{n \to \infty} \left| (-5) \cdot \frac{1}{n+1} \cdot (x+1) \right| \]

\[ = \lim_{n \to \infty} \frac{5}{n+1} \left| x+1 \right| = 0 \quad x \text{ is fixed and } n \to \infty \]

So we conclude: since \( p < 1 \) for all values of \( x \), the series converges for all \( x \).

**ROC:** \( \infty \)

**IOC:** \( (-\infty, \infty) \)

---

**Ex. 3**

Find ROC and IOC for the series

\[ \sum_{n=1}^{\infty} \frac{\sqrt{n^3+1}}{n^2 \cdot 3^n} \ (x-2)^n \]

**Solution:**
Use Root Test.

\[ f = \lim_{n \to \infty} \left| b_n \right|^\frac{1}{n} = \lim_{n \to \infty} \left| \frac{\sqrt{n^3 + 1}}{n^2 \cdot 3^n} (x-2)^n \right|^\frac{1}{n} \]

\[ = \lim_{n \to \infty} \frac{1}{2n} \frac{(n^3 + 1)^{\frac{2n}{n}}}{n^{2n} \cdot 3} (x-2) \]

\[ = \frac{|x-2|}{3} \lim_{n \to \infty} \left[ \frac{(n^3 + 1)^{\frac{1}{n}}}{(n^{\frac{1}{n}})^2} \right]^{\frac{1}{2}} = \frac{|x-2|}{3} \]

\[ \lim_{n \to \infty} n^{\frac{1}{n}} = 1 \]
\[ \lim_{n \to \infty} (n^3 + 1)^{\frac{1}{n}} = 1 \]

Let \( L = \lim_{n \to \infty} (n^3 + 1)^{\frac{1}{n}} \). Then

\[ \ln(L) = \lim_{n \to \infty} \frac{\ln(n^3 + 1)}{n} = \lim_{n \to \infty} \frac{3n^2}{n^3 + 1} = 0 \]

Since \( \ln(L) = 0 \), \( L = 1 \).

So we conclude that:

\[ f < 1 \iff (x-2) < 3 \], convergence
\[ p > 1: \quad |x - 2| > 3, \quad \text{divergence} \]
\[ p = 1: \quad |x - 2| = 3, \quad ??? \]

Test endpoints separately:

\[ x = -1: \quad \sum_{n=1}^{\infty} \sqrt{n^3+1} \cdot (-3)^n = \sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n^3+1}}{n^2} \]

Put \( a_n = \frac{\sqrt{n^3+1}}{n^2} = \sqrt{\frac{n^3+1}{n^4}} = \sqrt{\frac{1}{n} + \frac{1}{n^4}} \)

- \( a_n \geq 0 \) (obvious)
- \( \lim_{n \to \infty} a_n = \sqrt{0 + 0} = 0 \)
- \( \{a_n\} \) is decreasing
- \( \frac{1}{n} \) and \( \frac{1}{n^4} \) are decreasing
- \( \frac{1}{n} + \frac{1}{n^4} \) is decreasing
- \( \sqrt{\frac{1}{n} + \frac{1}{n^4}} \) is decreasing

So our series converges by AST.

\[ x = 5: \quad \sum_{n=1}^{\infty} \frac{\sqrt{n^3+1}}{n^2 \cdot 3^n} = \sum_{n=1}^{\infty} \frac{\sqrt{n^3+1}}{n^2} \]

\[ \frac{\sqrt{n^3+1}}{n^2} \sim \frac{\sqrt{n^3}}{n^2} \sim \frac{n^{3/2}}{n^2} \sim \frac{1}{n^{1/2}} \]
\[
\lim_{n \to \infty} \frac{\sqrt{n^3 + 1}}{n^2} \overset{1}{\div} \frac{1}{\sqrt{n}} = \lim_{n \to \infty} \sqrt{\frac{n^4 + n}{n^4}} = \sqrt{1} = 1
\]

Since \( \sum \frac{1}{\sqrt{n}} \) diverges by \( p \)-test \( (p=\frac{1}{2} \leq 1) \), our series diverges by LCT.

Final answer for power series:
- **ROC:** \( R = 3 \)
- **I0C:** \([-1, 5)\)

We can also use known power series to derive new power series. If \( |x| < 1 \),
\[
1 + x + x^2 + x^3 + \ldots = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}
\]
We can use the geometric series to derive other series. In the sequel,
\[
g(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \quad \text{Converges for } |x| < 1
\]
Ex. 4

Find a power series representation of 
\[ f(x) = \frac{1}{1 - 2x} \] with center \( c = 0 \). Find its ROC and IOC.

**Solution:**

We use the geometric series \( g(x) \). Observe that 
\[ f(x) = g(2x) \]

So then we have...

\[ f(x) = \frac{1}{1 - 2x} = g(2x) \quad g(x) \text{ converges for } |x| < 1 \]

\[ f(x) = \sum_{n=0}^{\infty} (2x)^n \quad \text{converges for } |2x| < 1 \]

\[ f(x) = \sum_{n=0}^{\infty} 2^n x^n = 1 + 2x + 4x^2 + 8x^3 + \ldots \]

We do not need to use Ratio Test.
to find the ROC or IOC! Why?

Our series for \( f(x) \) converges for 
\( |2x| < 1 \), or \( |x| < \frac{1}{2} \).

center: \( c = 0 \)

\( \text{ROC: } R = \frac{1}{2} \)

\( \text{IOC: } (- \frac{1}{2}, \frac{1}{2}) \)

Theorem #2:
Consider the power series 
\[
f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n
\]
Suppose \( f \) has an ROC of \( R > 0 \). (Possibly \( R = \infty \).) Then \( f \) is differentiable and integrable on \((c-R, c+R)\). The derivatives and integrals are obtained by term-by-term operations:

How do the ROC and IOC change by differentiating or integrating?
In general, the derivative $F'$ and function $F$ have the **same** ROC. (Same for antiderivatives.)

Differentiating a power series can only **lose** you convergence at one or both endpoints.

Integrating a power series can only **gain** you convergence at one or both endpoints.

**Ex:** $f(x) = \sum_{n=1}^{\infty} \frac{\sqrt{n^3+1}}{n^2 \cdot 3^n} (x-2)^n$

converges on $[-1, 5)$. Then $f$ is diff. and integrable on $(-1, 5)$.

- $f'(x) = \sum_{n=1}^{\infty} \frac{\sqrt{n^3+1}}{n^2 \cdot 3^n} \frac{d}{dx} (x-2)^n$

- $f'(x) = \sum_{n=1}^{\infty} \frac{\sqrt{n^3+1}}{n \cdot 3^n} (x-2)^{n-1}$
\[ \int_a^b f(x) \, dx = \sum_{n=1}^{\infty} \frac{\sqrt{n^3 + 1}}{n^2 \cdot 3^n} \int_a^b (x-2)^n \, dx \]

**Ex. 5**

Find a power series representation of \( f(x) = \frac{1}{(1-x)^2} \) with center \( c=0 \). Find the ROC and IOC.

**Solution:**

Recall: \( g(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \), \( |x| < 1 \)

Observe that \( g'(x) = f(x) \)

**Note:** We also have \( f(x) = g(x)^2 \).

So \( f(x) = (1 + x + x^2 + x^3 + \ldots)^2 = \ldots \), which is more difficult to compute than \( g'(x) \).

\[ g(x) = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \ldots \quad |x| < 1 \]
Differentiation can only lose you endpoint convergence, but we don't have endpoint convergence anyway!

So we have

\[
f(x) = \frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} nx^{n-1}, \quad |x| < 1
\]

Ex. 6

Find a power series representation for \( f(x) = \tan^{-1}(x) \). Find the ROC and IOC.

Solution:

Note: \[ \frac{d}{dx} \tan^{-1}(x) = \frac{1}{1+x^2} \] geometric series ??
Recall: \( g(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \), \( |x| < 1 \)

So we have (replacing \( x \) with \( -x^2 \))...

\[
\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n
\]

\[
\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}
\]

\[
\int \frac{1}{1+x^2} \, dx = \int \sum_{n=0}^{\infty} (-1)^n x^{2n} \, dx
\]

\[
\tan^{-1}(x) + C = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}
\]

\[
\tan^{-1}(x) + C = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots
\]

To find value of \( C \), substitute \( x = 0 \).
\[
\tan^{-1}(0) + C = 0 - 0 + 0 + \ldots \\
\implies C = 0
\]
Hence we have found
\[
\tan^{-1}(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}
\]
Where does this series converge?
Since the power series for \( \frac{1}{1+x^2} \) converges for \(-1 < x < 1\), the power series for \(\tan^{-1}(x)\) converges for \(-1 < x < 1\). So we check \(x = -1\) and \(x = 1\).

\[x = -1\]
\[
\sum_{n=0}^{\infty} (-1)^n \frac{(-1)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{2n+1}
\]
Converges by AST (let \( b_n = \frac{1}{2n+1} \))
- \( b_n > 0 \)
- \( b_n \to 0 \)
- \( \sum b_n \) is decreasing

\[x = 1\]
\[
\sum_{n=0}^{\infty} (-1)^n \frac{(1)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}
\]
Converges by AST (or it's just...
So the IOC of our series for \( \tan^{-1}(x) \) is \([-1, 1]\).

\[
\tan^{-1}(1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots
\]

\[
\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots
\]

**Ex. 7**

Determine a power series representation of \( f(x) = \frac{1}{8 + x^3} \) with center \( c = 0 \).

Find the ROC and IOC.

**Solution:**

Start with the fact:

\[
g(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad (|x| < 1)
\]

Observe that
\[
f(x) = \frac{1}{8} \cdot \frac{1}{1 + \frac{x^3}{8}} = \frac{1}{8} \cdot \frac{1}{1 - (-\frac{x^3}{8})}
\]

\[
f(x) = \frac{1}{8} \sum (-\frac{x^3}{8}) = \frac{1}{8} \sum_{n=0}^{\infty} (-\frac{x^3}{8})^n
\]

\[
f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{3n}}{8^{n+1}}
\]

(where does this converge?)

Recall that \( g(x) \) converges for \(|x| < 1\).

So \( f(x) \) converges for \(|-\frac{x^3}{8}| < 1\), or \(|x| < 2\). So we have:

\[\text{ROC: } 2\]

\[\text{TOC: } (-2,2)\]